Online Appendix

This Appendix provides supplementary material for "Directed Search on the Job, Heterogeneity, and Aggregate Fluctuations."

A Proof of Theorem 3

In order to keep the exposition as simple as possible, we prove the theorem for $N = 2$. The proof for $N \geq 3$ is similar.

Theorem 2 guarantees that there exists at least one BRE for the economy with ex-ante homogeneous workers of type s_i, $i = 1, 2$. Let $(\theta^i, R^i, m^i, U^i, J^i, c^i)$ be any BRE for the economy with ex-ante homogeneous types of type s_i. Let $(\theta, \phi, R, m, U, J, c)$ denote a candidate BRE for the economy with ex-ante heterogeneous agents. Fix an arbitrary $y \in Y$ and, without loss in generality, suppose that $\theta_1(\underline{x}, y) \geq \theta_2(\underline{x}, y)$. Now, choose the market tightness function, θ, and the distribution of applicants, ϕ, as follows. For all $(x_1, x_2) \in X \times (\underline{x}, \overline{x}]$, let $\theta(x_1, x_2) = \min\{\theta^1(x_1), \theta^2(x_2)\}$ and let $\phi(x_1, x_2) = (1, 0)$ if $\theta^1(x_1) < \theta^2(x_2)$ and $\phi(x_1, x_2) = (0, 1)$ if $\theta^1(x_1) > \theta^2(x_2)$. For all $x_1 \in X$, let $\theta(x_1, \underline{x}) = \theta^1(x_1)$ and $\phi(x_1, \underline{x}) = (1, 0)$. That is, for $x_2 > \underline{x}$, we set the tightness of submarket (x_1, x_2) to the minimum between the tightness of submarket x_1 in an economy with ex-ante homogeneous workers of type 1, and the tightness of submarket x_2 in an economy with ex-ante homogeneous workers of type 2. For $x_2 = \underline{x}$, we set the tightness of submarket (x_1, x_2) to be the tightness of submarket x_1 in an economy with ex-ante heomogeneous workers of type 1. Notice that, in the previous expressions, we have omitted the dependence of various functions on y. We shall do the same in the remainder of the proof.

Next, choose the search value function, R, the search policy function, m, the profit function, J, and the unemployment value function, U, as follows. For all $V \in X$ and $i = 1, 2$, let $R(s_i, V) = R^i(V)$. For all $V \in X$, let $m(s_1, V) = (m^1(V), \underline{x})$ and $m(s_2, V) = (\underline{x}, m^2(V))$. For all $(V, z) \in X \times Z$ and $i = 1, 2$, let $J(s_i, V, z) = J^i(V, z)$ and $c(s_i, V, z) = c^i(V, z)$. For $i = 1, 2$, let $U(s_i, y) = U^i(y)$. In words, the lifetime utility of worker s_i in an economy with ex-ante heterogeneous workers is set equal to the lifetime utility of a worker in an economy with ex-ante homogeneous workers of type s_i. Similarly, the profits of a firm from employing a worker s_i in an economy with ex-ante heterogeneous workers are equal to the profits of a firm in an economy with ex-ante homogeneous workers of type s_i.

Now, we verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium conditions (i)-(iv) and, hence, it is a BRE for the economy with ex-ante heteorogeneous workers. First, we verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium condition (iv). Consider a submarket $(x_1, x_2) \in X^2$ such that either $\theta^1(x_1) \leq \theta^2(x_2)$ or $x_2 = \underline{x}$. In this case, we have

$$\begin{aligned} & q(\theta(x_1, x_2)) \textstyle\sum_{i=1}^{2} \phi_i(x_1, x_2) J(s_i, x_i, z_0) = q(\theta^1(x_1)) J^1(x_1, z_0) \leq k, \\ & \theta(x_1, x_2) = \theta^1(x_1) \geq 0, \end{aligned} \tag{1}$$

with complementary slackness. The first line in (1) denotes as $\phi_i(x_1, x_2)$ the i-th component of the vector $\phi(x_1, x_2)$ and makes use of the equations $\phi(x_1, x_2) = (1, 0)$, $J(s_i, x_i, z_0) = J^i(x_i, z_0)$, the second line makes use of the equation $\theta(x_1, x_2) = \theta^1(x_1)$, and both lines use the fact that $(\theta^1, R^1, m^1, U^1, J^1, c^1)$ is a BRE. The inequalities in (1) imply that the equilibrium condition (iv) is satisfied for all $(x_1, x_2) \in X^2$ such that either $\theta^1(x_1) \leq \theta^2(x_2)$ or $x_2 = \underline{x}$. Using a similar argument, we can prove that the equilibrium condition (iv) is satisfied for all other submarkets.

Next, we verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium condition (i). Consider an arbitrary $x_1 \in X$. For all $x_2 \in (\underline{x}, \overline{x}]$, the tightness of submarket (x_1, x_2) is $\theta(x_1, x_2) \leq \min\{\theta^1(x_1), \theta^2(x_2)\}$. For $x_2 = \underline{x}$, the tightness of submarket (x_1, x_2) is $\theta(x_1, x_2) = \theta^1(x_1)$. Since these results hold for an arbitrary x_1, we have that

$$\begin{aligned} &\max_{(x_1,x_2)\in X^2} p(\theta(x_1, x_2))\,(x_1 - V) \\ &= \max_{x_1 \in X} p(\theta^1(x_1))(x_1 - V) \\ &= R^1(V) = R(s_1, V), \end{aligned} \tag{2}$$

where the third line makes use of the fact that $(\theta^1, R^1, m^1, U^1, J^1, c^1)$ is a BRE. Moreover, we have that

$$p(\theta(m(s_1, V)))\,(m_1(s_1, V) - V) = p(\theta^1(m^1(V)))(m^1(V) - V), \tag{3}$$

where $m_1(s_1, V)$ denotes the first component of the vector $m(s_1, V)$. Taken together, equations (2) and (3) imply that the equilibrium condition (i) is satisfied for all $V \in X$ and $i = 1$. Using a similar argument, we can prove that the equilibrium condition (i) is satisfied also for $i = 2$. Moreover, notice that the distribution of applicants ϕ across types is consistent with the worker's equilibrium search strategy m.

Finally, it is straightforward to verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium conditions (ii) and (iii). Hence, $(\theta, \phi, R, m, U, J, c)$ is a BRE for the economy with ex-ante heterogeneous agents.