

# Microstructure Bluffing with Nested Information: Mathematical Appendix

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**Lemma 1** *In any candidate non-bluffing equilibrium,*

**a** *following any first period buy (resp., sell) by the informed leader, the second period price*

*$p(h_2) = 1$  (resp.,  $p(h_2) = -1$ ) with the followers trading  $x_2^F(I) \geq 1 + x_2^F(N)$  (resp.,  $x_2^F(I) \leq -1 + x_2^F(N)$ ) in the second period;*

**b** *the first period order of the followers does not depend on their signal  $s$  and, without loss*

*of generality, we may take  $x_1^F(s) = 0$  for all  $s$ .*

**Proof of Lemma 1:** (a) Without loss of generality, we consider the case  $x_1^I > 0$ . In a non-bluffing equilibrium when  $s = I$ , following an order of  $x_1^I = z = x_1 - x_1^F > 0$ , the followers conclude that the leader's type is  $v = 1$ .

Suppose, contrary to the claim, that  $p(h_2) < 1$  for a positive measure of  $h_2$ , given  $z$  and  $s = I$ . Then such histories are generated by strictly positive probability conditional also on  $s = N$  and  $z$ . This also implies that  $E[p(h_2)|z, N] > 0$  so that followers will sell when

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$s = N$ , i.e.,  $x_2^F(N) < 0$  after such a first period history. It follows that the maximum possible second period net order flow conditional on  $s = N$  is equal to  $1 + x_2^F(N)$ , while the minimum possible order flow conditional on  $s = I$  is at least  $x_2^F(I)$ . For the market makers to put positive weight on both  $s = I$  and  $s = N$  after a positive measure of histories following  $z$ , we then need  $x_2^F(I) < 1 + x_2^F(N) < \Delta$ . But since  $E[p(h_2)|z, I] < 1$  by supposition, followers earn strictly positive profits when  $s = I$  and we cannot have  $x_2^F(I) < \Delta$ , yielding the desired contradiction. But then market makers will be able to discern the state  $s = I$  from the state  $s = N$  with probability 1 upon observing the second period order flow, implying that  $x_2^F(I) \geq 1 + x_2^F(N)$ .

(b) As shown in the proof of Lemma 2, the informed trader's first period strategy represented by the distribution  $Q(\cdot|v)$  must be atomless and admit a density  $q(\cdot|v)$  with support denoted by  $Z_v$  for all  $v \in \{-1, 1\}$  and we refer the reader to those arguments since they also apply here. Since the candidate equilibrium is non-bluffing  $z \geq 0$  (resp.,  $z \leq 0$ ) if  $z \in Z_1$  (resp.,  $z \in Z_{-1}$ ).

Let  $\delta = x_1^F(I) - x_1^F(N)$  and let  $z = x_1 - x_1^F(I) = x_1 - x_1^F(N) - \delta$  so that for any given first period order flow  $z = x_1^I(v) = x_1^N - \delta$ . Then, for all  $z \in [-1 - \delta, 1 - \delta] \cup Z_1 \cup Z_{-1}$ , the price as a function of  $z$  is

$$p(z) = \frac{\alpha[q(z|1) - q(z|-1)]}{\alpha q(z|1) + 2(1 - \alpha)g(z + \delta) + \alpha q(z|-1)}$$

using Bayes' Rule.

We wish to show that  $\delta = 0$  in any candidate non-bluffing equilibrium. Suppose to the contrary  $\delta \neq 0$  and focus on the case  $\delta > 0$  first.

**Case 1:**  $0 < \delta \leq 1$

In this case, the first period prices that can arise on the path of play in any non-bluffing equilibrium are given by

$$(1) \quad p(z) = \begin{cases} 1 & \text{if } z > 1 - \delta, z \in Z_1 \\ \frac{\alpha[q(z|1) - q(z|-1)]}{\alpha q(z|1) + 2(1-\alpha)g(\delta) + \alpha q(z|-1)} & \text{if } z \in [-1 - \delta, 1 - \delta] \\ -1 & \text{if } z < -1 - \delta, z \in Z_{-1} \end{cases}$$

and arbitrary otherwise.

Now the indifference condition for type  $v = 1$  of the leader says that for all  $z \in Z_1$ ,

$$z[1 - p(z)] = k_1 > 0$$

where the strict inequality follows from inspecting (1) and observing that the leader can always buy an order of size  $z \in [0, 1 - \delta)$  at a price  $p(z) \in [0, 1)$  for strictly positive expected profits. It then follows that  $\sup Z_1 \leq 1 - \delta$  and  $\inf Z_1 \geq k_1$  so that solving the indifference condition

$$z[1 - p(z)] = z \frac{2(1 - \alpha)g(z + \delta)}{\alpha q(z|1) + 2(1 - \alpha)g(z + \delta)} = k_1$$

for  $q(z|1)$  yields

$$q(z|1) = 2 \frac{1 - \alpha}{\alpha} g(z + \delta) \left[ \frac{z}{k_1} - 1 \right]; z \in (k_1, 1 - \delta]$$

showing also that  $q$  has convex support.

Similarly exploiting type  $v = -1$ 's indifference condition

$$z[-1 - p(z)] = k_{-1} > 0$$

we obtain

$$q(z|-1) = 2 \frac{1 - \alpha}{\alpha} g(z + \delta) \left[ -\frac{z}{k_{-1}} - 1 \right]; z \in [-1 - \delta, -k_{-1})$$

If we substitute the expressions for  $q(\cdot|1)$  and  $q(\cdot|-1)$  into expression (1) we see that the price

$$p(z) = \begin{cases} -1 - \frac{k_{-1}}{z} & \text{if } z \in [-1 - \delta, -k_{-1}) \\ 0 & \text{if } z \in [-k_{-1}, k_1] \\ 1 - \frac{k_1}{z} & \text{if } z \in (k_1, 1 - \delta] \end{cases}$$

for all  $z$  that can arise on the path of play in the candidate non-bluffing equilibrium. Furthermore, the constants  $k_1$  and  $k_{-1}$  are obtained from the identities

$$\int_{k_1}^{1-\delta} q(z|1)dz = 1 \text{ and } \int_{-(1+\delta)}^{-k_{-1}} q(z|-1)dz = 1$$

Using the fact that  $g$  is uniform, this yields

$$\frac{k_{-1}}{k_1} = \left[ \frac{1 + \delta - k_{-1}}{1 - \delta - k_1} \right]^2$$

allowing us to conclude that  $0 < k_{-1} - k_1 < 2\delta$ , a fact that we use later.

We now compute the expected first period price for each follower signal  $s \in \{I, N\}$ :

$$E[p(x_1)|I] = \frac{1 - \alpha}{2\alpha} \left[ \int_{-(1+\delta)}^{-k_{-1}} \left(-1 - \frac{k_{-1}}{z}\right) \left(-1 - \frac{z}{k_{-1}}\right) dz + \int_{k_1}^{1-\delta} \left(1 - \frac{k_1}{z}\right) \left(\frac{z}{k_1} - 1\right) dz \right]$$

and

$$E[p(x_1)|N] = \frac{1 - \alpha}{2\alpha} \left[ \int_{-(1+\delta)}^{-k_{-1}} \left(-1 - \frac{k_{-1}}{z}\right) dz + \int_{k_1}^{1-\delta} \left(1 - \frac{k_1}{z}\right) dz \right]$$

Performing the integrations, it is easy to verify that  $E[p(x_1)|I] + E[p(x_1)|N]$  equals

$$(2) \quad \frac{1 - \alpha}{2\alpha} \left[ -(k_{-1} - k_1 - 2\delta) - \frac{1}{2k_{-1}}(k_{-1}^2 - (1 + \delta)^2) + \frac{1}{2k_1}((1 - \delta)^2 - k_1^2) \right] > 0$$

where the inequality follows from recalling that  $k_{-1} - k_1 < 2\delta$ ,  $k_{-1} < 1 + \delta$  and  $k_1 < 1 - \delta$ .

Notice next that since  $\delta = x_1^F(I) - x_1^F(N) \leq 1 < \Delta$ , we must have  $x_1^F(s) \in (-\Delta, \Delta)$  for some  $s \in \{I, N\}$ . For such  $s$ , followers must be indifferent between trading or not, i.e., must

earn zero expected profits implying  $E[p(x_1)|s] = 0$ . Using (2), we then have  $E[p(x_1)|s'] > 0$  for  $s' \neq s$ . But then  $E[p(x_1)] = \alpha E[p(x_1)|I] + (1 - \alpha)E[p(x_1)|N] > 0$ , a contradiction with the fact that  $E[p(x_1)] = E[E[v|x_1]] = E[v] = 0$ , using the law of iterated expectations. This completes the proof for this case.

**Case 2:**  $\delta > 1$ .

In this case, it is easy to see that conditional on  $s = N$  followers know that the market makers cannot attach positive probability to the type  $v = 1$  of the leader upon observing any first period order flow that can arise. It follows that, given  $s = N$ ,  $p \leq 0$  with probability 1 and  $p < 0$  with positive probability on the path of play. But then  $E[p(x_1)|N] < 0$  so that followers earn strictly positive profits when  $s = N$  and  $x_1^F(N) = \Delta$ . But then we cannot have  $x_1^F(I) - x_1^F(N) = \delta > 0$ .

The proof for the case  $\delta < 0$  is symmetric, with the two types of the informed trader switching roles relative to the case  $\delta > 0$ . For the last part of the claim, notice that if  $\delta = 0$ , then  $k_1 = k_{-1}$  and  $E[p(x_1)|N] = 0 = E[p(x_1)|I]$  so that any non-informative submission strategy satisfying  $x_1^F(I) = x_1^F(N)$  can be part of such an equilibrium.

**Lemma 2** *In any candidate non-bluffing equilibrium, the informed leader plays an atomless mixed strategy in the first period, summarized by the densities*

$$q(x|1) = 2\frac{1-\alpha}{\alpha}g(x)\left[\frac{x}{k} - 1\right], \quad x \in (k, 1]$$

and

$$q(x|-1) = 2\frac{1-\alpha}{\alpha}g(x)\left[-\frac{x}{k} - 1\right], \quad x \in [-1, -k)$$

where  $k \in (0, 1)$  satisfies the identity  $\int_k^1 q(x|1)dx = 1$  and equals the informed leader's total trading profit.

**Proof of Lemma 2:** To see why the informed leader must use an atomless mixed strategy in period 1, first consider the case in which the informed leader buys (sells) with probability one whenever his type is  $v = 1$  (resp.,  $v = -1$ ). If there is a mass at any  $x_1 = x_1^I \neq 0$ , the trade reveals the existence and the information of the leader and thus market makers set the price equal to 1 (resp.,  $-1$ ) in both periods following such an order, implying that the leader earns zero profits in any equilibrium. But this contradicts the fact that he can always trade an amount on which his strategy does not put positive probability, but is in the support of noise, and so earn strictly positive profits. Similarly, a mass at  $x_1^I = 0$  reveals the existence of an informed leader and thus market makers set the price equal to 1 (resp.,  $-1$ ) in the second period when  $x_2^I > 0$  (resp.,  $x_2^I < 0$ ), eliminating all potential profits for the leader, again a contradiction. Therefore, the informed trader must play a mixed strategy without any mass in a non-bluffing equilibrium with  $q(x|v)$  the density representing this strategy. The derivation of the expressions for  $q(\cdot|1)$  and  $q(\cdot|-1)$  now follows from the arguments identical to those in the proof of Lemma 1b (in particular, case 1) with  $\delta$  set equal to zero. Furthermore, we must then have  $k_1 = k_{-1} = k \in (0, 1)$ , by the symmetry of  $g(x)$  around zero.