

Unpublished Appendix for
'OPTIMAL BELIEFS, ASSET PRICES, AND THE
PREFERENCE FOR SKEWED RETURNS'

by

Markus K. Brunnermeier, Christian Gollier, and Jonathan A. Parker

American Economic Review, 97(2), May 2007

A. Proof of Proposition 1

Define the set $\Pi = \left\{ \hat{\boldsymbol{\pi}} : 0 \leq \hat{\pi}_s \leq 1, \sum_{s=1}^S \hat{\pi}_s = 1 \right\}$. Optimal consumption choices given beliefs are continuous in probabilities on Π (Equation (2)). The objective for beliefs (Equation (3)) is thus continuous in beliefs on Π since L is linear in $\hat{\boldsymbol{\pi}}$ and continuous in c . Since Π is compact, a maximum exists on Π . To see that the optimum requires $0 < \hat{\pi}_s < 1$ for all s , note that for $\hat{\boldsymbol{\pi}}$ such that $\hat{\pi}_s = 0$ for at least one s , $c_s^* = 0$, and thus the objective for beliefs has value negative infinity and this cannot be optimal since the objective is finite on the interior of Π . ■

B. Proof of Proposition 2

To establish the results, we first prove four lemmas. Without loss of generality, we choose units so that

$$\sum_{s=1}^S p_s = 1.$$

This implies that $p_s/\pi_s = m = 1$ for all s for actuarially fair prices.

Lemma 1: The subjective belief of at most one state is biased upwards.

Proof of Lemma 1: The second-order condition, Equation (5), implies

$$\hat{\pi}_s^2 [\hat{\pi}_{s'} - \pi_{s'}] \leq \hat{\pi}_{s'}^2 [\pi_s - \hat{\pi}_s] \text{ for all } s \neq s'. \quad (\text{B.1})$$

Thus if beliefs about the likelihood of s' are biased upwards, so that $\hat{\pi}_{s'} - \pi_{s'} > 0$, then $\hat{\pi}_s - \pi_s < 0$ for all $s \neq s'$. ■

The proof of Lemma 1 also directly implies Lemma 2.

Lemma 2: If the subjective belief of one state is biased upwards, then the subjective beliefs of all other states are biased downwards.

Lemma 2 corresponds to result (i) in Proposition 2, except that we still need to prove that

rational expectation cannot be optimal except in the case in which prices are actuarially fair, probabilities are equal, and there are only two states. We first examine the case of actuarially unfair prices.

Lemma 3: If prices are not actuarially fair, rational expectations cannot be optimal.

Proof of Lemma 3: The first-order conditions (4) for rational expectations are

$$1 = \mu - 1 + \ln \frac{p_s}{\pi_s} \text{ for all } s$$

which cannot be satisfied for all s if $p_s/\pi_s \neq p_{s'}/\pi_{s'}$ for some s and s' . ■

Combining lemmata 2 and 3 implies that there is one, and only one, state whose probability is biased upwards when prices are actuarially unfair. In the remaining proof of result (i), we assume that prices are actuarially fair.

Lemma 4: If prices are actuarially fair, $p_s/\pi_s = 1$ for all s , then the investor biases his beliefs for all states with downward-biased subjective probabilities by a common factor: For one state, s' , $c_{s'} = \bar{c} \geq 1$ and $\hat{\pi}_{s'} = \bar{c}\pi_{s'} \geq \pi_{s'}$, and for all other states, $s \neq s'$, $c_s = \underline{c} \leq 1$ and $\hat{\pi}_s = \underline{c}\pi_s \leq \pi_s$, where (\underline{c}, \bar{c}) are the two solutions to

$$\frac{1}{\underline{c}} - \ln \frac{1}{\underline{c}} = \mu - 1 \tag{B.2}$$

with μ such that $\bar{c}\pi_{s'} + \underline{c} \sum_{s \neq s'} \pi_s = 1$.

Proof of Lemma 4: Equation (B.2) is the first-order conditions for optimal beliefs written in terms of consumption using Equation (2). If $\mu = 2$, $c_s = 1$ for all states, in which case beliefs are objective and the Lemma holds trivially ($\underline{c} = \bar{c}$). If $\mu > 2$, every first-order condition has the same two possible solution, so that $c_s = \hat{\pi}_s/p_s = \hat{\pi}_s/\pi_s$ is the same among all states whose likelihood is biased down. For these states let $\underline{c} := c_s^*(\hat{\boldsymbol{\pi}})$. Then Lemma 1 implies that there is at most one state perceived as more likely than it is, and let its consumption level be \bar{c} . We note $\mu < 2$ is not possible by Proposition 1. ■

We now prove that the one specific case has rational expectations and part (i) for actuarially fair prices by using these Lemmas to analyze the program given by Equation (3).

Let π denote the probability of the state whose probability is biased upwards, and let \bar{c} be the consumption level in that state, which is denoted s' . By Lemma 3, we know that the consumption level is a constant \underline{c} in all other states. Using the fact that $\hat{\pi} = \bar{c}\pi$, we can rewrite the Lagrangian objective for beliefs as a function of only \bar{c} and π :

$$\begin{aligned}\mathcal{L} &= \sum_{s=1}^S c_s \pi_s \ln c_s + \sum_{s=1}^S \pi_s \ln c_s - \mu \left[\sum_{s=1}^S c_s \pi_s - 1 \right] \\ &= \pi \bar{c} \ln \bar{c} + (1 - \pi) \underline{c} \ln \underline{c} + \pi \ln \bar{c} + (1 - \pi) \ln \underline{c} - \mu [\bar{c}\pi + (1 - \pi) \underline{c} - 1] \\ &= \pi (\bar{c} + 1) \ln \bar{c} + (2 - \pi (\bar{c} + 1)) \ln \left(\frac{1 - \pi \bar{c}}{1 - \pi} \right) := U(\bar{c}, \pi)\end{aligned}$$

where the last step imposes the constraint by substituting in $\underline{c} = \frac{1 - \pi \bar{c}}{1 - \pi}$. $U(\bar{c}, \pi)$ is maximized over the choice of state to bias upward and consumption level in that state. We proceed in two steps.

Suppose that we know the state, and thus its objective probability π , whose belief is biased upwards. We determine the optimal bias \bar{c} in the following way. Notice first that $U(1, \pi) = 0$ and that

$$\frac{\partial U}{\partial \bar{c}}(\bar{c}, \pi) = \pi \ln \frac{\bar{c}(1 - \pi)}{1 - \pi \bar{c}} + \pi \frac{1 - \bar{c}}{\bar{c}(1 - \pi \bar{c})}.$$

Evaluating $U_{\bar{c}}$ at $\bar{c} = 1$ and $\bar{c} = (1 - \pi)/\pi$, we obtain that

$$\frac{\partial U}{\partial \bar{c}}(1, \pi) = 0,$$

and

$$\frac{\partial U}{\partial \bar{c}}\left(\frac{1 - \pi}{\pi}, \pi\right) = 2\pi \ln \frac{1 - \pi}{\pi} + \frac{2\pi - 1}{1 - \pi}.$$

The second derivative of U with respect to \bar{c} is:

$$\frac{\partial^2 U}{\partial \bar{c}^2}(\bar{c}, \pi) = -2\pi^2 \frac{(\bar{c} - 1)(\bar{c} - \frac{1}{2\pi})}{\bar{c}^2(1 - \pi\bar{c})^2}. \quad (\text{B.3})$$

When $\pi = 1/2$, the right-hand side of the above equality is negative, implying that U is concave in \bar{c} . Combining this with $U_{\bar{c}}(1, 1/2) = 0$ directly implies that $\bar{c} = 1$ (and so $\hat{\pi} = \pi$) is optimal if $\pi = 1/2$. This sheds light on the special case with only two states that are equally likely. In that case, the choice of the state whose probability would be biased upwards is arbitrary, and we have just shown that it is optimal not to distort beliefs. We now argue that in all other cases, $\hat{\pi} \neq \pi$.

First, we show that if we consider $\pi < 1/2$, then $\bar{c} > 1/2\pi$ so that $\hat{\pi} > 1/2 > \pi$. When π is less than $1/2$, we see from (B.3) that U is alternatively concave, convex and concave (in \bar{c}) over the intervals $]0, 1]$, $[1, 1/2\pi]$ and $[1/2\pi, 1/\pi]$. Combining this with $U(1, 1/2) = U_{\bar{c}}(1, 1/2) = 0$ implies that the optimal solution has a \bar{c} larger than $1/2\pi$.

Second, we show that given π , the optimal \bar{c} has $\bar{c} \leq (1 - \pi)/\pi$, or equivalently that $\hat{\pi} \leq 1 - \pi$. This follows from

$$q(\pi) = \frac{\partial U}{\partial \bar{c}}\left(\frac{1 - \pi}{\pi}, \pi\right) = 2\pi \ln \frac{1 - \pi}{\pi} + \frac{2\pi - 1}{1 - \pi}$$

being negative when π is less than $1/2$.¹ This implies that $\bar{c} \leq (1 - \pi)/\pi$ or $\hat{\pi} \leq 1 - \pi$ which implies that the state perceived as more likely than it is necessarily has $\pi \leq 1/2$. Combined with the first result, we know that if the optimal $\pi \neq 1/2$, we have $\hat{\pi}_s \neq \pi_s$ for all s (“for all s ” follows from Lemma 2).

¹This is because $q(1/2) = 0$, and

$$q'(\pi) = 2 \ln \frac{1 - \pi}{\pi} + \frac{2\pi - 1}{(1 - \pi)^2},$$

$$q''(\pi) = \frac{2(2\pi - 1)}{\pi(1 - \pi)^3},$$

so that $q'(1/2) = 0$ and q'' has the same sign than $\pi - 0.5$. This implies that q has the same sign than $\pi - 0.5$.

What we still need to show is that $\pi \neq 1/2$ when there are more than 2 states with one state having probability $1/2$. We do this by showing that the function $V(\pi) = \max_{\bar{c}} U(\bar{c}, \pi)$ is symmetric and U-shaped with a minimum at $1/2$. Thus, as long as there exists a state with probability different from $1/2$, $\hat{\pi} \neq \pi$. V is symmetric around $\pi = 1/2$ from the definition of V . By the envelop theorem, we have that

$$V'(\pi) = \frac{\pi (\bar{c} - \frac{1-\pi}{\pi}) (1 - \bar{c})^2}{\bar{c}(1 - \pi)(1 - \pi\bar{c})},$$

where \bar{c} maximizes $U(\bar{c}, \pi)$. Suppose that π is less than $1/2$. We have seen above that it implies that \bar{c} is larger than $1/2$ but smaller $(1 - \pi)/\pi$, yielding $V'(\pi) < 0$. This shows that the optimal state to bias upwards is the objectively least likely one. Except the case $\pi_1 = \pi_2 = 1/2$, this state has a π less than $1/2$, which implies that $\underline{c} < 1 < \bar{c}$. This concludes the proof of result (i) in Proposition 2.

The above proof also directly implies Lemma 5, used in the proof of Lemma 6.

Lemma 5: If prices are actuarially fair, $p_s/\pi_s = 1$ for all s , then $\hat{\pi}_{s'} \geq 1/2$ where s' is the state whose probability is biased upwards.

Proposition 2(ii) follows directly from the first-order conditions, Equation (4). Given that one selects the solution with $\pi_s/\hat{\pi}_s > 1$, the left-hand side is increasing in $\pi_s/\hat{\pi}_s$ and the right-hand side is increasing in price-probability ratio, p_s/π_s . ■

C. Proof of Proposition 3

Proposition 2 implies the pattern of belief distortion but does not specify which is the state that has its probability biased upwards.

The proof of part (i) relies on the function $V(\pi)$ defined in the proof of Proposition 2(i) as the value of holding beliefs that are optimistic about state s' where π is the probability associated with this state. The proof of Proposition 2(i) shows that this function, $V(\pi)$, is

symmetric and U-shaped with a minimum at $1/2$. Further, the proof of Proposition 2(i) shows that the optimal s' is necessarily such that $\pi \leq 1/2$. Thus, $V(\pi)$ is maximized by choosing to bias upwards the probability of (one of) the smallest probability state(s).

To prove part (ii), we show that the local maximum in which subjective probabilities satisfy the first-order conditions and the budget constraint with an arbitrary optimistic state is dominated by the same set of subjective probabilities in which the subjective probability of this state and the cheapest state are interchanged.

Let states 1 to \bar{s} be the least expensive states, $p_s = p_{s'} < p_{s''}$ for $s', s \leq \bar{s}$ and $s'' > \bar{s}$. Let $\hat{\boldsymbol{\pi}}^*(s'')$ denote the vector of subjective probabilities that satisfy the first-order conditions, sum to one, and have $\hat{\pi}_{s''} > \pi_{s''}$ and $\hat{\pi}_s < \pi_s$ for all $s \neq s''$ and let $\mathcal{L}(\hat{\boldsymbol{\pi}}^*(s''), \mathbf{p})$ denote the associated value of the objective for beliefs:

$$\mathcal{L}(\hat{\boldsymbol{\pi}}^*(s''), \mathbf{p}) := \sum_{s=1}^S \hat{\pi}_s^*(s'') \ln \left(\frac{\hat{\pi}_s^*(s'')}{p_s} \right) + \sum_{s=1}^S \pi_s \ln \left(\frac{\hat{\pi}_s^*(s'')}{p_s} \right)$$

where \mathbf{p} denotes the vector of prices.

Consider taking $\hat{\boldsymbol{\pi}}^*(s'')$ for some $s'' > \bar{s}$ and switching the subjective probability for state s'' ($\hat{\pi}_{s''}^*(s'')$) with the subjective probability of some state $s' \leq \bar{s}$ ($\hat{\pi}_{s'}^*(s'')$),

$$(\hat{\pi}_1^*(s''), \dots, \hat{\pi}_{s''}^*(s''), \dots, \hat{\pi}_{s'}^*(s''), \dots, \hat{\pi}_{s'}^*(s''), \dots, \hat{\pi}_{s'}^*(s'')) := \hat{\boldsymbol{\pi}}^{Switch}(s'')$$

This is feasible because it is still the case that probabilities sum to one and as a result the budget constraint is also satisfied. For notational simplicity, for the moment, let $\hat{\pi}_s^* = \hat{\pi}_s^*(s'')$. Since well-being differs only in states s' and s'' ,

$$\begin{aligned} \mathcal{L}(\hat{\boldsymbol{\pi}}^{Switch}(s''), \mathbf{p}) - \mathcal{L}(\hat{\boldsymbol{\pi}}^*(s''), \mathbf{p}) &= (\hat{\pi}_{s''}^* + \pi) \ln \frac{\hat{\pi}_{s''}^*}{p_{s'}} - (\hat{\pi}_{s'}^* + \pi) \ln \frac{\hat{\pi}_{s'}^*}{p_{s'}} \\ &\quad + (\hat{\pi}_{s'}^* + \pi) \ln \frac{\hat{\pi}_{s'}^*}{p_{s''}} - (\hat{\pi}_{s''}^* + \pi) \ln \frac{\hat{\pi}_{s''}^*}{p_{s''}} \\ &= (\hat{\pi}_{s''}^* - \hat{\pi}_{s'}^*) \ln \frac{p_{s''}}{p_{s'}} > 0 \end{aligned}$$

where the sign follows from the initial assumptions that $p_{s'} < p_{s''}$ and that optimism is focussed on state s' so that $\hat{\pi}_{s''}^*(s'') > \pi > \hat{\pi}_{s'}^*(s'')$.

Since $\hat{\pi}^{Switch}(s'')$ is not optimally chosen for the situation in which the investor biases upward his beliefs about state s' , these probabilities may not satisfy the first-order conditions and so are weakly worse than those that are optimally chosen conditional on being optimistic about state s' :

$$\mathcal{L}(\hat{\pi}^*(s'), \mathbf{p}) \geq \mathcal{L}(\hat{\pi}^{Switch}(s''), \mathbf{p})$$

Thus we have,

$$\mathcal{L}(\hat{\pi}^*(s'), \mathbf{p}) \geq \mathcal{L}(\hat{\pi}^{Switch}(s''), \mathbf{p}) > \mathcal{L}(\hat{\pi}^*(s''), \mathbf{p}) \text{ for all } s'' > \bar{s} \text{ and } s' \leq \bar{s}$$

which completes the proof of part (ii).

For part (iii), we make a similar argument. Let states 1 to \bar{s} be the least expensive and lowest probability states, $p_s = p_{s'} < p_{s''}$ and $\pi_s = \pi_{s'} < \pi_{s''}$ for $s', s \leq \bar{s}$ and $s'' > \bar{s}$. Consider taking $\hat{\pi}^*(s'')$ for some $s'' > \bar{s}$ and switching the subjective probability for state s'' ($\hat{\pi}_{s'}^*(s'')$) with the subjective probability of some state $s' \leq \bar{s}$ ($\hat{\pi}_{s'}^*(s'')$) and let this vector be denoted $\hat{\pi}^{Switch}(s'')$. Since well-being differs only in states s' and s'' ,

$$\begin{aligned} \mathcal{L}(\hat{\pi}^{Switch}(s''), \mathbf{p}) - \mathcal{L}(\hat{\pi}^*(s''), \mathbf{p}) &= (\hat{\pi}_{s''}^* + \pi_{s''}) \ln \frac{\hat{\pi}_{s''}^*}{p_{s'}} - (\hat{\pi}_{s'}^* + \pi_{s'}) \ln \frac{\hat{\pi}_{s'}^*}{p_{s'}} \\ &\quad + (\hat{\pi}_{s'}^* + \pi_{s'}) \ln \frac{\hat{\pi}_{s'}^*}{p_{s''}} - (\hat{\pi}_{s''}^* + \pi_{s''}) \ln \frac{\hat{\pi}_{s''}^*}{p_{s''}} \\ &= (\hat{\pi}_{s''}^* - \hat{\pi}_{s'}^*) \ln \frac{p_{s''}}{p_{s'}} + (\pi_{s''} - \pi_{s'}) \ln \frac{p_{s''}}{p_{s'}} \end{aligned}$$

where the sign follows from the initial assumptions that $\pi_{s''} > \pi_{s'}$, $p_{s'} < p_{s''}$ and that optimism is focussed on state s' so that $\hat{\pi}_{s''}^*(s'') > \pi > \hat{\pi}_{s'}^*(s'')$.

Analogously to part (ii), since $\hat{\pi}^{Switch}(s'')$ is not optimally chosen for the situation in

which the investor biases upward his beliefs about state s' :

$$\mathcal{L}(\hat{\pi}^*(s'), \mathbf{p}) \geq \mathcal{L}(\hat{\pi}^{Switch}(s''), \mathbf{p})$$

Thus we have,

$$\mathcal{L}(\hat{\pi}^*(s'), \mathbf{p}) \geq \mathcal{L}(\hat{\pi}^{Switch}(s''), \mathbf{p}) > \mathcal{L}(\hat{\pi}^*(s''), \mathbf{p}) \text{ for all } s'' > \bar{s} \text{ and } s' \leq \bar{s}$$

which completes the proof of part (iii).

For part (iv) of Proposition 3, we first note that, for any problem, there is a lower bound placed on μ by the requirement of real solutions to the first-order conditions (Proposition 1). Thus,

$$\mu \geq \underline{\mu} := \max_s \left\{ 2 - \ln \frac{p_s}{\pi_s} \right\} \quad (\text{C.1})$$

Second, for any problem, there is an upper bound placed on μ by the requirement that the $\hat{\pi}_{s'} < 1$ in the solution to the first-order condition for the probability that is positively-biased state. Thus, for state s' to have $\hat{\pi}_{s'} > \pi_{s'}$, we require

$$\mu < \bar{\mu}(s') := 1 + \pi_{s'} - \ln p_{s'}.$$

Consider first \underline{m} . As one decreases $p_{s'}$, $\underline{\mu}$ increases (at least once $p_{s'}/\pi_{s'}$ is the minimum), $\bar{\mu}(s')$ increases, and $\bar{\mu}(s)$ for $s \neq s'$ does not change. Thus, there is an $\underline{m}_{s'}$ such that for $p_{s'} = \underline{m}_{s'}\pi_{s'}$, $\bar{\mu}(s') > \underline{\mu}$ and $\bar{\mu}(s') < \underline{\mu}$ for all $s \neq s'$. Thus, the agent must be optimistic about state s' . Then $\underline{m} = \min_{s'} \{\underline{m}_{s'}\}$.

In terms of \bar{m} , as one increases $p_{s'}$, there is an \bar{m} such that for $p_{s'} = \bar{m}\pi_{s'}$, $\bar{\mu}(s') < \underline{\mu}$ and $\bar{\mu}(s') > \underline{\mu}$ for some $s \neq s'$. Thus, the agent must be pessimistic about state s' . Then $\bar{m} = \max_{s'} \{\bar{m}_{s'}\}$. ■

D. Proof of Proposition 4

Proof by construction in the text.

E. Proof of Proposition 5

This Proposition is not trivial because, while an increase in the price of a given state decreases the demand for the asset that pays off in that state, an increase in the price of a state for which $\hat{\pi}_s^* < \pi_s$ decreases the demands for the assets that pay off in all other states that are viewed with pessimism. Following the proof of Proposition 5, we prove two Lemmata that characterize these price effects in Section F.

We start with the following Lemma.

Lemma 6: In any equilibrium with actuarially fair prices:

$$\sum_{s=1}^S \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right) > 0. \quad (\text{E.1})$$

or equivalently

$$\pi_{s'} \left(\frac{\bar{c}^2}{\bar{c} - 1} \right) + \sum_{s \neq s'} \pi_s \left(\frac{\underline{c}^2}{\underline{c} - 1} \right) > 0 \quad (\text{E.2})$$

Proof: By Lemma 4, there are two consumption levels such that $\hat{\pi}_s/\pi_s = \underline{c}$ for all $s \neq s'$, and $\hat{\pi}_{s'}/\pi_{s'} = \bar{c}$. Since subjective probabilities sum to one, $\sum_{s \neq s'} \hat{\pi}_s = 1 - \hat{\pi}_{s'}$, and the budget constraint at fair prices implies

$$\begin{aligned} 1 &= \bar{c}\pi_{s'} + \sum_{s \neq s'} \underline{c}\pi_s \\ \underline{c} &= \frac{1 - \pi_{s'}\bar{c}}{1 - \pi_{s'}} \\ \underline{c} &= \frac{1 - \hat{\pi}_{s'}}{1 - \pi_{s'}}. \end{aligned}$$

Using these relationships, our term of interest becomes:

$$\begin{aligned}
\sum_{s=1}^S \frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} &= \frac{\hat{\pi}_{s'}}{1 - \frac{\pi_{s'}}{\hat{\pi}_{s'}}} + \sum_{s \neq s'} \hat{\pi}_s \left(\frac{1}{1 - \underline{c}^{-1}} \right) \\
&= \frac{\hat{\pi}_{s'}^2}{\hat{\pi}_{s'} - \pi_{s'}} + (1 - \hat{\pi}_{s'}) \frac{1}{1 - \frac{1 - \pi_{s'}}{1 - \hat{\pi}_{s'}}} \\
&= \frac{\hat{\pi}_{s'}^2}{\hat{\pi}_{s'} - \pi_{s'}} + \frac{(1 - \hat{\pi}_{s'})^2}{1 - \hat{\pi}_{s'} - 1 + \pi_{s'}} \\
&= \frac{2\hat{\pi}_{s'} - 1}{(\hat{\pi}_{s'} - \pi_{s'})}
\end{aligned}$$

which is positive if $\hat{\pi}_{s'} > 1/2$. This follows from Lemma 5. ■

For notational simplicity, let a index the states $s \leq \underline{s}$ so that $\pi_s = \pi_a$ and $C_s = C_a$ for $s \leq \underline{s}$, and let b index the $S - \underline{s}$ states with $\pi_s = \pi_b$ and $C_s = C_b$. Let a' be the state, less than or equal to \underline{s} , that is viewed as more likely than it is; this state differs across investors but all a -states are symmetric so we can impose that $p_{a'} = p_a$.

To prove Proposition 5, we write the conditions for the initial equilibrium with fair prices, totally differentiate the system for a small increase in the aggregate endowment in the a -states, dC_a , imposing that $dp_b = 0$, show that $dp_a > 0$, and check that $dC_b < 0$. Because there is a discrete interval between π_a and π_b and because the wellbeing functions evaluated for different choices of s' (the state such that $\hat{\pi}_s > \pi_s$) are continuous in prices, small enough changes in p_a and p_b do not change the relative rankings of the wellbeing as a function of s' . Thus, locally the pattern of a' across investors remains unchanged.

E1. Equilibrium conditions at fair prices

The exogenous variables are S , \underline{s} , π_a . Given these three, π_b is given by the fact that probabilities sum to one

$$\underline{s}\pi_a + (S - \underline{s})\pi_b = 1.$$

For actuarially fair prices that (normalized) sum to one, we have

$$\begin{aligned} p_a &= \pi_a \\ p_b &= \pi_b. \end{aligned}$$

Optimal beliefs are given by the investor first-order conditions,

$$\begin{aligned} \frac{\pi_{a'}}{\hat{\pi}_{a'}} - \ln \frac{\pi_{a'}}{\hat{\pi}_{a'}} &= \mu - 1 + \ln \frac{p_a}{\pi_a} \\ \frac{\pi_a}{\hat{\pi}_a} - \ln \frac{\pi_a}{\hat{\pi}_a} &= \mu - 1 + \ln \frac{p_a}{\pi_a} \text{ for } a \neq a' \\ \frac{\pi_b}{\hat{\pi}_b} - \ln \frac{\pi_b}{\hat{\pi}_b} &= \mu - 1 + \ln \frac{p_b}{\pi_b} \end{aligned} \tag{E.3}$$

(note that at fair prices, the last two imply and $\frac{\pi_a}{\hat{\pi}_a} = \frac{\pi_b}{\hat{\pi}_b}$), and the fact that subjective probabilities sum to one,

$$\hat{\pi}_{a'} + (\underline{s} - 1) \hat{\pi}_a + (S - \underline{s}) \hat{\pi}_b = 1. \tag{E.4}$$

These equations can be used to solve for $\hat{\pi}_{a'}$, $\hat{\pi}_a$, $\hat{\pi}_b$ and μ , which we know exist and are unique. Note that $\underline{c} = \frac{\hat{\pi}_a}{\pi_a} = \frac{\hat{\pi}_b}{\pi_b}$ and $\bar{c} = \frac{\hat{\pi}_{a'}}{\pi_{a'}}$.

Finally, the two remaining exogenous variables that deliver fair prices in equilibrium, C_a and C_b , are calculated from market clearing conditions:

$$\begin{aligned} C_a &= \frac{1}{\underline{s}} \frac{\hat{\pi}_{a'}}{p_a} + \left(1 - \frac{1}{\underline{s}}\right) \frac{\hat{\pi}_a}{p_a} \\ C_b &= \frac{\hat{\pi}_b}{p_b}. \end{aligned} \tag{E.5}$$

We have three exogenous variables that are chosen to generate an initial fair-prices equilibrium, π_b , C_a , and C_b , six endogenous variables (μ , $\hat{\pi}_{a'}$, $\hat{\pi}_b$, $\hat{\pi}_a$, p_a , p_b), and nine equations.

E2. Totally differentiated equilibrium conditions

We totally differentiate the system (E.3), (E.4), and (E.5) allowing C_a and C_b to vary and imposing $dp_b = 0$:

$$\begin{aligned}\frac{\hat{\pi}_{a'} - \pi_{a'}}{\hat{\pi}_{a'}^2} d\hat{\pi}_{a'} &= d\mu + \frac{1}{p_a} dp_a \\ \frac{\hat{\pi}_a - \pi_a}{\hat{\pi}_a^2} d\hat{\pi}_a &= d\mu + \frac{1}{p_a} dp_a \text{ for } a \neq a' \\ \frac{\hat{\pi}_b - \pi_b}{\hat{\pi}_b^2} d\hat{\pi}_b &= d\mu + 0\end{aligned}\tag{E.6}$$

$$\begin{aligned}d\hat{\pi}_{a'} + (\underline{s} - 1) d\hat{\pi}_a + (S - \underline{s}) d\hat{\pi}_b &= 0 \\ p_a^2 dC_a &= \frac{1}{\underline{s}} (p_a d\hat{\pi}_{a'} - \hat{\pi}_{a'} dp_a) + \left(1 - \frac{1}{\underline{s}}\right) (p_a d\hat{\pi}_a - \hat{\pi}_a dp_a) \\ dC_b &= \frac{d\hat{\pi}_b}{p_b} - 0\end{aligned}\tag{E.7}$$

Now we want to study dp_a , dC_b , and dC_b around an equilibrium with fair prices, i.e., with

$$\frac{p_a}{\pi_a} = \frac{p_b}{\pi_b} = 1.$$

E3. Signing dp_a , dC_a and dC_b around a fair price equilibrium

Set $dC_a = 1$ (so that implicitly dx is dx/dC_a), and re-write the equations using $\bar{c} = \frac{\hat{\pi}_{a'}}{\pi_{a'}}$ and

$C_b = \frac{\hat{\pi}_b}{p_b}$ and replace prices with probabilities:

$$\begin{aligned}\frac{\bar{c} - 1}{\bar{c}^2} \frac{1}{\pi_a} d\hat{\pi}_{a'} &= d\mu + \frac{1}{\pi_a} dp_a \\ \frac{C_b - 1}{C_b^2} \frac{1}{\pi_a} d\hat{\pi}_a &= d\mu + \frac{1}{\pi_a} dp_a \text{ for } a \neq a' \\ \frac{C_b - 1}{C_b^2} \frac{1}{\pi_b} d\hat{\pi}_b &= d\mu \\ d\hat{\pi}_{a'} + (\underline{s} - 1) d\hat{\pi}_a + (S - \underline{s}) d\hat{\pi}_b &= 0 \\ \frac{1}{\underline{s}} (d\hat{\pi}_{a'} - \bar{c} dp_a) + \left(1 - \frac{1}{\underline{s}}\right) (d\hat{\pi}_a - C_b dp_a) &= \pi_a \\ p_b dC_b &= d\hat{\pi}_b\end{aligned}$$

Now using the first three equations (the first-order conditions) to eliminate beliefs in the third-to-last and the second-to-last equations gives:

$$\begin{aligned} (\pi_a \alpha + \pi_b \beta) d\mu + \alpha dp_a &= 0 \\ \alpha \pi_a d\mu + \left(\frac{\bar{c}}{\bar{c} - 1} + (\underline{s} - 1) \frac{C_b}{C_b - 1} \right) dp_a &= \underline{s} \pi_a \end{aligned} \tag{E.8}$$

where

$$\begin{aligned} \alpha &= \frac{\bar{c}^2}{\bar{c} - 1} + (\underline{s} - 1) \frac{C_b^2}{C_b - 1} \\ \beta &= (S - \underline{s}) \frac{C_b^2}{C_b - 1} \end{aligned}$$

Note that $\beta < 0$ since $C_b - 1 = \hat{\pi}_b / \pi_b - 1$. In the current notation, the inequality of Lemma 6 is

$$\pi_a \left(\frac{\bar{c}^2}{\bar{c} - 1} \right) + (\underline{s} - 1) \pi_b \left(\frac{\underline{c}^2}{\underline{c} - 1} \right) + (S - \underline{s}) \pi_b \left(\frac{\underline{c}^2}{\underline{c} - 1} \right) = \pi_a \alpha + \pi_b \beta > 0$$

Together these two inequalities imply that $\alpha > 0$. Thus $d\mu$ and dp_a have opposite signs.

Combining equations (E.8) gives

$$dp_a = \frac{\underline{s} \pi_a}{\frac{\bar{c}}{\bar{c} - 1} + (\underline{s} - 1) \frac{C_b}{C_b - 1} - \frac{\alpha^2 \pi_a}{\pi_a \alpha + \pi_b \beta}}$$

To show that $dp_a < 0$, we note that the numerator is positive and that

$$\frac{\bar{c}}{\bar{c} - 1} + (\underline{s} - 1) \frac{C_b}{C_b - 1} = \alpha - \bar{c} - (\underline{s} - 1) C_b$$

so that the sign of the denominator is given by the sign of:

$$\begin{aligned}
& \left(\frac{\bar{c}}{\bar{c}-1} + (\underline{s}-1) \frac{C_b}{C_b-1} \right) (\pi_a \alpha + \pi_b \beta) - \alpha^2 \pi_a \\
&= (\alpha - \bar{c} - (\underline{s}-1) C_b) (\pi_a \alpha + \pi_b \beta) - \alpha^2 \pi_a \\
&= -\bar{c} (\pi_a \alpha + \pi_b \beta) - (\underline{s}-1) C_b (\pi_a \alpha + \pi_b \beta) + \pi_b \beta \alpha < 0
\end{aligned}$$

From equation (E.8), we have that $d\mu > 0$, and from equations (E.6), we have that $d\hat{\pi}_b < 0$, and so from equation (E.7), we have $dC_b < 0$.

Thus, reversing signs, for a small reduction in aggregate risk – a decrease in C_a and an increase in C_b – such that the expected returns on the less-skewed assets do not change, $dp_b = 0$, the expected returns on the more skewed assets rises, $dp_a < 0$. ■

F. Lemmata on the effects of prices on demands

Lemma 7: (Law of demand for fair prices) For $\pi_s/p_s = 1$ for all s , $c_s^*(\hat{\boldsymbol{\pi}}^*(\mathbf{p}), \mathbf{p}) = c_s^*(\hat{\boldsymbol{\pi}}_s^*(\mathbf{p}), p_s)$ is decreasing in p_s .

Proof of Lemma 7: Let t be the (only) state for which price increases. If $\hat{\pi} > \pi$ and switches to $\hat{\pi} < \pi$, then we have our result. Otherwise, using Equation (2), the change in portfolio for a small change in p_t is

$$\frac{dc_t}{dp_t} = \frac{1}{p_t} \frac{d\hat{\pi}_t}{dp_t} - \frac{\hat{\pi}_t}{(p_t)^2}. \tag{F.1}$$

We thus want to show that

$$\frac{d\hat{\pi}_t}{dp_t} < \frac{\hat{\pi}_t}{p_t}.$$

Totally differentiating each first-order condition (Equations (4)), gives

$$\frac{d\hat{\pi}_t}{dp_t} = \left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t - \pi_t} \right) \left(\frac{d\mu}{dp_t} + \frac{1}{p_t} \right) \quad (\text{F.2})$$

$$\frac{d\hat{\pi}_s}{dp_t} = \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right) \frac{d\mu}{dp_t} \text{ for all } s \neq t. \quad (\text{F.3})$$

Summing across all states and imposing that $\sum_{s=1}^S d\pi_s/dp_t = 0$ gives

$$\frac{d\mu}{dp_t} = - \frac{\left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t - \pi_t} \right) \frac{1}{p_t}}{\sum_{s=1}^S \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right) p_t}$$

which can be plugged into equation (F.2) to give

$$\frac{d\hat{\pi}_t}{dp_t} = \frac{\hat{\pi}_t}{p_t} \left(\frac{\hat{\pi}_t}{\hat{\pi}_t - \pi_t} \right) \left(1 - \frac{\left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t - \pi_t} \right)}{\sum_{s=1}^S \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right)} \right).$$

Thus, we have our result iff

$$1 > \left(\frac{\hat{\pi}_t}{\hat{\pi}_t - \pi_t} \right) \left(\frac{\sum_{s \neq t} \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right)}{\sum_{s=1}^S \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right)} \right).$$

If $\hat{\pi}_t > \pi_t$, then $\sum_{s \neq t} \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right) < 0$, and our result follows if

$$\sum_{s=1}^S \left(\frac{\hat{\pi}_s}{1 - \frac{\pi_s}{\hat{\pi}_s}} \right) > 0$$

which is true by Lemma 6.

If $\hat{\pi}_t < \pi_t$, then our result also follows if this inequality is satisfied because that implies $\sum_{s \neq t} \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right) > 0$. ■

Lemma 8: (Cross-price effects for fair prices) For $\pi_s/p_s = 1$ for all s and t such that $t \neq s$, $c_s^*(\hat{\pi}_s^*(\mathbf{p}), p_s)$ is increasing in p_t if $\hat{\pi}_s^* > \pi_s$ or $\hat{\pi}_t^* > \pi_t$, otherwise it is decreasing in p_t

as long as it remains true that $\hat{\pi}_s^* < \pi_s$.

Proof of Lemma 8: If the investor switches to become optimistic about state s as we increase the price of the state previously viewed with optimism, then we have our result. Increasing the price of another state cannot cause the investor to switch their optimism from the state viewed with optimism.

So under the assumption that the investor remains optimistic about the same state, using Equation (2), for any $s \neq t$, the change in portfolio for a small change in p_t is

$$\frac{dc_s}{dp_t} = \frac{1}{p_s} \frac{d\hat{\pi}_s}{dp_t}.$$

We thus the sign of $\frac{dc_s}{dp_t}$ is the same as that of $\frac{d\hat{\pi}_s}{dp_t} > 0$.

Totally differentiating each first-order condition (equations (4)), gives

$$\begin{aligned} \frac{d\hat{\pi}_t}{dp_t} &= \left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t - \pi_t} \right) \left(\frac{d\mu}{dp_t} + \frac{1}{p_t} \right) \\ \frac{d\hat{\pi}_s}{dp_t} &= \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right) \frac{d\mu}{dp_t} \text{ for all } s \neq t. \end{aligned} \tag{F.4}$$

Summing across all states and imposing that $\sum_{s=1}^S d\pi_s/dp_t = 0$ gives

$$\frac{d\mu}{dp_t} = - \frac{\left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t - \pi_t} \right) \frac{1}{p_t}}{\sum_{s=1}^S \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right) p_t}$$

which can be plugged into equation (F.4) to give

$$\frac{d\hat{\pi}_s}{dp_t} = - \frac{\left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right) \left(\frac{\hat{\pi}_t^2}{\hat{\pi}_t - \pi_t} \right) \frac{1}{p_t}}{\sum_{s=1}^S \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right) p_t}.$$

From Lemma 6, we have that $\sum_{s=1}^S \left(\frac{\hat{\pi}_s^2}{\hat{\pi}_s - \pi_s} \right) > 0$, thus $\frac{d\hat{\pi}_s}{dp_t} > 0$ if $\hat{\pi}_s > \pi_s$ or $\hat{\pi}_t > \pi_t$ otherwise it is negative. ■