Supplementary Material for

"Ascending Auctions: Uniqueness and Robustness to Strategic Uncertainty"

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The appendix contains the proofs of Proposition 1 and 2.

Proof of Proposition 1. The uniqueness result is an immediate consequence of Theorem 1 and Theorem 3 in general incomplete information environments in Bergemann, D. and Morris, S. (2009): "Robust Implementation in Direct Mechanisms," *Review of Economic Studies*, **76**, 1175-1206. ■

The proof of Proposition 2 assumes that the payoff types of the bidders are distinct: $\theta_1 < \theta_2 < \cdots < \theta_I$. In addition we allow the exit time of each agent to be in the interval $[0, I-1]$ rather than $[0,1]$. With a larger set of feasible exit times we avoid the possibility of corner solutions in the best response functions. The resulting set of rationalizable exit times will nonetheless be in $[0,1]$. Proposition 2 continues to hold without distinct payoff types and with exit times restricted to the unit interval. The only consequence is a longer proof caused by the necessity of case distinctions due to corner solutions.

Proof of Proposition 2. The proof proceeds by induction on the number j of bidders still to leave the auction. We show that the best response of bidder $I-j$ to the reports of the bidders exiting before him is given by:

$$\beta_{I-j}\left(t_1, t_2, ..., t_{I-j-1}\right) = \theta_{I-j} + \frac{\gamma}{1+j\gamma} \sum_{i=1}^{I-j-1} \left(\theta_i - t_i\right). \tag{1}$$

We begin with the final bidder I and thus $j = 0$. We showed earlier in (6) that the best response function of bidder I is indeed given by:

$$\beta_I\left(t_1, t_2, ..., t_{I-1}\right) = \theta_I + \gamma \sum_{i=1}^{I-1} \left(\theta_i - t_i\right).$$

We now proof the general inductive step. With the outcome function defined by (4), (9) and (10), we can write the payoff of agent $I-j$ for $j > 0$ as follows:

$$\varepsilon\left(\left(\theta_{I-j} + \gamma \sum_{i \neq I-j} \theta_i\right) t_{I-j} - \frac{1}{2} w_{I-j} t_{I-j}^2 - \gamma t_{I-j} \left(\sum_{i \neq I-j} t_i\right)\right).$$

We can rewrite the payoff function of bidder $I-j$ by separating the bidders who exited before and those who will exit after $I-j$:

$$\varepsilon\left(\left(\theta_{I-j}+\gamma\sum_{i\neq I-j}\theta_i\right)t_{I-j}-\frac{1}{2}w_{I-j}t_{I-j}^2-\gamma t_{I-j}\left(\sum_{i<I-j}t_i+\sum_{i>I-j}t_i\right)\right).$$

By hypothesis, the inductive step holds for all $k<j$. For all $i>I-j$, we can therefore replace the report by the best response given by (1):

$$\varepsilon\left(\left(\theta_{I-j}+\gamma\sum_{i\neq I-j}\theta_i\right)t_{I-j}-\frac{1}{2}w_{I-j}t_{I-j}^2-\gamma t_{I-j}\left(\sum_{i<I-j}t_i+\sum_{i>I-j}\beta_i\left(t_1,...t_{i-1}\right)\right)\right). \quad (2)$$

In particular, as the inductive step holds for all $i>I-j$, we can rewrite the best response of every agent $i>I-j$ as follows:

$$\beta_i\left(t_1,t_2,...,t_{I-j}\right)=\theta_i+\frac{\gamma}{1+j\gamma}\sum_{k=1}^{I-j}\left(\theta_k-t_k\right).$$

We can now insert $\beta_i\left(t_1,t_2,...,t_{I-j}\right)$ for all $i>I-j$ into (2) to get:

$$\begin{aligned}&\varepsilon\left(\theta_{I-j}+\gamma\sum_{i\neq I-j}\theta_i\right)t_{I-j}\\&-\varepsilon\left(\frac{1}{2}w_{I-j}t_{I-j}^2+\gamma t_{I-j}\left(\sum_{i<I-j}t_i+\sum_{i>I-j}\left(\theta_i+\frac{\gamma}{1+j\gamma}\sum_{k=1}^{I-j}\left(\theta_k-t_k\right)\right)\right)\right).\end{aligned}$$

We find the best response of bidder $I-j$ by determining the optimal exit time t_{I-j} in response to past exit times and differentiate the above payoff with respect to the exit time t_{I-j}:

$$\begin{aligned}\theta_{I-j}+\gamma\sum_{i\neq I-j}\theta_i-w_{I-j}t_{I-j}&\\-\gamma\left(\sum_{i<I-j}t_i+\sum_{i>I-j}\left(\theta_i+\frac{\gamma}{1+j\gamma}\sum_{k=1}^{I-j}\left(\theta_k-t_k\right)\right)-\frac{j\gamma}{1+j\gamma}t_{I-j}\right)&=0.\end{aligned}$$

We can collect terms to obtain:

$$\begin{aligned}\left(\theta_{I-j}+\left(\gamma-\frac{j\gamma^2}{1+(j-1)\gamma}\right)\sum_{i<I-j}\left(\theta_i-t_i\right)\right)-w_{I-j}t_{I-j}&\\-\gamma\left(+\frac{j\gamma}{1+(j-1)\gamma}\theta_{I-j}-\frac{j\gamma}{1+(j-1)\gamma}2t_{I-j}\right)&=0,\end{aligned}$$

or

$$
\begin{aligned}
& \left(\theta_{I-j}\left(1-\frac{j\gamma^2}{1+(j-1)\,\gamma}\right)+\left(\gamma-\frac{j\gamma^2}{1+(j-1)\,\gamma}\right)\sum_{i<I-j}(\theta_i-t_i)\right) \\
= & \left(w_{I-j}-\frac{2j\gamma^2}{1+(j-1)\,\gamma}\right)t_{I-j}.
\end{aligned}
$$

We can then solve for t_{I-j} and find that:

$$
t_{I-j}=\theta_{I-j}+\frac{\gamma}{1+j\gamma}\sum_{i=1}^{I-j-1}(\theta_i-t_i)\,,
$$

thus proving the inductive step.■