Appendix of "Monetary Policy Rules and Macroeconomic Stability: Some new evidence"

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This note contains proofs of the identification results given in section 4 of the paper.

1 Identification of the non-inertial policy rule

The model is given by the following equations:

$$\pi_t = \beta E_t \pi_{t+1} + \lambda x_t \tag{1}$$

$$y_t = E_t y_{t+1} - \sigma (r_t - E_t \pi_{t+1}) + g_t \tag{2}$$

$$r_t = \psi_\pi E_t \pi_{t+1} + \psi_x x_t + \varepsilon_{r,t} \tag{3}$$

$$x_t = y_t - z_t$$

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t} \tag{4}$$

$$g_t = \rho_g g_{t-1} + \varepsilon_{g,t}. \tag{5}$$

It is convenient to express eqs. (1) and (2) in terms of the output gap x_t. Given eq. (4), eq. (2) will be

$$x_t = E_t x_{t+1} - \sigma (r_t - E_t \pi_{t+1}) + g_t - z_t (1 - \rho_z). \tag{6}$$

Define $Y_t = (\pi_t, x_t)'$, and $X_t = (z_t, g_t)'$ and the vector of innovations $\varepsilon_t = (\varepsilon_{z,t}, \varepsilon_{g,t}, \varepsilon_{r,t})'$. From the Taylor rule, I derive

$$r_t - E_t \pi_{t+1} = (\psi_\pi - 1) E_t \pi_{t+1} + \psi_x x_t + \varepsilon_{r,t}$$

and I substitute this into eq. (6) to obtain

$$x_t = E_t x_{t+1} - \sigma (\psi_\pi - 1) E_t \pi_{t+1} - \sigma \psi_x x_t - \sigma \varepsilon_{r,t} + g_t - z_t (1 - \rho_z) .$$

The model for Y_t can be written in the form

$$\begin{aligned} B_0 (\theta) Y_t &= B_1 (\theta) E_t Y_{t+1} + C (\theta) X_t + c (\theta) \varepsilon_{r,t} \\ X_t &= D (\theta) X_{t-1} + (\varepsilon_{z,t}, \varepsilon_{g,t})' \end{aligned} \tag{7}$$

where $B_0, B_1(\theta)$, $C(\theta)$ and $c(\theta)$ are given by

$$B_0 (\theta) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 + \sigma \psi_x \end{pmatrix}, \; B_1 (\theta) = \begin{pmatrix} \beta & 0 \\ -\sigma (\psi_\pi - 1) & 1 \end{pmatrix} \tag{8}$$

$$C (\theta) = \begin{pmatrix} 0 & 0 \\ -(1 - \rho_z) & 1 \end{pmatrix}, .c (\theta) = \begin{pmatrix} 0 \\ -\sigma \end{pmatrix} \tag{9}$$

and $D(\theta)$ is a diagonal matrix with ρ_z and ρ_g along its diagonal.

1.1 Determinate solution

When the Taylor principle is satisfied, the equilibrium is determinate, that is, there exists a unique non-explosive solution to eq. (7). This solution will be of the form

$$Y_t = A (\theta) X_t + f (\theta) \varepsilon_{r,t} \tag{10}$$

which is a stable process for any matrix A, since X_t is stable.

To verify that this is a solution, derive $E_t Y_{t+1} = A(\theta) D(\theta) X_t$, substitute this into (7) to get

$$B_0 (\theta) A (\theta) X_t + B_0 (\theta) f (\theta) \varepsilon_{r.t} = [B_1 (\theta) A (\theta) D (\theta) + C (\theta)] X_t + c (\theta) \varepsilon_{r,t}$$

and observe that there exists a matrix

$$A(\theta) = \begin{pmatrix} a_{\pi z} & a_{\pi g} \\ a_{xz} & a_{xg} \end{pmatrix}$$

such that the above equation is satisfied for all X_t, i.e., $A(\theta)$ solves the equation $B_0(\theta) A(\theta)$ $= B_1(\theta) A(\theta) D(\theta) + C(\theta)$, or

$$\begin{pmatrix} 1 & -\lambda \\ 0 & 1+\sigma\psi_x \end{pmatrix} \begin{pmatrix} a_{\pi z} & a_{\pi g} \\ a_{xz} & a_{xg} \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ -\sigma(\psi_\pi - 1) & 1 \end{pmatrix} \begin{pmatrix} a_{\pi z}\rho_z & a_{\pi g}\rho_g \\ a_{xz}\rho_z & a_{xg}\rho_g \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -(1-\rho_z) & 1 \end{pmatrix}.$$

This yields the set of equations

$$\begin{aligned} a_{\pi z} - \lambda a_{xz} - \beta\rho_z a_{\pi z} &= 0 \\ a_{\pi g} - \lambda a_{gx} - \beta\rho_g a_{\pi g} &= 0 \\ a_{xz}(\sigma\psi_x + 1) - \rho_z a_{xz} + \sigma\rho_z(\psi_\pi - 1) a_{\pi z} &= -1 + \rho_z \\ a_{gx}(\sigma\psi_x + 1) - \rho_g a_{gx} + \sigma\rho_g(\psi_\pi - 1) a_{\pi g} &= 1. \end{aligned}$$

Solving these equations, we obtain:

$$\begin{aligned} a_{\pi z} &= \lambda a_{xz}/(1-\beta\rho_z) \\ a_{\pi g} &= \lambda a_{gx}/(1-\beta\rho_g) \end{aligned}$$

$$\begin{aligned} a_{xz}(\sigma\psi_x + 1 - \rho_z) &= \rho_z - \sigma\rho_z(\psi_\pi - 1) a_{\pi z} - 1 \Rightarrow \\ a_{xz}(\sigma\psi_x + 1 - \rho_z) &= \rho_z - \sigma\rho_z(\psi_\pi - 1)\lambda a_{xz}/(1-\beta\rho_z) - 1 \Rightarrow \\ a_{xz} &= \frac{(\rho_z - 1)(1-\beta\rho_z)}{(\sigma\psi_x + 1 - \rho_z)(1-\beta\rho_z) + \sigma\rho_z(\psi_\pi - 1)\lambda} \end{aligned}$$

$$\begin{aligned} a_{xg}(\sigma\psi_x + 1 - \rho_g) &= 1 - \sigma\rho_g(\psi_\pi - 1) a_{\pi g} \Rightarrow \\ a_{xg}(\sigma\psi_x + 1 - \rho_g) &= 1 - \sigma\rho_g(\psi_\pi - 1)\lambda a_{gx}/(1-\beta\rho_g) \Rightarrow \\ a_{xg} &= \frac{(1-\beta\rho_g)}{(\sigma\psi_x + 1 - \rho_g)(1-\beta\rho_g) + \sigma\rho_g(\psi_\pi - 1)\lambda} \end{aligned}$$

So,

$$A = \begin{pmatrix} \lambda & \lambda \\ 1-\beta\rho_z & 1-\beta\rho_g \end{pmatrix} \begin{pmatrix} \frac{(\rho_z-1)}{(\sigma\psi_x+1-\rho_z)(1-\beta\rho_z)+\sigma\rho_z(\psi_\pi-1)\lambda} & 0 \\ 0 & \frac{1}{(\sigma\psi_x+1-\rho_g)(1-\beta\rho_g)+\sigma\rho_g(\psi_\pi-1)\lambda} \end{pmatrix}.$$

A necessary condition for the identification of eq. (3) is that there is no linear combination of the endogenous regressors, π_{t+1} and x_t, that is uncorrelated with variables known at time $t-1$. Dropping the explicit dependence of $A(\theta)$ on θ for simplicity

$$\pi_t = a_{\pi z} z_t + a_{\pi g} g_t + f_{\pi r} \varepsilon_{r,t} \tag{11}$$

$$x_t = a_{xz} z_t + a_{xg} g_t + f_{xr} \varepsilon_{r,t} \tag{12}$$

it follows that

$$\begin{pmatrix} E_{t-1}\pi_{t+1} \\ E_{t-1}x_t \end{pmatrix} = \underbrace{\begin{pmatrix} a_{\pi z}\rho_z & a_{\pi g}\rho_g \\ a_{xz} & a_{xg} \end{pmatrix} \begin{pmatrix} \rho_z & 0 \\ 0 & \rho_g \end{pmatrix}}_{F(\theta)} \begin{pmatrix} z_{t-1} \\ g_{t-1} \end{pmatrix}. \tag{13}$$

Thus, the rank condition for identification is that

$$\det \begin{pmatrix} a_{\pi z}\rho_z & a_{\pi g}\rho_g \\ a_{xz} & a_{xg} \end{pmatrix} \det \begin{pmatrix} \rho_z & 0 \\ 0 & \rho_g \end{pmatrix} \neq 0.$$

But

$$\begin{aligned} &\det \begin{pmatrix} a_{\pi z}\rho_z & a_{\pi g}\rho_g \\ a_{xz} & a_{xg} \end{pmatrix} \\ &= \det \begin{pmatrix} \lambda\rho_z & \lambda\rho_g \\ 1-\beta\rho_z & 1-\beta\rho_g \end{pmatrix} \det \begin{pmatrix} \frac{(\rho_z-1)}{(\sigma\psi_x+1-\rho_z)(1-\beta\rho_z)+\sigma\rho_z(\psi_\pi-1)\lambda} & 0 \\ 0 & \frac{1}{(\sigma\psi_x+1-\rho_g)(1-\beta\rho_g)+\sigma\rho_g(\psi_\pi-1)\lambda} \end{pmatrix} \\ &\propto \lambda(\rho_z-\rho_g)(\rho_z-1) \end{aligned}$$

So, the rank condition for identification is

$$\lambda(\rho_z-\rho_g)(\rho_z-1)\rho_z\rho_g \neq 0.$$

2 An alternative DSGE model with cost push shocks

An alternative model of the transmission mechanism is found in Clarida Gali Gertler (1999, Equations 2.1 through 2.4)

$$\pi_t = \beta E_t \pi_{t+1} + \lambda x_t + u_t \tag{14}$$

$$x_t = E_t x_{t+1} - \sigma \left(r_t - E_t \pi_{t+1} \right) + g_t \tag{15}$$

where x_t is the output gap as above, u_t and g_t are cost-push and demand shocks, respectively, whose laws of motion are given by

$$u_t = \rho u_{t-1} + \hat{u}_t$$

$$g_t = \mu g_{t-1} + \hat{g}_t$$

with $0 \leq \rho, \mu \leq 1$ and $\hat{u}_t, \hat{g}_t$ are iid random variables with zero mean and variances σ_u^2 and σ_g^2, respectively. The system is closed by the non-inertial forward-looking Taylor rule (3). The model can be written compactly in the form (7), with the coefficient matrices $B_0(\theta)$ and $B_1(\theta)$ given by (8), $C(\theta) = I_2$, the identity matrix of dimension 2. Following the arguments in the last section, the conditions for determinacy are exactly as in the model given by (1), (2) and (3), and the determinate solution will be of the form (10), but with a different coefficient matrix $A(\theta)$, which can be found by solving the equation

$$B_0(\theta) A(\theta) = B_1(\theta) A(\theta) D(\theta) + I_2$$

where $D(\theta)$ is a diagonal matrix with elements ρ and μ along the diagonal. This equation is given by

$$\begin{pmatrix} 1 & -\lambda \\ 0 & 1 + \sigma\psi_x \end{pmatrix} \begin{pmatrix} a_{\pi u} & a_{\pi g} \\ a_{xu} & a_{xg} \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ -\sigma(\psi_\pi - 1) & 1 \end{pmatrix} \begin{pmatrix} a_{\pi u}\rho & a_{\pi g}\mu \\ a_{xu}\rho & a_{xg}\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} a_{\pi u} - \lambda a_{xu} - \beta\rho a_{\pi u} & a_{\pi g} - \lambda a_{xg} - \beta\mu a_{\pi g} \\ a_{xu}(\sigma\psi_x + 1) - \rho a_{xu} + \sigma\rho(\psi_\pi - 1) a_{\pi u} & a_{xg}(\sigma\psi_x + 1) - \mu a_{xg} + \sigma\mu(\psi_\pi - 1) a_{\pi g} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which yields the four equations:

$$
\begin{aligned}
(1-\beta\rho)\, a_{\pi u} &= 1+\lambda a_{xu} \\
(1-\beta\mu)\, a_{\pi g} &= \lambda a_{xg} \\
(\sigma\psi_x+1-\rho)\, a_{xu} &= -\sigma\rho\,(\psi_\pi-1)\, a_{\pi u} \\
(\sigma\psi_x+1-\mu)\, a_{xg} &= 1-\sigma\mu\,(\psi_\pi-1)\, a_{\pi g}
\end{aligned}
$$

So,

$$
\begin{aligned}
(\sigma\psi_x+1-\mu)\,(1-\beta\mu)\, a_{\pi g} = \lambda-\lambda\sigma\mu\,(\psi_\pi-1)\, a_{\pi g} \Rightarrow \\
a_{\pi g} = \frac{\lambda}{(\sigma\psi_x+1-\mu)\,(1-\beta\mu)+\lambda\sigma\mu\,(\psi_\pi-1)}
\end{aligned}
$$

$$
a_{xg} = \frac{(1-\beta\mu)}{(\sigma\psi_x+1-\mu)\,(1-\beta\mu)+\lambda\sigma\mu\,(\psi_\pi-1)}
$$

$$
\begin{aligned}
(\sigma\psi_x+1-\rho)\,(1-\beta\rho)\, a_{xu} = -\sigma\rho\,(\psi_\pi-1)\,(1+\lambda a_{xu}) \Rightarrow \\
a_{xu} = \frac{-\sigma\rho\,(\psi_\pi-1)}{(\sigma\psi_x+1-\rho)\,(1-\beta\rho)+\lambda\sigma\rho\,(\psi_\pi-1)}
\end{aligned}
$$

$$
a_{\pi u} = \frac{\sigma\psi_x+1-\rho}{(\sigma\psi_x+1-\rho)\,(1-\beta\rho)+\lambda\sigma\rho\,(\psi_\pi-1)}.
$$

The rank condition for identification of the forward-looking policy rule (3) is determined by the rank of the matrix $F(\theta)$ in the regression

$$
\begin{pmatrix} E_{t-1}\pi_{t+1} \\ E_{t-1}x_t \end{pmatrix} = \underbrace{\begin{pmatrix} a_{\pi u}\rho & a_{\pi g}\mu \\ a_{xu} & a_{xg} \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & \mu \end{pmatrix}}_{F(\theta)} \begin{pmatrix} u_{t-1} \\ g_{t-1} \end{pmatrix}
$$

Now,

$$
\begin{aligned}
\det \begin{pmatrix} a_{\pi u}\rho & a_{\pi g}\mu \\ a_{xu} & a_{xg} \end{pmatrix} &= \det \begin{pmatrix} \frac{(\sigma\psi_x+1-\rho)\rho}{(\sigma\psi_x+1-\rho)(1-\beta\rho)+\lambda\sigma\rho(\psi_\pi-1)} & \frac{\lambda\mu}{(\sigma\psi_x+1-\mu)(1-\beta\mu)+\lambda\sigma\mu(\psi_\pi-1)} \\ \frac{-\sigma\rho(\psi_\pi-1)}{(\sigma\psi_x+1-\rho)(1-\beta\rho)+\lambda\sigma\rho(\psi_\pi-1)} & \frac{(1-\beta\mu)}{(\sigma\psi_x+1-\mu)(1-\beta\mu)+\lambda\sigma\mu(\psi_\pi-1)} \end{pmatrix} \\
&\propto \rho\left[(\sigma\psi_x+1-\rho)\,(1-\beta\mu)+\lambda\mu\sigma\,(\psi_\pi-1)\right]
\end{aligned}
$$

which, given the sign restrictions in the parameters, is zero if and only if $\rho = 0$. So, the rank condition does not fail when the autocorrelation of the demand and cost push shocks is the same (provided the demand and cost push shocks are not highly correlated), or when the Phillips curve is flat, unlike the model in Clarida Gali and Gertler (2000) and Lubik and Schorfheide (2004).