Appendix

Proof of Lemma 1: We begin showing that the monotonicity condition, $h_{ij} \geq h_{i,j+1}$ for $j \leq m-1$, is necessary for incentive compatibility. Consider type (i,j) and a deviation to $(i,\hat{\jmath})$:

$$h_{ij} \cdot (i-j) - T_{ij} \geq h_{i\hat{\jmath}} \cdot (i-j) - T_{i\hat{\jmath}}.$$

Now, consider type $(i,\hat{\jmath})$ (interchanging type and message) and a deviation to (i,j) :

$$h_{i\hat{\jmath}} \cdot (i-\hat{\jmath}) - T_{i\hat{\jmath}} \geq h_{ij} \cdot (i-\hat{\jmath}) - T_{ij}.$$

Rearranging, we have

$$(h_{ij} - h_{i\hat{\jmath}})(\hat{\jmath} - j) \geq 0$$

which proves the claim.

Next we argue that $v_{ij} = v_{im} + \sum_{k=j+1}^{m} h_{ik}$ and h_{ij} non-increasing in j are sufficient for incentive compatibility. To ease notation, define

$$V(i,\hat{\jmath},j) \equiv h_{i\hat{\jmath}} \cdot (i-j) - T_{i\hat{\jmath}}.$$

The condition $v_{ij} = v_{im} + \sum_{k=j+1}^{m} h_{ik}$ results from imposing the right-wards adjacent incentive constraints with equality and solving recursively. To see this, suppose the right-ward adjacent constraint holds with equality. Then,

$$\begin{aligned} h_{ij} \cdot (i-j) - T_{ij} &= h_{i,j+1} \cdot (i-j) - T_{i,j+1} \\ &= h_{i,j+1} \cdot (i-(j+1)) - T_{i,j+1} + h_{i,j+1}. \end{aligned}$$

So, $V(i,j,j) = V(i,j+1,j+1) + h_{i,j+1}$. Applying this logic repeatedly and solving recursively, gives expression (7). We wish to show that (7) and monotonicity of h_{ij} jointly imply that any deviation from truth-telling is suboptimal. Notice that the excess income that type (i,j) obtains from mimicking type (i,l) is given by

$$\begin{aligned} V(i,l,j) &= h_{il} \cdot (i-j) - T_{il} \\ &= h_{il} \cdot (i-l) - T_{il} - (j-l) \cdot h_{il} \\ &= V(i,l,l) - (j-l) \cdot h_{il}. \end{aligned}$$

Thus, $V(i,j,j) \geq V(i,l,j)$ for any l and j if

$$V(i,m,m) + \sum_{k=j+1}^{m} h_{ik} \geq V(i,m,m) + \sum_{k=l+1}^{m} h_{ik} - (j-l) \cdot h_{il}.$$

Consider first any $l > j$. We can write the comparison as

$$
\begin{aligned}
& V(i,m,m) + \sum_{k=l+1}^{m} h_{ik} + h_{i,j+1} + ... + h_{il} \\
\geq \; & V(i,m,m) + \sum_{k=l+1}^{m} h_{ik} - (j-l) \cdot h_{il}
\end{aligned}
$$

Cancelling equal terms on both sides we can simplify the condition to

$$h_{i,j+1} + ... + h_{il} \geq (l-j) \cdot h_{il}.$$

Since the number of terms on each side is the same, and h_{ij} is non-increasing in j, the inequality is satisfied. The proof for the case where $l < j$ is similar and therefore omitted.

Consider now the participation constraints. From the right-wards adjacent incentive constraints, $V(i,j,j) = V(i,j+1,j) \geq V(i,j+1,j+1)$, and from the participation constraint of type (i,m), $V(i,m,m) \geq 0$, all the participation constraints are satisfied.

Next, we show that all the incentive constraints must hold with equality. To see this, suppose there is a type (i,j) such that

$$V(i,j,j) > V(i,j+1,j+1) + h_{i,j+1}.$$

Then we can change the incentive system as follows. We can find $\varepsilon_1, \varepsilon_2 > 0$ to change the taxes to

$$\tilde{T}_{ij} = T_{ij} + \varepsilon_1 \quad \text{and} \quad \tilde{T}_{i,j+1} = T_{i,j+1} - \varepsilon_2.$$

The effect is to reduce type $(i,j)'s$ after-tax excess income and to increase type $(i,j+1)'s$ after-tax excess income. Let $\tilde{V}(i,j,j)$ and $\tilde{V}(i,j+1,j+1)$, respectively, denote the resulting after-tax excess incomes. Recall that $p_j(i)$ denotes the conditional probability that the non-market productivity takes value j conditional on i. Since we do not change the allocation of types' (i,j) and $(i,j+1)$ working time, we have to respect the condition $p_j(i) \cdot \varepsilon_1 = p_{j+1}(i) \cdot \varepsilon_2$. By construction, $\begin{pmatrix} V(i,j,j) \\ V(i,j+1,j+1) \end{pmatrix}$ can be viewed as generated from $\begin{pmatrix} \tilde{V}(i,j,j) \\ \tilde{V}(i,j+1,j+1) \end{pmatrix}$ by a mean-preserving spread. Since $U(\cdot)$ is concave, the latter gives the objective function a higher value.

Finally, we can recover the taxes collected from the allocation of work time and the optimal after-tax excess incomes using the relation $v_{ij} = h_{ij} \cdot (i-j) - T_{ij}$ and (7). We obtain

$$T_{ij} = h_{ij} \cdot (i-j) - v_{im} - \sum_{k=j+1}^{m} h_{ik} \tag{20}$$

Substituting condition (20) into the objective function and the resource constraint gives the representation of the problem in the lemma.

Proof of Proposition 1: The proof is given in two parts. In the first part, we characterize the optimal allocation. In the second part, we use the structure of the optimal allocation to derive the budget constraint.

Part i: the structure of the allocation

The Lagrangian for our problem takes the form

$$\begin{aligned} L_i &= \sum_{j=1}^{m} p_j(i) \cdot U\left(j + v_{im} + \sum_{k=j+1}^{m} h_{ik}\right) \\ &\quad + \lambda_i \cdot \left(\sum_{j=1}^{m} p_j(i) \cdot \left(h_{ij} \cdot (i-j) - v_{im} - \sum_{k=j+1}^{m} h_{ik}\right) - T\right). \end{aligned}$$

For notational ease in this proof, let the marginal utility of type (i, j) be

$$u_{ij} \equiv \frac{d}{dc} U\left(j + v_{im} + \sum_{k=j+1}^{m} h_{ik}\right).$$

The derivative of L_i with respect to h_{i1} is equal to

$$\frac{\partial L_i}{\partial h_{i1}} = \lambda_i \cdot (i-1) \cdot p_1(i)$$

which implies directly that $h^*_{i1} = 1$ since $i - 1 \geq 0$.

The derivative of L_i with respect to h_{iz} is equal to

$$\frac{\partial L_i}{\partial h_{iz}} = \sum_{j=1}^{z-1} p_j(i) \cdot u_{ij} + \lambda_i \cdot (p_z(i) \cdot (i-z) - P_{z-1}(i)).$$

In what follows, we will make repeated use of a convenient transformation. Define

$$E[u_{ij} | j \leq z-1] \equiv \sum_{j=1}^{z-1} \frac{p_j(i)}{P_{z-1}(i)} \cdot u_{ij}.$$

We prove that our problem admits an interior solution for at most one h_{iz}. The derivative of L_i with respect to h_{iz} for $z > 1$ is proportional to

$$\frac{\frac{\partial L_i}{\partial h_{iz}}}{P_{z-1}(i)} = E[u_{ij} | j \leq z-1] + \lambda_i \cdot \left(\frac{p_z(i)}{P_{z-1}(i)} \cdot (i-z) - 1\right). \tag{21}$$

Suppose (21) admits an interior solution for $z = t$, so the first-order condition holds:

$$E[u_{ij} | j \leq t-1] = \lambda_i \cdot \left(1 - \frac{p_t(i)}{P_{t-1}(i)} \cdot (i-t)\right).$$

$E[u_{ij} | j \leq z-1]$ is non-increasing in z. To see this, note that incomes are non-decreasing in the value of non-market time, since by incentive compatibility

$$j+1+\sum_{k=j+2}^{m} h_{ik}-\left(j+\sum_{k=j+1}^{m} h_{ik}\right)=1-h_{i,j+1}\geq 0. \tag{22}$$

Hence, to prove our claim, it suffices to show that the expression $(p_z(i)/P_{z-1(i)})\cdot(i-z)$ is decreasing in z, because that implies that the first-order condition cannot hold for $z>t$. So we want to show that

$$\frac{p_z(i)}{P_{z-1}(i)}\cdot(i-z)>\frac{p_{z+1}(i)}{P_z(i)}\cdot(i-(z+1)).$$

Let $a\equiv i-z$. With these definitions, the condition is equivalent to

$$a\frac{p_{i-a}(i)}{P_{i-a-1}(i)}>(a-1)\frac{p_{i-a+1}(i)}{P_{i-a}(i)}.$$

The condition is trivially satisfied for $a=1$; so assume that $a>1$. Multiplying both sides by $\frac{P_{i-a}(i)}{p_{i-a+1}(i)}\cdot\frac{P_{i-a-1}(i)}{p_{i-a}(i)}$ and rearranging we have the equivalent condition

$$\frac{P_{i-a-1}(i)}{p_{i-a}(i)}>a\cdot\left(\frac{P_{i-a-1}(i)}{p_{i-a}(i)}-\frac{P_{i-a}(i)}{p_{i-a+1}(i)}\right). \tag{23}$$

Notice that the condition is trivially satisfied if the inverse hazard rate is non-decreasing in j, because that makes the expression on the right-hand side become negative, while the expression on the left-hand side is strictly positive. However, suppose the inverse hazard rate is decreasing so that the right-hand side is strictly positive. In that case, (23) is still satisfied, provided that

$$\frac{P_{i-a-1}(i)}{p_{i-a}(i)}\geq\frac{P_{i-1}(i)}{p_i(i)}+a\cdot\left(\frac{P_{i-a-1}(i)}{p_{i-a}(i)}-\frac{P_{i-a}(i)}{p_{i-a+1}(i)}\right).$$

Rearranging, we have

$$\frac{P_{i-a-1}(i)}{p_{i-a}(i)}+a\left(\frac{P_{i-a}(i)}{p_{i-a+1}(i)}-\frac{P_{i-a-1}(i)}{p_{i-a}(i)}\right)\geq\frac{P_{i-1}(i)}{p_i(i)}. \tag{24}$$

Assumption 1 implies that $\frac{P_j(i)}{p_{j-1}(i)}-\frac{P_{j-1}(i)}{p_j(i)}$ is non-increasing, which in turn implies that (24) is satisfied. Hence, the solution for $z>t$ is $h^*_{iz}=0$.

Part ii: derivation of the resource constraint

Using the particular allocation, the tax paid by the marginal type satisfies $h^*_{it}\cdot(i-t)=T^*_{it}$ because this type's participation constraint is binding. The excess income of type $(i,t-1)$ satisfies $V(i,t-1,t-1)=V(i,t,t-1)=h^*_{it}$, so his total income is equal to $t-1+h^*_{it}$. The taxes he pays satisfy the relation $i-(t-1)-T^*_{i,t-1}=h^*_{it}$, so

$$T^*_{i,t-1}=i-((t-1)+h^*_{it}).$$

Since the marginal utilities of all inframarginal types are the same, all their incomes are the same, so the taxes paid by all inframarginal types are the same. Summing the taxes together we obtain the expression in the text:

$$T = P_{t-1}(i) \cdot (i - ((t-1) + h_{it}^*)) + p_t(i) \cdot h_{it}^* \cdot (i - t).$$

Let $T^{\max} \equiv \max_{t \leq i} (i - (t-1)) P_{t-1}(i)$ and $\tau \equiv \arg\max_{t \leq i} (i - (t-1)) P_{t-1}(i)$. Given Assumption 1, τ exists and is unique. We can compute t and h_{it}^* as follows, generalizing the procedure described in the text. For $x \in \mathbb{N}_+ \cup 0$ and T satisfying

$$xP_{i-a} < T \leq (x+1) P_{i-(x+1)}$$

we set

$$t = i - x.$$

This solution is feasible if and only if $t \geq \tau$. The time spent by the marginal type in the formal sector satisfies

$$h_{it}^* = \frac{T - (x+1) P_{i-(x+1)}(i)}{xp_{i-x}(i) - P_{i-(x+1)}(i)}.$$

Proof of Lemma 2: In contrast to Lemma 1, the left-wards adjacent constraints must bind whenever the left-wards neighbor is working in the formal sector. Imposing these constraints and solving recursively, we find that

$$v_{ij} = v_{i1} - \sum_{k=1}^{j-1} h_{ik} \tag{25}$$

for any type who is supposed to be included in the redistribution program.

It can be shown that the left-ward adjacent incentive constraints plus monotonicity imply that there is no profitable deviation from truth-telling. Since this is standard, it is omitted. Second, following the same proof as in Lemma 1, one can show that the adjacent constraints must be tight for all types that work in the formal sector. To avoid repetition, this step is omitted as well.

If the government wishes to include type (i, s), then

$$v_{is} = v_{i1} - \sum_{k=1}^{s-1} h_{ik} \geq 0.$$

The participation constraint of type (i, s) implies that all types (i, j) for $j < s$ also want to participate. On the other hand, the exclusion constraint for type $(i, s+1)$,

$$v_{i,s+1} = v_{i1} - \sum_{k=1}^{s} h_{ik} \leq 0$$

implies that all types (i, j) for $j > s+1$ are also excluded. In particular, if type $(i, s+2)$ mimics the marginal type (i,s) he obtains a net excess income of

$$V(i,s,s+2) = V(i,1,1) - \sum_{k=1}^{s} h_{ik} - h_{is} \leq -h_{is} < 0.$$

An analogous argument can be given for any type (i,j) for $j > s+1$.

Finally, using

$$v_{ij} = \begin{cases} v_{i1} - \sum\limits_{k=1}^{j-1} h_{ik} & \text{for } j \leq s \\ 0 & \text{otherwise,} \end{cases}$$

one can recover the subsidies paid to types (i,j) for $j \leq s$. Substituting the resulting expressions into the objective and the resource constraint gives the representation of the problem in the lemma.

Proof of Proposition 2: The proof is given in two parts. In part i, we derive the structure of the allocation. In part ii) we use this structure to derive the representation of the resource constraint.

Part i: structure of the allocation

The Lagrangian function for our problem takes the form

$$\begin{aligned} L_i(s) \quad = \quad & \sum_{j=1}^{s} p_j(i) \cdot U\left(j + v_{i1} - \sum_{k=1}^{j-1} h_{ik}\right) + \sum_{j=s+1}^{m} p_j(i) \cdot U(j) \\ & + \lambda_i(s) \cdot \left(\sum_{j=1}^{s} p_j(i) \cdot \left(h_{ij} \cdot (i-j) - \left(v_{i1} - \sum_{k=1}^{j-1} h_{ik} \right) \right) + S \right) \\ & + \alpha \left(v_{i1} - \sum_{k=1}^{s-1} h_{ik} \right) - \beta \left(v_{i1} - \sum_{k=1}^{s} h_{ik} \right). \end{aligned}$$

To ease notation in this proof, we define the marginal utility of type (i,j) as

$$u_{ij} \equiv \frac{d}{dc} U\left(j + v_{i1} - \sum_{k=1}^{j-1} h_{ik}\right).$$

We begin by stating the derivatives of the objective function with respect to the relevant choice variables. The derivative with respect to h_{iz} for $z < s$ is equal to

$$\frac{\partial L_i(s)}{\partial h_{iz}} = -\sum_{j=z+1}^{s} p_j(i) \cdot u_{ij} + \lambda_i(s) \cdot \left(p_z(i) \cdot (i-z) + \sum_{j=z+1}^{s} p_j(i) \right) - \alpha + \beta. \tag{26}$$

The derivative with respect to h_{is} is equal to

$$\frac{\partial L_i(s)}{\partial h_{is}} = \lambda_i(s) \cdot p_s(i) \cdot (i-s) + \beta. \tag{27}$$

The derivative with respect to v_{i1} is equal to

$$\frac{\partial L_1(s)}{\partial v_{i1}} = \sum_{j=1}^{s} p_j(i) \cdot u_{ij} - \lambda_i(s) \cdot \sum_{j=1}^{s} p_j(i) + \alpha - \beta. \tag{28}$$

We analyze the case where $S < P_{m-1}(m-i)$, because -as we show in Sections II.C an II.D - the case $S > P_{m-1}(m-i)$ cannot be part of an overall optimum. At the optimum, for $S > 0$ we must have $i - s < 0$. This implies by (27) that $\beta > 0$. To see this, suppose that $\beta = 0$. Then, by (27), we would have $h^*_{is} = 0$. But then, type $(i, s+1)$ is not excluded, so the allocation is not incentive compatible. Next, notice that $\alpha = 0$ at the optimum. If both β and α were strictly positive, then - since both constraints must hold with equality - we would have again that $h^*_{is} = 0$, which means that effectively type $(i, s-1)$ is the marginal type. Finally, at any optimum both h_{is} and v_{i1} must be at stationary points, so $\frac{\partial L_i(s)}{\partial h_{is}} = 0$ and $\frac{\partial L_1(s)}{\partial v_{i1}} = 0$. From (28) we have the first-order condition for v_{i1}

$$\sum_{j=1}^{s} p_j(i) \cdot u_{ij} - \lambda_i(s) \cdot \sum_{j=1}^{s} p_j(i) = \beta. \tag{29}$$

Substituting (29) into the condition (27), we obtain

$$\frac{\partial L_i(s)}{\partial h_{is}} = \sum_{j=1}^{s} p_j(i) \cdot u_{ij} + \lambda_i(s) \cdot \left(p_s(i) \cdot (i-s) - \sum_{j=1}^{s} p_j(i) \right). \tag{30}$$

Substituting (29) into (26), we obtain

$$\begin{aligned} \frac{\partial L_i(s)}{\partial h_{iz}} &= -\sum_{j=z+1}^{s} p_j(i) \cdot u_{ij} + \lambda_i(s) \cdot \left(p_z(i) \cdot (i-z) + \sum_{j=z+1}^{s} p_j(i) \right) \\ &\quad + \sum_{j=1}^{s} p_j(i) \cdot u_{ij} - \lambda_i(s) \cdot \sum_{j=1}^{s} p_j(i) \\ &= \sum_{j=1}^{z} p_j(i) \cdot u_{ij} + \lambda_i(s) \cdot \left(p_z(i) \cdot (i-z) - \sum_{j=1}^{z} p_j(i) \right). \end{aligned} \tag{31}$$

Dividing by $P_z(i)$, we can write both (30) and (31) as

$$\frac{\frac{\partial L_i(s)}{\partial h_{iz}}}{P_z(1)} = \sum_{j=1}^{z} \frac{p_j(i)}{P_z(i)} \cdot u_{ij} + \lambda_i(s) \cdot \left(\frac{p_z(i)}{P_z(i)} \cdot (i-z) - 1 \right) \tag{32}$$

for $z \leq s$. From our derivation above, the right-hand side of (32) is equal to zero at $z = s$, so

$$\sum_{j=1}^{s} \frac{p_j(i)}{P_s(i)} \cdot u_{ij} + \lambda_i(s) \cdot \left(\frac{p_s(i)}{P_s(i)} \cdot (i-s) - 1 \right) = 0. \tag{33}$$

To prove our proposition, it suffices to show that (33), in conjunction with Assumption 1 implies that

$$\sum_{j=1}^{z} \frac{p_j(i)}{P_z(i)} \cdot u_{ij} + \lambda_i(s) \cdot \left(\frac{p_z(i)}{P_z(i)} \cdot (i-z) - 1 \right) > 0$$

for all $z < s$. Letting $E[u_{ij}| j \le z] \equiv \sum_{j=1}^{z} \frac{p_j(i)}{P_z(i)} \cdot u_{ij}$ this inequality can be written as

$$E[u_{ij}| j \le z] > \lambda_i(s) \cdot \left(1 - \frac{p_z(i)}{P_z(i)} (i-z) \right)$$

for all $z < s$. We note that type (i, z) receives a weakly higher total income than type $(i, z-1)$, since

$$z + v_{i1} - \sum_{k=1}^{z-1} h_{ik} - \left(z - 1 + v_{i1} - \sum_{k=1}^{z-2} h_{ik} \right) = 1 - h_{i,z-1} \ge 0.$$

Therefore, $E[u_{ij}| j \le z]$ is non-increasing in z. Hence, $E[u_{ij}| j \le z] \ge E[u_{ij}| j \le s]$ for all $z < s$. To complete the argument, it suffices to show that $(1 - (p_z(i)/P_z(i)) \cdot (i-z)) < (1 - (p_s(i)/P_s(i)) \cdot (i-s))$ for all $z < s$. This is equivalent to

$$\frac{p_z(i)}{P_z(i)} \cdot (z-i) < \frac{p_s(i)}{P_s(i)} \cdot (s-i). \tag{34}$$

It is easy to see that this condition is verified for all z such that $z \le i$. We now prove that, under Assumption 1, the condition is also verified for any z such that $i < z < s$.

In particular, we show that Assumption 1 implies that for all $z \le s$

$$\frac{p_{z-1}(i)}{P_{z-1}(i)} \cdot (z-1-i) < \frac{p_z(i)}{P_z(i)} \cdot (z-i)$$

which in turn implies (34). To see this, it proves convenient to normalize this monotonicity condition around i. Let $a \equiv z - i$. With that definition, the condition is equivalent to

$$(a-1) \cdot \frac{p_{i+a-1}(i)}{P_{i+a-1}(i)} < a \cdot \frac{p_{i+a}(i)}{P_{i+a}(i)}.$$

This condition is trivially satisfied for $a = 1$. So consider the case where $a > 1$. Manipulating this condition the same way as we did in the case of taxation, we have the equivalent condition that

$$\frac{P_{i+a}(i)}{p_{i+a}(i)} > a \cdot \left(\frac{P_{i+a}(i)}{p_{i+a}(i)} - \frac{P_{i+a-1}(i)}{p_{i+a-1}(i)} \right). \tag{35}$$

Finally, notice that Assumption 1 implies condition (35). To see this, observe simply that Assumption 1 implies that

$$\frac{P_{z+a}(i)}{p_{z+a}(i)} \ge \frac{P_z(i)}{p_z(i)} + a \cdot \left(\frac{P_{z+a}(i)}{p_{z+a}(i)} - \frac{P_{z+a-1}(i)}{p_{z+a-1}(i)} \right) \tag{36}$$

for any z and any $a \geq 0$. Since $\frac{P_z(i)}{p_z(i)} > 0$, (36) implies (35).

Part ii: Derivation of the Resource Constraint

With a binding exclusion constraint we have $v_{i1} - \sum_{k=1}^{s} h_{ik} = 0$. Therefore, the excess income of all types who receive subsidies are given by

$$v_{ij} = v_{i1} - \sum_{k=1}^{j-1} h_{ik} = \sum_{k=1}^{s} h_{ik} - \sum_{k=1}^{j-1} h_{ik} = \sum_{k=j}^{s} h_{ik}.$$

We can calculate the individual subsidies, $S_{ij} = -T_{ij}$, using the relation

$$v_{ij} = h_{ij} \cdot (i - j) + S_{ij}.$$

Hence,

$$S_{ij} = \sum_{k=j}^{s} h_{ik} - h_{ij} \cdot (i - j).$$

Using the structure of the allocation, we get

$$S_{ij}^{*} = \begin{cases} s - i + h_{is}^{*} & for\ j < s \\ h_{is}^{*} \cdot (s + 1 - i) & for\ j = s \\ 0 & for\ j > s. \end{cases}$$

Summing these individual subsidies up we obtain

$$P_{s-1}(i) \cdot (s - i + h_{is}^{*}) + p_s(i) \cdot h_{is}^{*} \cdot (s + 1 - i) = S. \tag{37}$$

The marginal type is chosen optimally if s is as large as possible. If s can still be increased, this means that we can find a Pareto improvement as follows. By raising s, fewer types are excluded. All types that are included receive the same level of income. Hence, by raising s we raise all the incomes of all types that are included. The incomes of those who are and remain excluded are unchanged.

Generalizing the procedure described in the text, one can check that for $x \in \mathbb{N}_{+} \cup 0$ and for $S \in \left(xP_{i+(x-1)}(i), (x+1) P_{i+x}(i)\right]$ the marginal type is

$$s = i + x$$

and h_{is}^{*} is determined by the condition

$$h_{is}^{*} = \frac{S - xP_{i+x-1}(i)}{xp_{i+x}(i) + P_{i+x}(i)}.$$

Proof of Proposition 4: Suppose the contrapositive were true and there were a productivity group i'' that is subsidized and a productivity group i' that is taxed, where $i'' > i'$. Based on this assumption, we will construct a budget balanced, incentive compatible redistribution scheme between these two groups. It follows that the initial allocation was not optimal.

The idea of the redistribution scheme is as follows. Given $i' \geq 1$, there is in each productivity group a set of individuals with low opportunity costs of time who will work full time at the optimal allocation. In groups that are taxed, the right-wards incentive constraints are tight. It follows that the marginal type, who works part time, has a strict preference for his own allocation relative to mimicking his left-ward neighbor who works full time. To ease notation in this proof, we shall write $t(i')$ for $t(i', T_{i'})$, $s(i'')$ for $s(i'', T_{i''})$, and, accordingly, $h_{s(i)}$ and $h_{t(i)}$.

To see these arguments formally, recall that the optimal allocation satisfies

$$V\left(i', t\left(i'\right) - 1, t\left(i'\right) - 1\right) = V\left(i', t\left(i'\right), t\left(i'\right) - 1\right) = V\left(i', t\left(i'\right), t\left(i'\right)\right) + h_{t(i')}.$$

Hence, we can write

$$V\left(i', t\left(i'\right), t\left(i'\right)\right) = V\left(i', t\left(i'\right) - 1, t\left(i'\right) - 1\right) - h_{t(i')}.$$

If the marginal type mimics his left-wards neighbor, he obtains excess income

$$V\left(i', t\left(i'\right) - 1, t\left(i'\right)\right) = V\left(i', t\left(i'\right) - 1, t\left(i'\right) - 1\right) - 1.$$

But then it follows that

$$\begin{aligned} V\left(i', t\left(i'\right) - 1, t\left(i'\right)\right) &= V\left(i', t\left(i'\right) - 1, t\left(i'\right) - 1\right) - 1 \\ &< V\left(i', t\left(i'\right) - 1, t\left(i'\right) - 1\right) - h_{t(i')} = V\left(i', t\left(i'\right), t\left(i'\right)\right). \end{aligned}$$

Hence, we can decrease the taxes paid by all individuals who work full time by an identical amount, say $\varepsilon_{i'}$, without violating incentive compatibility.

In the group that is subsidized, we can decrease the subsidies paid to all individuals who work full time by an amount $\varepsilon_{i''}$ without affecting incentive compatibility and the exclusion constraint. To see this, recall that we have imposed the left-wards constraint for the marginal type so that

$$V\left(i'', s\left(i''\right), s\left(i''\right)\right) = V\left(i'', s\left(i''\right) - 1, s\left(i''\right)\right) = V\left(i'', s\left(i''\right) - 1, s\left(i''\right) - 1\right) - 1$$

Hence,

$$V\left(i'', s\left(i''\right) - 1, s\left(i''\right) - 1\right) = V\left(i'', s\left(i''\right), s\left(i''\right)\right) + 1.$$

If type $(i'', s(i'') - 1)$ mimics his right-wards neighbor, then he would obtain an excess income of

$$V\left(i'', s\left(i''\right), s\left(i''\right) - 1\right) = V\left(i'', s\left(i''\right), s\left(i''\right)\right) + h_{s(i'')}.$$

Hence, it follows that

$$V\left(i'', s\left(i''\right) - 1, s\left(i''\right) - 1\right) = V\left(i'', s\left(i''\right), s\left(i''\right)\right) + 1 > V\left(i'', s\left(i''\right), s\left(i''\right)\right) + h_{s(i'')} = V\left(i'', s\left(i''\right), s\left(i''\right) - 1\right).$$

Choose $\varepsilon_{i''}$ and $\varepsilon_{i'}$ such that

$$p_{i''} \cdot P_{s(i'')-1}\left(i''\right) \cdot \varepsilon_{i''} + p_{i'} \cdot P_{t(i')-1}\left(i'\right) \cdot \varepsilon_{i'} = 0.$$

By construction, the new allocation and the initial allocation generate the same expected level of income for all groups together. However, the distributions differ by a mean preserving spread. Hence, the new allocation is preferred.

The following definition and Lemma are used in the proof of Proposition 5.

Definition *The parameters i and j are affiliated if for any $j > j'$ and any integer $b > 0$*

$$\frac{p_{ij'}}{p_{ij}} - \frac{p_{i+b,j'}}{p_{i+b,j}} \geq 0. \tag{38}$$

The parameters $-i$ and j are affiliated if for any $j > j'$ and any integer $b > 0$

$$\frac{p_{ij'}}{p_{ij}} - \frac{p_{i+b,j'}}{p_{i+b,j}} \leq 0. \tag{39}$$

We say that the degree of affiliation is non-decreasing in i if for any $j > j'$ and any integer $b > 0$

$$\frac{p_{ij'}}{p_{ij}} - \frac{p_{i+b,j'}}{p_{i+b,j}} \geq \frac{p_{i'j'}}{p_{i'j}} - \frac{p_{i'+b,j'}}{p_{i'+b,j}} \text{ for any } i > i'. \tag{40}$$

Lemma 3 *If the degree of affiliation is non-decreasing in i then for any j $\frac{P_j(i)}{p_j(i)} \geq \frac{1}{2}\frac{P_j(i-1)}{p_j(i-1)} + \frac{1}{2}\frac{P_j(i+1)}{p_j(i+1)}$ for all $1 < i < n$.*

Proof of Lemma 3: With $b = 1$ and $i' = i - 1$ we have from (40)

$$\frac{p_{ij'}}{p_{ij}} - \frac{p_{i+1,j'}}{p_{i+1,j}} \geq \frac{p_{i-1,j'}}{p_{i-1,j}} - \frac{p_{i,j'}}{p_{i,j}}.$$

Multiplying the first ratio by $\frac{p_i}{p_i}$, the second by $\frac{p_{i+1}}{p_{i+1}}$, and so on, we can write

$$\frac{p_i}{p_i} \cdot \frac{p_{ij'}}{p_{ij}} - \frac{p_{i+1}}{p_{i+1}} \cdot \frac{p_{i+1,j'}}{p_{i+1,j}} \geq \frac{p_{i-1}}{p_{i-1}} \cdot \frac{p_{i-1,j'}}{p_{i-1,j}} - \frac{p_i}{p_i} \cdot \frac{p_{ij'}}{p_{ij}}.$$

Substituting for $\frac{p_{ij'}}{p_i} = p_{j'}(i)$, and for analogous terms in the remaining ratios, we have

$$\frac{p_{j'}(i)}{p_j(i)} - \frac{p_{j'}(i+1)}{p_j(i+1)} \geq \frac{p_{j'}(i-1)}{p_j(i-1)} - \frac{p_{j'}(i)}{p_j(i)}.$$

Summing for $j' = 1, ..., j-1$, we can write

$$\sum_{j'=1}^{j-1} \frac{p_{j'}(i)}{p_j(i)} - \sum_{j'=1}^{j-1} \frac{p_{j'}(i+1)}{p_j(i+1)} \geq \sum_{j'=1}^{j-1} \frac{p_{j'}(i-1)}{p_j(i-1)} - \sum_{j'=1}^{j-1} \frac{p_{j'}(i)}{p_j(i)}.$$

Performing this summation, we have

$$\frac{P_{j-1}(i)}{p_j(i)} - \frac{P_{j-1}(i+1)}{p_j(i+1)} \geq \frac{P_{j-1}(i-1)}{p_j(i-1)} - \frac{P_{j-1}(i)}{p_j(i)}.$$

Adding $\frac{p_j(i)}{p_j(i)} - \frac{p_j(i+1)}{p_j(i+1)} = 0$ on the left-hand side and $\frac{p_j(i-1)}{p_j(i-1)} - \frac{p_j(i)}{p_j(i)} = 0$ on the right-hand side, we obtain

$$\frac{P_j(i)}{p_j(i)} - \frac{P_j(i+1)}{p_j(i+1)} \geq \frac{P_j(i-1)}{p_j(i-1)} - \frac{P_j(i)}{p_j(i)}.$$

Rearranging this condition, we have

$$\frac{P_j(i)}{p_j(i)} \geq \frac{1}{2}\frac{P_j(i-1)}{p_j(i-1)} + \frac{1}{2}\frac{P_j(i+1)}{p_j(i+1)}.$$

Proof of Proposition 6: Part i) The observable variables to the government are h, w, T, and x, where $x = 1$ if $w \leq i$ and $x = 0$ if $w > i$. That is, the government can observe whether or not the individual selects into a task he or she is unable to do. As a result, the taxes paid by the individual can be contingent on x. Consider any mechanism with strategy set Σ for the individuals. Let σ denote a generic element of Σ and let $\sigma^*(i,j)$ denote the optimal strategy of individual (i,j); $h(\sigma^*(i,j))$, $w(\sigma^*(i,j))$, $T\big(\sigma^*(i,j), x(\mathbf{w}(\sigma^*(i,j)), i)\big)$, and $x(\mathbf{w}(\sigma^*(i,j)), i)$ are the observable variables associated to each pure strategy. An incentive compatible mechanism satisfies for all (i,j) and for all σ' the incentive constraint

$$\begin{aligned}(41)\qquad & \mathbf{h}(\sigma^*(i,j)) \cdot (x(\mathbf{w}(\sigma^*(i,j)), i) \cdot \mathbf{w}(\sigma^*(i,j)) - j) - \mathbf{T}(\sigma^*(i,j), x(\mathbf{w}(\sigma^*(i,j)), i)) \\ & \geq \mathbf{h}(\sigma') \cdot \left(x\left(\mathbf{w}(\sigma'), i\right) \cdot \mathbf{w}(\sigma') - j\right) - \mathbf{T}\left(\sigma', x\left(\mathbf{w}(\sigma'), i\right)\right)\end{aligned}$$

and the participation constraint

$$(42)\qquad \mathbf{h}(\sigma^*(i,j)) \cdot (x(\mathbf{w}(\sigma^*(i,j)), i) \cdot \mathbf{w}(\sigma^*(i,j)) - j) - \mathbf{T}(\sigma^*(i,j), x(\mathbf{w}(\sigma^*(i,j)), i)) \geq 0,$$

where $\sigma' = \sigma^*(i',j')$ for any (i',j').

Clearly, exactly the same choices can be induced by a mechanism where the agent is asked to announce his type and is given incentives to do so truthfully. More specifically, define the direct tax functions $T \equiv \mathbf{T} \circ \sigma^*$ for $x = 1$ and $T^- \equiv \mathbf{T} \circ \sigma^*$ for $x = 0$ and the functions $h \equiv h \circ \sigma^*$ and $w \equiv w \circ \sigma^*$. Let $h_{\hat{\imath}\hat{\jmath}}$, $w_{\hat{\imath}\hat{\jmath}}$, $T_{\hat{\imath}\hat{\jmath}}$, $T^-_{\hat{\imath}\hat{\jmath}}$, and $x(w_{\hat{\imath}\hat{\jmath}}, i)$ denote generic values of the observables. Using these definitions, we can write (41) equivalently as the pair of constraints for each i, j

$$(43)\qquad h_{ij} \cdot (1 \cdot w_{ij} - j) - T_{ij} \geq h_{\hat{\imath}\hat{\jmath}} \cdot (1 \cdot w_{\hat{\imath}\hat{\jmath}} - j) - T_{\hat{\imath}\hat{\jmath}} \quad \forall (\hat{\imath}, \hat{\jmath}) \;\; s.t. \; w_{\hat{\imath}\hat{\jmath}} \leq i$$

and

$$(44)\qquad h_{ij} \cdot (1 \cdot w_{ij} - j) - T_{ij} \geq h_{\hat{\imath}\hat{\jmath}} \cdot (0 \cdot w_{\hat{\imath}\hat{\jmath}} - j) - T^-_{\hat{\imath}\hat{\jmath}} \quad \forall (\hat{\imath}, \hat{\jmath}) \;\; s.t. \; w_{\hat{\imath}\hat{\jmath}} > i.$$

Allocating an individual to a task he or she is unable to do cannot be desirable, since the individual foregoes production opportunities in the informal sector and does not produce anything in the formal sector. In other words, any optimal mechanism must induce $w(\sigma^*(i,j)) \leq i$ for all i. Hence, the function T^- is relevant only off equilibrium path and we can set $T^-_{\hat{\imath}\hat{\jmath}} = 0$ for all $(\hat{\imath}, \hat{\jmath})$ *s.t.* $w_{\hat{\imath}\hat{\jmath}} > i$, which implies that the right-hand side of (44) is non-positive. By (42) the left-hand side of (44) is non-negative. Hence, constraint (44) cannot be binding at the optimum, and there is no loss in generality writing the incentive constraints as (43) and the participation constraint as

$h_{ij} \cdot (w_{ij} - j) - T_{ij} \geq 0.$

Part ii) From (43) the relevant incentive constraint for an arbitrary allocation rule w_{ij} is that for all (i, j)

$$h_{ij} \cdot (w_{ij} - j) - T_{ij} \geq h_{\hat{\imath}\hat{\jmath}} \cdot (w_{\hat{\imath}\hat{\jmath}} - j) - T_{\hat{\imath}\hat{\jmath}} \quad \forall (\hat{\imath}, \hat{\jmath}) \;\; s.t. w_{\hat{\imath}\hat{\jmath}} \leq i. \tag{45}$$

Adjust the allocation for all (i, j) as follows: change w_{ij} to $\tilde{w}_{ij} = i$, leave the allocation of working times unchanged, $h_{ij} = \tilde{h}_{ij}$, and change the taxes from T_{ij} to $\tilde{T}_{ij} = T_{ij} + h_{ij} \cdot (i - w_{ij})$. Substituting the adjusted rules, $\left\{h_{ij}, \tilde{w}_{ij}, \tilde{T}_{ij}\right\}$ for all i, j into (45) , we find that

$$h_{ij} \cdot (i - j) - \tilde{T}_{ij} \geq h_{\hat{\imath}\hat{\jmath}} \cdot (\hat{\imath} - j) - \tilde{T}_{\hat{\imath}\hat{\jmath}} \quad \forall (\hat{\imath}, \hat{\jmath}) \;\; s.t. w_{\hat{\imath}\hat{\jmath}} \leq i. \tag{46}$$

By construction, the left-hand side of (45) and (46) are equal for all (i, j) and the right-hand sides of (45) and (46) are equal for all $\hat{\imath}, \hat{\jmath}$. The incentive constraint under the new allocation takes the form

$$h_{ij} \cdot (i - j) - \tilde{T}_{ij} \geq h_{\hat{\imath}\hat{\jmath}} \cdot (\hat{\imath} - j) - \tilde{T}_{\hat{\imath}\hat{\jmath}} \quad \forall (\hat{\imath}, \hat{\jmath}) \;\; s.t. \hat{\imath} \leq i, \tag{47}$$

because $\tilde{w}_{ij} = i$. (47) is implied by (46) . The reason is that feasibility of the initial allocation requires that the rule w_{ij} satisfies $w_{ij} \leq i$ for all i, j. Hence, the set $\{\hat{\imath}, \hat{\jmath} : \hat{\imath} \leq i\}$ is a subset of the set $\{\hat{\imath}, \hat{\jmath} : w_{\hat{\imath}\hat{\jmath}} \leq i\}$. So, the new rule is incentive compatible. Finally consider budget balancedness and individual rationality. Since the initial allocation was budget balanced, the new allocation runs a surplus, which can be distributed in lump sum fashion to all individuals. This redistribution does not affect any participation constraints.