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A Dynamic Theory of Public Spending, Taxation and Debt:

Technical Appendix

Abstract

This appendix provides complete and detailed proofs of the results presented in the paper "A Dynamic Theory of Public Spending, Taxation and Debt."

In this appendix we provide complete and detailed proofs of the results presented in the paper "A Dynamic Theory of Public Spending, Taxation and Debt." The numbering of the results is the same as in the paper.

1 Proof of Proposition 1

From the analysis in the text we know that if $A \leq A^o(b, \underline{x})$ the planner would select a tax rate of 0, a public good level of $g_S(A)$, a debt level of $\underline{x}$, and transfer $B(0, g_S(A), \underline{x}; b)$ to the citizens. Since $A^o(\underline{x}, \underline{x}) = \overline{A}$, it follows that once the planner has selected the debt level $\underline{x}$ the economy enters a deterministic steady state in which the debt level is $\underline{x}$, the tax rate is 0, the public good level is $g_S(A)$, and citizens receive $\rho(-\underline{x}) - pg_S(A)$ in transfers. Thus, it only remains to show that whatever the initial debt level, the planner will eventually select the debt level $\underline{x}$ with probability one. We will establish this following the proof of Proposition 4 below. ■

2 Definition of political equilibrium

As background for the analysis to follow, we need to provide a more precise definition of political equilibrium. An equilibrium is described by a collection of proposal functions: $\{r_\tau(b, A), g_\tau(b, A), x_\tau(b, A), s_\tau(b, A)\}_{\tau=1}^T$. Here $r_\tau(b, A)$ is the income tax rate that is proposed at round τ when the state is (b, A); $g_\tau(b, A)$ is the level of the public good; $x_\tau(b, A)$ is the new level of public debt, and $s_\tau(b, A)$ is a transfer the proposer offers to the districts of $q-1$ randomly selected representatives.[1]

Any remaining surplus revenues are used to finance a transfer for the proposer's own district. Following the notation used in Section 4, we will sometimes drop the subscript and refer to the first round policy proposal as $\{r(b, A), g(b, A), x(b, A), s(b, A)\}$.

The collection of proposal functions $\{r_\tau(b, A), g_\tau(b, A), x_\tau(b, A), s_\tau(b, A)\}_{\tau=1}^T$ is an *equilibrium* if at each proposal round τ and all states (b, A), the prescribed proposal maximizes the proposer's payoff subject to the incentive constraint of getting the required number of affirmative votes and the appropriate feasibility constraints. To state this formally, let $v_\tau(b, A)$ denote the legislators' value function at round τ which describes the expected future payoff of a legislator at the beginning

[1] It should be clear that there is no loss of generality in assuming that the proposer only offers transfers to $q-1$ representatives.

of a period in which the state is (b, A). Again, following the notation of Section 4, we refer to the value function at round 1 as $v(b, A)$. Then, for each proposal round τ and all states (b, A), the proposal $(r_\tau(b, A),\ g_\tau(b, A),\ x_\tau(b, A),\ s_\tau(b, A))$ must solve the problem

$$\max_{(r,g,x,s)} u(w(1-r), g; A) + B(r, g, x; b) - (q-1)\, s + \delta E v(x, A')$$

$$s.t. \quad u(w(1-r), g; A) + s + \delta E v(x, A') \geq v_{\tau+1}(b, A),$$

$$B(r, g, x; b) \geq (q-1)s, \quad s \geq 0 \ \&\ x \in [\underline{x}, \overline{x}].$$

The first constraint is the incentive constraint and the remainder are feasibility constraints. The formulation reflects the assumption that on the equilibrium path, the proposal made in the first proposal round is accepted.

As noted in the text, the legislators' round one value function is defined recursively by (17). To understand this recall that a legislator is chosen to propose in round one with probability $1/n$. If chosen to propose, he obtains a payoff in that period of

$$u(w(1-r(b,A)), g(b,A); A) + B(r(b,A), g(b,A), x(b,A); b) - (q-1)s(b,A).$$

If he is not chosen to propose, but is included in the minimum winning coalition, he obtains $u(w(1-r(b,A)), g(b,A); A) + s(b,A)$ and if he is not included he obtains just $u(w(1-r(b,A)), g(b,A); A)$. The probability that he will be included in the minimum winning coalition, conditional on not being chosen to propose, is $(q-1)/(n-1)$. Taking expectations, the pork barrel transfers $s(b, A)$ cancel and the period payoff is as described in (17).

Once we have the round one value function, the other value functions can be readily derived. For all proposal rounds $\tau = 1, .., T-1$ the expected future payoff of a legislator if the round τ proposal is rejected is

$$v_{\tau+1}(b, A) = u\left(w(1 - r_{\tau+1}(b,A)), g_{\tau+1}(b,A); A\right) + \tfrac{B(r_{\tau+1}(b,A), g_{\tau+1}(b,A), x_{\tau+1}(b,A); b)}{n}$$

$$+ \delta E v(x_{\tau+1}(b,A), A').$$

This reflects the assumption that the round $\tau + 1$ proposal will be accepted. Recall that if the round T proposal is rejected, the assumption is that a legislator is appointed to choose a default tax rate, public goods level, level of debt and a uniform transfer. Thus,

$$v_{T+1}(b, A) = \max_{(r,g,x)} \left\{ u(w(1-r), g; A) + \frac{B(r, g, x; b)}{n} + \delta E v(x, A') : B(r, g, x; b) \geq 0 \ \&\ x \in [\underline{x}, \overline{x}] \right\}.$$

3 Proof of Lemma 1

We begin by establishing the claim made in the text that, given that utility is transferable, the proposer is effectively making decisions to maximize the collective utility of q legislators under the assumption that they get to divide any surplus revenues among their districts.

Lemma A.1: *Let* $\{r_\tau(b,A), g_\tau(b,A), x_\tau(b,A), s_\tau(b,A)\}_{\tau=1}^{T}$ *be an equilibrium with associated value function* $v(b,A)$. *Then, for all states* (b,A), *the tax rate-public good-public debt triple* $(r_\tau(b,A),$ $g_\tau(b,A),$ $x_\tau(b,A))$ *proposed in any round* τ *solves the problem*

$$\max_{(r,g,x)} u(w(1-r),g;A) + \frac{B(r,g,x;b)}{q} + \delta Ev(x,A')$$
$$s.t. \quad B(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \,.$$

Moreover, the transfer to coalition members is given by

$$s_\tau(b,A) = v_{\tau+1}(b,A) - u(w(1-r_\tau(b,A), g_\tau(b,A);A) - \delta Ev(x_\tau(b,A),A').$$

Proof: We begin with proposal round T. Let (b,A) be given. Multiplying the objective function through by q, we need to show that if (r_T, s_T, g_T, x_T) solves the round T proposer's problem when the state is (b,A), (r_T, g_T, x_T) solves the problem

$$\begin{array}{c} \max_{(r,g,x)} q[u(w(1-r),g;A) + \delta Ev(x,A')] + B(r,g,x;b) \\ s.t. \quad B(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{array}, \tag{A.1}$$

and $s_T = v_{T+1}(b,A) - u(w(1-r_T), g_T;A) - \delta Ev(x_T,A')$. Recall that the round T proposer's problem is:

$$\max_{(r,g,x,s)} u(w(1-r),g;A) + B(r,g,x;b) - (q-1)\,s + \delta Ev(x,A)$$
$$s.t. \quad u(w(1-r),g;A) + s + \delta Ev(x,A) \geq v_{T+1}(b,A),$$
$$B(r,g,x;b) \geq (q-1)s, \quad s \geq 0 \ \& \ x \in [\underline{x}, \overline{x}].$$

It is easy to see that $s_T = v_{T+1}(b,A) - \delta Ev(x_T,A') - u(w(1-r_T),g_T;A)$, for if this were not the case it would follow from the definition of $v_{T+1}(b,A)$ that $s_T > 0$ and we could create a preferred proposal by just reducing s_T. It follows that we can write the proposer's payoff as

$$q\,[u(w(1-r_T),g_T;A) + \delta Ev(x_T,A')] + B(r_T,g_T,x_T;b).$$

Now suppose that (r_T, g_T, x_T) does not solve problem (A.1). Let (r', g', x') solve problem (A.1) and $s' = v_{T+1}(b, A) - u(w(1-r'), g'; A) - \delta Ev(x', A')$. Then, the proposer's payoff under the proposal (r', g', x', s') is $q\left[u(w(1-r'), g'; A) + \delta Ev(x', A')\right] + B(r', g', x'; b)$. By construction, the incentive constraint is satisfied and, by definition of $v_{T+1}(b, A)$, $s' \geq 0$. Moreover, $x' \in [\underline{x}, \overline{x}]$. Finally, note that

$$\begin{aligned} B(r', g', x'; b) - (q-1)s' &= (q-1)[u(w(1-r'), g'; A) + \delta Ev(x', A')] + B(r', g', x'; b) \\ & \quad -(q-1)v_{T+1}(b, A) \geq 0 \end{aligned}$$

where the last inequality follows from the fact that (r', g', x') solves problem (A.1) and the definition of $v_{T+1}(b, A)$. It follows that (r', g', x', s') is feasible for the proposer's problem and yields a higher payoff than (r_T, g_T, x_T, s_T) - a contradiction.

Now consider the round $T-1$ proposer's problem

$$\begin{aligned} \max_{(r,g,x,s)} \; & u(w(1-r), g; A) + B(r, g, x; b) - (q-1)\, s + \delta Ev(x, A') \\ s.t. \quad & u(w(1-r), g; A) + s + \delta Ev(x, A') \geq v_T(b, A), \\ & B(r, g, x; b) \geq (q-1)s, \quad s \geq 0 \;\&\; x \in [\underline{x}, \overline{x}]. \end{aligned} \tag{A.2}$$

From what we know about the round T proposer's problem,

$$v_T(b, A) = u(w(1-r_T), g_T; A) + \frac{B(r_T, g_T, x_T; b)}{n} + \delta Ev(x_T, A'),$$

where (r_T, g_T, x_T) solves problem (A.1).

We need to show that if $(r_{T-1}, s_{T-1}, g_{T-1}, x_{T-1})$ is the solution to the round $T-1$ proposer's problem, $(r_{T-1}, g_{T-1}, x_{T-1})$ solves problem (A.1) and

$$s_{T-1} = v_T(b, A) - u(w(1-r_{T-1}), g_{T-1}; A) - \delta Ev(x_{T-1}, A').$$

The result would follow from our earlier argument if we could show that

$$s_{T-1} = v_T(b, A) - u(w(1-r_{T-1}), g_{T-1}; A) - \delta Ev(x_{T-1}, A'),$$

so suppose that $s_{T-1} > v_T(b, A) - u(w(1-r_{T-1}), g_{T-1}; A) - \delta Ev(x_{T-1}, A')$. Then it must be the case that $s_{T-1} = 0$, or we could obtain a preferred proposal by simply reducing s_{T-1}. It follows that

$$v_T(b, A) < u(w(1-r_{T-1}), g_{T-1}; A) + \delta Ev(x_{T-1}, A'). \tag{A.3}$$

This implies that $(r_{T-1}, g_{T-1}, x_{T-1})$ solves

$$\max_{(r,g,x)} u(w(1-r), g; A) + B(r, g, x; b) + \delta E v(x, A)$$
$$s.t. \;\; B(r, g, x; b) \geq 0 \;\&\; x \in [\underline{x}, \overline{x}].$$

Now consider the proposal $(r_T, g_T, x_T, \frac{B(r_T, g_T, x_T; b)}{n})$. Clearly, this proposal satisfies all the constraints of the proposer's problem. The payoff to the proposer under this policy is

$$q[u(w(1-r_T), g_T; A) + \delta E v(x_T, A')] + B(r_T, g_T, x_T; b) - (q-1)v_T(b, A).$$

From (A.3), this payoff is strictly larger than

$$q[u(w(1-r_T), g_T; A) + \delta E v(x_T, A)] + B(r_T, x_T, g_T; b)$$
$$-(q-1)[u(w(1-r_{T-1}), g_{T-1}; A) + \delta E v(x_{T-1}, A)].$$

The payoff to the proposer under the optimal policy $(r_{T-1}, g_{T-1}, x_{T-1})$ is

$$u(w(1-r_{T-1}), g_{T-1}; A) + B(r_{T-1}, x_{T-1}, g_{T-1}; b) + \delta E v(x_{T-1}, A).$$

Thus, it must be the case that

$$\begin{aligned} & u(w(1-r_{T-1}), g_{T-1}; A) + B(r_{T-1}, x_{T-1}, g_{T-1}; b) + \delta E v(x_{T-1}, A') \\ > \;& q[u(w(1-r_T), g_T; A) + \delta E v(x_T, A')] + B(r_T, x_T, g_T; b) \\ & -(q-1)[u(w(1-r_{T-1}), g_{T-1}; A) + \delta E v(x_{T-1}, A')], \end{aligned}$$

implying that

$$\begin{aligned} & q[u(w(1-r_{T-1}), g_{T-1}; A) + \delta E v(x_{T-1}, A')] + B(r_{T-1}, x_{T-1}, g_{T-1}; b) \\ > \;& q[u(w(1-r_T), g_T; A) + \delta E v(x_T, A')] + B(r_T, x_T, g_T; b). \end{aligned}$$

This contradicts the fact that (r_T, g_T, x_T) solves problem (A.1).

Application of the same logic to proposal rounds $\tau = T-2, ..., 1$ implies the lemma. ■

Using this result, we can prove:

Lemma A.2: *Let $\{r_\tau(b, A),\; g_\tau(b, A),\; x_\tau(b, A),\; s_\tau(b, A)\}_{\tau=1}^{T}$ be an equilibrium with associated value function $v(b, A)$. Then, there exists some debt level x^* such that for any proposal round τ if*

$A \geq A^*(b, x^*)$

$$(r_\tau(b,A), g_\tau(b,A), x_\tau(b,A)) = \arg\max \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x; A') \\ \\ B(r,g,x;b) \geq 0 \ \&\ x \in [\underline{x}, \overline{x}] \end{array} \right\}$$

and $s_\tau(b, A) = 0$, while if $A < A^*(b, x^*)$

$$(r_\tau(b,A), g_\tau(b,A), x_\tau(b,A)) = (r^*, g^*(A), x^*)$$

and

$$s_\tau(b,A) = \left\{ \begin{array}{cc} \frac{B(r^*,g^*(A),x^*;b)}{n} & \text{if } \tau = 1, ..., T-1 \\ \\ v_{T+1}(b,A) - u(w(1-r^*, g^*(A); A) - \delta E v(x^*, A') & \text{if } \tau = T \end{array} \right. .$$

Proof: The argument in Section 4.1 of the paper together with Lemma A.1 implies that for any proposal round τ if $A \geq A^*(b, x^*)$

$$(r_\tau(b,A), g_\tau(b,A), x_\tau(b,A)) = \arg\max \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ \\ B(r,g,x;b) \geq 0 \ \&\ x \in [\underline{x}, \overline{x}] \end{array} \right\},$$

while if $A < A^*(b, x^*)$

$$(r_\tau(b,A), g_\tau(b,A), x_\tau(b,A)) = (r^*, g^*(A), x^*).$$

Turning to the equilibrium transfers, it is clear that, since there are no surplus revenues when $A \geq A^*(b, x^*)$, transfers are zero. If $A < A^*(b, x^*)$ it follows that for all proposal rounds $\tau = 1, ..., T-1$ we have that

$$v_{\tau+1}(b,A) = u(w(1-r^*), g^*(A); A) + \frac{B(r^*, g^*(A), x^*; b)}{n} + \delta E v(x^*, A').$$

Thus, Lemma A.1 implies that the transfers to coalition members are given by:

$$s_\tau(b,A) = \left\{ \begin{array}{cc} B(r^*, g^*(A), x^*; b)/n & \tau = 1, ..., T-1 \\ \\ v_{T+1}(b,A) - u(w(1-r^*, g^*(A); A) - \delta E v(x^*, A') & \tau = T \end{array} \right. .$$

■

Lemma 1 now follows immediately from Lemma A.2.

4 Properties of the equilibrium policy functions

In this section we establish some properties of the equilibrium (and optimal) policy functions that are mentioned in the text and that will be used in the following proofs.

We first show that when $A \geq A^*(b, x^*)$, the tax rate, public debt level and the level of the public good depend positively on the value of the public good (A), the tax rate and level of public debt depend positively on the current level of debt (b) and the level of the public good depends negatively on b. From Lemma A.2 we know that when $A \geq A^*(b, x^*)$, the equilibrium tax rate-public good-public debt triple $(r_\tau(b, A), g_\tau(b, A), x_\tau(b, A))$ solve

$$\begin{gathered} \max_{(r,g,x)} u(w(1-r), g; A) + \tfrac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ s.t. \quad B(r, g, x; b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{gathered} .$$

Moreover, from the discussion in the text, $(r_\tau(b, A), g_\tau(b, A), x_\tau(b, A))$ is implicitly defined by equations (10), (11) and (12) in Section 3.1 (with the appropriate equilibrium value function).

Lemma A.3: *Let $b \in [\underline{x}, \overline{x}]$ and let $A_0, A_1 \in [\underline{A}, \overline{A}]$ be such that $A^*(b, x^*) < A_0 < A_1$. Then, it is the case that $g_\tau(b, A_0) < g_\tau(b, A_1)$ and $r_\tau(b, A_0) < r_\tau(b, A_1)$. Moreover, it is also the case that $x_\tau(b, A_0) \leq x_\tau(b, A_1)$ with strict inequality if $x_\tau(b, A_0) < \overline{x}$.*

Proof of Lemma A.3: We begin by showing that $g_\tau(b, A_0) < g_\tau(b, A_1)$. Let $\varphi(A'; b, A)$ be the value of the objective function for the problem when the state is A' and the policies are those that are optimal given state (b, A); that is,

$$\varphi(A'; b, A) = u(w(1 - r_\tau(b, A), g_\tau(b, A); A') + \frac{B(r_\tau(b, A), g_\tau(b, A), x_\tau(b, A); b)}{n} + \delta E v(x_\tau(b, A), A').$$

Then, we have that $\varphi(A_0; b, A_0) > \varphi(A_0; b, A_1)$ and $\varphi(A_1; b, A_1) > \varphi(A_1; b, A_0)$ (the strict inequality follows from the fact that the problem has a unique solution).

Moreover, using the definition of the indirect utility function $u(w(1-r), g; A)$ (see equation (3) in Section 2) and letting $\Delta A = A_1 - A_0$, we can write $\varphi(A_0; b, A_0) = \varphi(A_1; b, A_0) - \Delta A g_\tau(b, A_0)$ and $\varphi(A_0; b, A_1) = \varphi(A_1; b, A_1) - \Delta A g_\tau(b, A_1)$. Since $\varphi(A_0; b, A_0) > \varphi(A_0; b, A_1)$, this means that $\varphi(A_1; b, A_0) - \Delta A g_\tau(b, A_0) > \varphi(A_1; b, A_1) - \Delta A g_\tau(b, A_1)$, and hence

$$\Delta A[g_\tau(b, A_0) - g_\tau(b, A_1)] < \varphi(A_1; b, A_0) - \varphi(A_1; b, A_1) < 0.$$

Since $\Delta A > 0$, this implies that $g_\tau(b, A_0) < g_\tau(b, A_1)$ as required.

We next show that $r_\tau(b, A_0) < r_\tau(b, A_1)$. Suppose to the contrary that $r_\tau(b, A_0) \geq r_\tau(b, A_1)$. Then the first order condition for x (i.e., (11)) and the concavity of $Ev(\cdot, A)$ imply that $x_\tau(b, A_0) \geq x_\tau(b, A_1)$. But then it follows that

$$B(r_\tau(b, A_0), g_\tau(b, A_0), x_\tau(b, A_0); b) > B(r_\tau(b, A_1), g_\tau(b, A_1), x_\tau(b, A_1); b) = 0$$

which is a contradiction.

Finally, we show that $x_\tau(b, A_0) \leq x_\tau(b, A_1)$ with strict inequality if $x_\tau(b, A_0) < \overline{x}$. This follows immediately from the first order condition for x and the concavity of v given that $r_\tau(b, A_0) < r_\tau(b, A_1)$. ■

Lemma A.4: *Let $b_0, b_1 \in [\underline{x}, \overline{x}]$ be such that $b_0 < b_1$ and let $A \in [\underline{A}, \overline{A}]$ be such that $A^*(b_0, x^*) < A$. Then, it is the case that $r_\tau(b_0, A) < r_\tau(b_1, A)$ and $g_\tau(b_0, A) > g_\tau(b_1, A)$. Moreover, it is also the case that $x_\tau(b_0, A) \leq x_\tau(b_1, A)$ with strict inequality if $x_\tau(b_0, A) < \overline{x}$.*

Proof of Lemma A.4: We first show that $r_\tau(b_0, A) < r_\tau(b_1, A)$. Suppose to the contrary that $r_\tau(b_0, A) \geq r_\tau(b_1, A)$. Then the first order conditions for g and x (i.e., (10) and (11)) and the concavity of $Ev(\cdot, A)$ imply that $g_\tau(b_0, A) \leq g_\tau(b_1, A)$ and $x_\tau(b_0, A) \geq x_\tau(b_1, A)$. But then it follows that

$$B(r_\tau(b_0, A), g_\tau(b_0, A), x_\tau(b_0, A); b_0) > B(r_\tau(b_1, A), g_\tau(b_1, A), x_\tau(b_1, A); b_1) = 0$$

which is a contradiction.

The fact that $g_\tau(b_0, A) > g_\tau(b_1, A)$ follows immediately from the first order condition for g and the fact that $r_\tau(b_0, A) < r_\tau(b_1, A)$. In addition, the fact that $x_\tau(b_0, A) \leq x_\tau(b_1, A)$ with strict inequality if $x_\tau(b_0, A) < \overline{x}$ follows immediately from the first order condition for x (i.e., (11)), the concavity of $Ev(\cdot, A)$ and the fact that $r_\tau(b_0, A) < r_\tau(b_1, A)$. ■

5 Proof of Proposition 2

We begin by proving the existence of an equilibrium. The proof is divided into seven steps.

Step 1: Let F denote the set of all real valued functions $v(\cdot, \cdot)$ defined over the compact set $[\underline{x}, \overline{x}] \times [\underline{A}, \overline{A}]$. Let F^* be the subset of these functions that are continuous and concave in x for all A. For any $z \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$ and $v \in F^*$ consider the maximization problem

$$\begin{array}{ll} \max_{(r,g,x)} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ s.t. \quad B(r,g,x;b) \geq 0,\ r \geq r^*,\ g \leq g^*(A) \ \&\ x \in [z, \overline{x}] \end{array}.$$

For all $\mu > 0$, let

$$X_z^{\mu}(v) = \arg\max_x \{\frac{x}{\mu} + \delta E v(x, A') : x \in [z, \overline{x}]\}$$

and let $x_z^{\mu}(v)$ be the largest element of the compact set $X_z^{\mu}(v)$. Notice that $x_z^{\mu}(v)$ is non-increasing in μ.

Suppose that (r, g, x) is a solution to the maximization problem. It is straightforward to show that (i) if $A \leq A^*(b, x_z^n(v))$ then $(r, g) = (r^*, g^*(A))$ and $x \in X_z^n(v) \cap \{x : B(r^*, g^*(A), x; b) \geq 0\}$; (ii) if $A \in (A^*(b, x_z^n(v)), A^*(b, x_z^q(v))]$ then $(r, g) = (r^*, g^*(A))$ and $B(r^*, g^*(A), x; b) = 0$; and (iii) if $A > A^*(b, x_z^q(v))$ (r, g, x) is uniquely defined and the budget constraint is binding. Moreover, $r > r^*$ and $g < g^*(A)$. Note that in all cases the tax rate and public good level are uniquely defined.

Step 2: For any $z \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$, define the operator $T_z : F^* \to F$ as follows:

$$T_z(v)(b, A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ B(r,g,x;b) \geq 0,\ r \geq r^*,\ g \leq g^*(A) \ \&\ x \in [z, \overline{x}] \end{array} \right\}.$$

It can be verified that $T_z(v) \in F^*$ and that T_z is a contraction. Thus, there exists a unique fixed point $v_z(b, A)$ which is continuous and concave in b for all A. This fixed point satisfies the functional equation

$$v_z(b, A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v_z(x, A') \\ B(r,g,x;b) \geq 0,\ r \geq r^*,\ g \leq g^*(A) \ \&\ x \in [z, \overline{x}] \end{array} \right\}.$$

Let (b, A) be given and let (r, g, x) denote an optimal policy. By Step 1, we have that (i) if $A \leq A^*(b, x_z^n(v_z))$ then $(r, g) = (r^*, g^*(A))$ and $x \in X_z^n(v_z) \cap \{x : B(r^*, g^*(A), x; b) \geq 0\}$; (ii) if $A \in (A^*(b, x_z^n(v_z)), A^*(b, x_z^q(v_z))]$ then $(r, g) = (r^*, g^*(A))$ and $B(r^*, g^*(A), x; b) = 0$; and (iii) if $A > A^*(b, x_z^q(v_z))$ (r, g, x) is uniquely defined and the budget constraint is binding. Moreover, $r > r^*$ and $g < g^*(A)$. Again, in all cases the tax rate and public good level is uniquely defined. Let these be given by $(r_z(b, A), g_z(b, A))$ - these are also continuous functions on the state space.

Step 3: For any $z \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$, the expected value function $Ev_z(\cdot, A)$ is strictly concave on the set $\{b \in [\underline{x}, \overline{x}] : A^*(b, x_z^q(v_z)) < \overline{A}\}$.

Proof: It suffices to show that for any $v \in F^*$, the function $ET_z(v)(\cdot, A)$ is strictly concave on the set $\{b \in [\underline{x}, \overline{x}] : A^*(b, x_z^q(v)) < \overline{A}\}$. Since $T_z(v) \in F^*$, we know already that the function $T_z(v)(\cdot, A)$ is concave for all A. We now show that for all A, the function $T_z(v)(\cdot, A)$ is strictly concave on $\{b \in [\underline{x}, \overline{x}] : A^*(b, x_z^q(v)) < A\}$. In this case, the budget constraint is strictly binding and $g_z(b, A) < g^*(A)$, $r_z(b, A) > r^*$. We can therefore write:

$$T_z(v)(b, A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta Ev\left(x, A'\right) \\ B(r, g, x; b) \geq 0 \ \& \ x \in [z, \overline{x}] \end{array} \right\}$$

Take two points b_1 and b_2 in the set $\{b \in [\underline{x}, \overline{x}] : A^*(b, x_z^q(v)) < \overline{A}\}$ and assume that $b_1 < b_2$. Let λ be a point in the interval $[0, 1]$. Define (r_i, g_i, x_i) to be the optimal policies associated with (b_i, A) for $i = 1, 2$ (as noted above these are unique). Let $b_\lambda = \lambda b_1 + (1-\lambda)\, b_2$, $r_\lambda = \lambda r_1 + (1-\lambda)\, r_2$, $g_\lambda = \lambda g_1 + (1-\lambda)\, g_2$ and $x_\lambda = \lambda x_1 + (1-\lambda)\, x_2$. Since $v\left(x, A'\right) + x/n$ is weakly concave in x, $u(w(1-r), g; A) + [R(r) - pg]/n$ is strictly concave in (r, g), and $(r_1, g_1, x_1) \neq (r_2, g_2, x_2)$, we have that:

$$\begin{aligned} \lambda T_z(v)(b_1, A) + (1-\lambda) T_z(v)(b_2, A) &= \lambda \left[\begin{array}{c} u(w(1-r_1), g_1; A) \\ + \frac{B(r_1,g_1,x_1;b_1)}{n} + \delta Ev\left(x_1, A'\right) \end{array} \right] \\ &\quad +(1-\lambda) \left[\begin{array}{c} u(w(1-r_2), g_2; A) \\ + \frac{B(r_2,g_2,x_2;b_2)}{n} + \delta Ev\left(x_2, A'\right) \end{array} \right] \\ &< u(w(1-r_\lambda), g_\lambda; A) + \frac{B(r_\lambda, g_\lambda, x_\lambda; b_\lambda)}{n} + \delta Ev\left(x_\lambda, A'\right) \end{aligned}$$

Since $R(r)$ is concave in r, we have that $B(r_\lambda, g_\lambda, x_\lambda; b_\lambda) > 0$ and, in addition, $x_\lambda \in [z, \overline{x}]$. Therefore:

$$\begin{aligned} & u(w(1-r_\lambda), g_\lambda; A) + \frac{B(r_\lambda, g_\lambda, x_\lambda; b_\lambda)}{n} + \delta Ev\left(x_\lambda, A'\right) \\ \leq \ & \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b_\lambda)}{n} + \delta Ev\left(x, A'\right) \\ B(r, g, x; b_\lambda) \geq 0 \ \& \ x \in [z, \overline{x}] \end{array} \right\} = T_z(v)(b_\lambda, A). \end{aligned}$$

We conclude that $\lambda T_z(v)(b_1, A) + (1-\lambda) T_z(v)(b_2, A) < T_z(v)(b_\lambda, A)$ as required.

Now take any two points b_1 and b_2 in the set $\{b \in [\underline{x}, \overline{x}] : A^*(b, x_z^q(v)) < \overline{A}\}$ and assume that $b_1 < b_2$. Then, we have that

$$\begin{aligned}
& \lambda ET_z(v)(b_1, A) + (1-\lambda)ET_z(v)(b_2, A) \\
= \; & \lambda\{\int_{\underline{A}}^{A^*(b_1, x_z^q(v))} T_z(v)(b_1, A)dG(A) + \int_{A^*(b_1, x_z^q(v))}^{\overline{A}} T_z(v)(b_1, A)dG(A)\} \\
& +(1-\lambda)\{\int_{\underline{A}}^{A^*(b_2, x_z^q(v))} T_z(v)(b_2, A)dG(A) + \int_{A^*(b_2, x_z^q(v))}^{\overline{A}} T_z(v)(b_2, A)dG(A)\} \\
= \; & \int_{\underline{A}}^{A^*(b_1, x_z^q(v))} [\lambda T_z(v)(b_1, A) + (1-\lambda)T_z(v)(b_2, A)]dG(A) \\
& + \int_{A^*(b_1, x_z^q(v))}^{\overline{A}} [\lambda T_z(v)(b_1, A) + (1-\lambda)T_z(v)(b_2, A)]dG(A) \\
< \; & \int_{\underline{A}}^{A^*(b_1, x_z^q(v))} T_z(v)(b_\lambda, A)dG(A) + \int_{A^*(b_1, x_z^q(v))}^{\overline{A}} T_z(v)(b_\lambda, A)dG(A) = ET_z(v)(b_\lambda, A)
\end{aligned}$$

Step 4: For any $z \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$, let

$$M(z) = \arg\max\{\frac{x}{q} + \delta Ev_z(x, A) : x \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]\}.$$

Then there exists $z^* \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$ such that $z^* \in M(z^*)$.

Proof: The result follows from *Kakutani's Fixed Point Theorem* if $M(z)$ is non-empty, upper hemi-continuous, and convex and compact-valued. We have:

Claim: $M(z)$ is non-empty, upper hemi-continuous, and convex and compact-valued.

Proof: Let F_z denote the set of all bounded and continuous real valued functions $\varphi(\cdot,\cdot,\cdot)$ defined over the compact set $[\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}] \times [\underline{x}, \overline{x}] \times [\underline{A}, \overline{A}]$. Define the operator:

$$\Psi(\varphi)(z, b, A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E\varphi(z, x, A') \\ B(r, g, x; b) \geq 0,\ g \leq g^*(A),\ r \geq r^* \ \&\ x \in [z, \overline{x}] \end{array} \right\}$$

It is easy to verify that Ψ maps F_z into itself and is a contraction. Thus, it has a unique fixpoint $\varphi^* = \Psi(\varphi^*)$ which belongs to F_z. Now note that for any $z \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$, $v_z(b, A) = \varphi^*(z, b, A)$. To see this, note that for any given z, $\varphi^*(z, b, A) \in F^*$, so we can define $T_z(\varphi^*(z, b, A))$. The definition of φ^*, however, implies $T_z(\varphi^*(z, b, A)) = \varphi^*(z, b, A)$. Since T_z has a unique fixpoint, it must be that $v_z(b, A) = \varphi^*(z, b, A)$.

Given this, we conclude that $v_z(b, A)$ is continuous in z and the *Theorem of the Maximum* then implies that $M(z)$ is non-empty, upper hemi-continuous, and compact-valued. Convexity of $M(z)$ follows from the fact that $Ev_z(x, A)$ is weakly concave. ■

Step 5: Let z^* be such that $z^* \in M(z^*)$. Then, $x^q_{z^*}(v_{z^*}) = z^*$.

Proof: By definition, $x^q_{z^*}(v_{z^*})$ is the largest element in the set $X^q_{z^*}(v_{z^*})$. By construction, z^* belongs to the set

$$M(z^*) = \arg\max\{\frac{x}{q} + \delta Ev_{z^*}(x, A) : x \in [\frac{R(r^*) - pg^*(\overline{A})}{\rho}, \overline{x}]\}.$$

Since z^* obviously satisfies the constraint that $x \geq z^*$, it must be the case that $z^* \in X^q_{z^*}(v_{z^*})$. If $z^* \neq x^q_{z^*}(v_{z^*})$, then it must be the case that $z^* < x^q_{z^*}(v_{z^*})$ and that

$$\frac{x^q_{z^*}(v_{z^*})}{q} + \delta Ev_{z^*}(x^q_{z^*}(v_{z^*}), A) = \frac{z^*}{q} + \delta Ev_{z^*}(z^*, A).$$

This implies that the expected value function $Ev_{z^*}(\cdot, A)$ is linear on the interval $[z^*, x^q_{z^*}(v_{z^*})]$. However, we know that

$$x^q_{z^*}(v_{z^*}) > z^* \geq \frac{R(r^*) - pg^*(\overline{A})}{\rho}$$

which implies that $pg^*(\overline{A}) + \rho x^q_{z^*}(v_{z^*}) > R(r^*)$, and hence that $A^*(x^q_{z^*}(v_{z^*}), x^q_{z^*}(v_{z^*})) < \overline{A}$. By continuity, therefore, there must exist an interval $[x', x^q_{z^*}(v_{z^*})] \subset [z^*, x^q_{z^*}(v_{z^*})]$ such that for all x in this interval $A^*(x, x^q_{z^*}(v_{z^*})) < \overline{A}$. But by Step 3, the expected value function $Ev_{z^*}(\cdot, A)$ is strictly concave on the interval $[x', x^q_{z^*}(v_{z^*})]$ - a contradiction.

Step 6: Let z^* be such that $z^* \in M(z^*)$. Then, the function $v_{z^*}(\cdot, A)$ is differentiable for all b such that $A \neq A^*(b, z^*)$. Moreover:

$$\frac{\partial v_{z^*}(b, A)}{\partial x} = \begin{cases} -(\frac{1-r_{z^*}(b,A)}{1-r_{z^*}(b,A)(1+\varepsilon)})(\frac{1+\rho}{n}) & \text{if } A > A^*(b, z^*) \\ -(\frac{1+\rho}{n}) & \text{if } A < A^*(b, z^*) \end{cases}.$$

Proof: Let $A \in [\underline{A}, \overline{A}]$ and let x_o be given. By Step 5, we know that $x^q_{z^*}(v_{z^*}) = z^*$ which immediately implies that $x^n_{z^*}(v_{z^*}) = z^*$. Suppose first that $A < A^*(x_o, z^*)$. Then, we have that in a neighborhood of x_o that

$$v_{z^*}(x, A) = u(w(1 - r^*), g^*(A); A) + \frac{B(r^*, g^*(A), z^*; x)}{n} + \delta Ev_{z^*}(z^*, A').$$

Thus, it is immediate that the value function $v_{z^*}(x, A)$ is differentiable at x_o and that

$$\frac{\partial v_{z^*}(x_o, A)}{\partial x} = -(\frac{1 + \rho}{n}).$$

Now suppose that $A > A^*(x_o, z^*)$. Then, we know that the budget constraint is binding, and that the constraints $r \geq r^*$ and $g \leq g^*(A)$ are not binding. Thus, we have that in a neighborhood of x_o that

$$v_{z^*}(x, A) = \max_{(r,g,y)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,y;x)}{n} + \delta E v_{z^*}(y, A') \\ B(r, g, y; x) \geq 0 \ \& \ x \in [z^*, \overline{x}] \end{array} \right\}.$$

Define the function

$$g(x) = \frac{R(r_{z^*}(x_o, A)) + x_{z^*}(x_o, A) - (1+\rho)x}{p}$$

and let

$$\eta(x) = u(w(1 - r_{z^*}(x_o, A)), g(x); A) + \frac{B(r_{z^*}(x_o, A), g(x), x_{z^*}(x_o, A); x)}{n} + \delta E v_{z^*}(x_{z^*}(x_o, A), A').$$

Notice that $(r_{z^*}(x_o, A), g(x), x_{z^*}(x_o, A))$ is a feasible policy when the initial debt level is x so that in a neighborhood of x_o we have that $v_{z^*}(x, A) \geq \eta(x)$. Moreover, $\eta(x)$ is twice continuously differentiable with derivatives

$$\eta'(x) = -\alpha A g(x)^{\alpha-1}(\tfrac{1+\rho}{p})$$

$$\eta''(x) = -(1-\alpha)\alpha A g(x)^{\alpha-2}(\tfrac{1+\rho}{p})^2 < 0$$

The second derivative property implies that $\eta(x)$ is strictly concave. It follows from Theorem 4.10 of Stokey and Lucas (1989) that $v_{z^*}(x, A)$ is differentiable at x_o with derivative $\frac{\partial v_{z^*}(x_o,A)}{\partial x} = \eta'(x_o) = -\alpha A g_{z^*}(x_o, A)^{\alpha-1}(\frac{1+\rho}{p})$. To complete the proof note that $(r_{z^*}(x_o, A), g_{z^*}(x_o, A))$ must solve the problem:

$$\max_{(r,g)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x_{z^*}(x_o,A);x_o)}{n} \\ B(r, g, x_{z^*}(x_o, A); x_o) \geq 0 \end{array} \right\},$$

which implies that $\alpha n A g_{z^*}(x_o, A)^{\alpha-1} = p[\frac{1-r_{z^*}(x_o,A)}{1-r_{z^*}(x_o,A)(1+\varepsilon)}]$. Thus, we have that

$$\frac{\partial v_{z^*}(x_o, A)}{\partial x} = -[\frac{1 - r_{z^*}(x_o, A)}{1 - r_{z^*}(x_o, A)(1+\varepsilon)}](\frac{1+\rho}{n}).$$

Step 7: Let z^* be such that $z^* \in M(z^*)$. Then, the following constitutes an equilibrium. For each proposal round τ

$$(r_\tau(b, A), g_\tau(b, A), x_\tau(b, A)) = (r_{z^*}(b, A), g_{z^*}(b, A), x_{z^*}(b, A));$$

for proposal rounds $\tau = 1, ..., T-1$

$$s_\tau(b, A) = B(r_{z^*}(b, A), g_{z^*}(b, A), x_{z^*}(b, A); b)/n;$$

and for proposal round T

$$s_T(b; A) = v_{T+1}(b, A) - u(w(1 - r_{z^*}(b, A)), g_{z^*}(b, A); A) - \delta E v_{z^*}(x_{z^*}(b, A), A');$$

where

$$v_{T+1}(b, A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,x,g;b)}{n} + \delta E v_{z^*}(x, A') \\ \\ \text{s.t. } B(r, x, g; b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{array} \right\}.$$

Proof: Given these proposals, the legislators' round one value function is given by $v_{z^*}(b, A)$. This follows from the fact that

$$\begin{aligned} v(b, A) &= u(w(1 - r_{z^*}(b, A)), g_{z^*}(b, A); A) + \frac{B(r_{z^*}(b, A), g_{z^*}(b, A), x_{z^*}(b, A); b)}{n} \\ &\quad + \delta E v_{z^*}(x_{z^*}(b, A), A') = v_{z^*}(b, A). \end{aligned}$$

Similarly, the round $\tau = 2, ..., T$ legislators' value function $v_\tau(b, A)$ is given by $v_{z^*}(b, A)$. It follows from Steps 3 and 4 that the value function $v_{z^*}(b, A)$ has the properties required for an equilibrium to be well-behaved. Thus, we need only show: (i) that for proposal rounds $\tau = 1, .., T-1$ the proposal

$$(r_{z^*}(b, A), g_{z^*}(b, A), x_{z^*}(b, A), \frac{B(r_{z^*}(b, A), g_{z^*}(b, A), x_{z^*}(b, A); b)}{n})$$

solves the problem

$$\max_{(r,g,x,s)} u(w(1-r), g; A) + B(r, g, x; b) - (q-1)\, s + \delta E v_{z^*}(x; A')$$

$$s.t. \quad u(w(1-r), g; A) + s + \delta E v_{z^*}(x; A') \geq v_{z^*}(b; A),$$

$$B(r, g, x; b) \geq (q-1)s, \quad s \geq 0 \ \& \ x \in [\underline{x}, \overline{x}],$$

and (ii) that for proposal round T the proposal

$$(r_{z^*}(b, A), g_{z^*}(b, A), x_{z^*}(b, A), v_{T+1}(b, A) - u(w(1 - r_{z^*}(b, A)), g_{z^*}(b, A); A) - \delta E v_{z^*}(x_{z^*}(b, A), A'))$$

solves the problem

$$\max_{(r,g,x,s)} u(w(1-r), g; A) + B(r, g, x; b) - (q-1)\, s + \delta E v_{z^*}(x; A')$$

$$s.t. \quad u(w(1-r), g; A) + s + \delta E v_{z^*}(x; A') \geq v_{T+1}(b, A),$$

$$B(r, g, x; b) \geq (q-1)s, \quad s \geq 0 \ \& \ x \in [\underline{x}, \overline{x}],$$

We show only (i) - the argument for (ii) being analogous.

Consider some proposal round $\tau = 1, ..., T-1$. Let (b, A) be given. To simplify notation, let

$$(\widehat{r}, \widehat{g}, \widehat{x}, \widehat{s}) = (r_{z^*}(b, A), g_{z^*}(b, A), x_{z^*}(b, A), \frac{B(r_{z^*}(b, A), g_{z^*}(b, A), x_{z^*}(b, A); b)}{n}).$$

It is clear from the argument in the text that $(\widehat{r}, \widehat{g}, \widehat{x})$ solves the problem

$$\max_{(r,g,x)} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{q} + \delta E v_{z^*}(x; A')$$

$$s.t. \quad B(r, g, x; b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}],$$

and that $\widehat{s} = v_{z^*}(b, A) - u(w(1-\widehat{r}), \widehat{g}; A) - \delta E v_{z^*}(\widehat{x}; A')$. Suppose that $(\widehat{r}, \widehat{g}, \widehat{x}, \widehat{s})$ does not solve the round τ proposer's problem. Then there exist some (r', g', x', s') which achieves a higher value of the proposer's objective function. We know that $s' \geq v_{z^*}(b; A) - u(w(1-r'), g'; A) - \delta E v_{z^*}(x'; A')$. Thus, we have that the value of the proposer's objective function satisfies

$$\begin{aligned} & u(w(1-r'), g'; A) + B(r', g', x'; b) - (q-1) s' + \delta E v_{z^*}(x'; A') \\ \leq \ & q[u(w(1-r'), g'; A) + \delta E v_{z^*}(x'; A')] + B(r', g', x'; b). \end{aligned}$$

But since $B(r', g', x'; b) \geq 0$, we know that

$$\begin{aligned} & q[u(w(1-r'), g'; A) + \delta E v_{z^*}(x'; A')] + B(r', g', x'; b) \\ \leq \ & q[u(w(1-\widehat{r}), \widehat{g}; A) + \delta E v_{z^*}(\widehat{x}; A')] + B(\widehat{r}, \widehat{g}, \widehat{x}; b). \end{aligned}$$

But the right hand side of the inequality is the value of the proposer's objective function under the proposal $(\widehat{r}, \widehat{g}, \widehat{x}, \widehat{s})$. This therefore contradicts the assumption that (r', g', x', s') achieves a higher value for the proposer's problem. ■

We now turn to proving that the equilibrium is unique. Let $\{r^0_\tau(b, A),\ g^0_\tau(b, A),\ x^0_\tau(b, A),\ s^0_\tau(b, A)\}_{\tau=1}^T$ and $\{r^1_\tau(b, A), g^1_\tau(b, A), x^1_\tau(b, A), s^1_\tau(b, A)\}_{\tau=1}^T$ be two equilibria with associated round one value functions $v^0(b, A)$ and $v^1(b, A)$. Let x^*_0 and x^*_1 be the debt levels chosen in the BAU regimes of the two equilibria and suppose that $x^*_0 \leq x^*_1$. We will demonstrate that it must be the case that $x^*_0 = x^*_1$. To do this, we will show that the assumption that $x^*_0 < x^*_1$ results in a contradiction.

As in the proof of existence, define the operator $T_z : F^* \rightarrow F$ as follows:

$$T_z(v)(b, A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ \\ B(r, g, x; b) \geq 0,\ g \leq g^*(A),\ r \geq r^* \ \&\ x \in [z, \overline{x}] \end{array} \right\}.$$

We know that $T_z(v) \in F^*$ and that T_z is a contraction. Moreover, for $i \in \{0,1\}$, we have that $T_{x_i^*}(v^i) = v^i$.

Now let $v \in F^*$ be such that for all b, $v(\cdot, A)$ is differentiable at b for almost all A and for each $i \in \{0,1\}$ consider the sequence of functions $\langle v_k^i \rangle_{k=1}^{\infty}$ defined inductively as follows: $v = T_{x_i^*}(v)$, and $v_{k+1}^i = T_{x_i^*}(v_k^i)$. Notice that since T_{z_i} is a contraction, $\langle v_k^i \rangle_{k=1}^{\infty}$ converges to v^i. We now establish the following result:

Claim: Let $\rho' \in (0, \rho)$. Then, for all k and for any $x \in [x_1^*, \overline{x}]$ we have that

$$-E(\frac{\partial v_k^1(x,A)}{\partial b}) > -E(\frac{\partial v_k^0(x - \frac{x_1^* - x_0^*}{1+\rho'}, A)}{\partial b}).$$

Proof: The proof proceeds via induction. Consider first the claim for $k = 1$. Recall from Step 1 of the existence part of the proof of Proposition 2 that if (r, g, x) is a solution to the problem

$$\begin{array}{c} \max u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ \text{s.t. } B(r,g,x;b) \geq 0,\ g \leq g^*(A),\ r \geq r^*,\ x \in [z, \overline{x}] \end{array},$$

then: (i) if $A \leq A^*(b, x_z^n(v))$ then $(r,g) = (r^*, g^*(A))$ and $x \in X_z^n(v) \cap \{x : B(r^*, g^*(A), x; b) \geq 0\}$; (ii) if $A \in (A^*(b, x_z^n(v)), A^*(b, x_z^q(v))]$ then $(r,g) = (r^*, g^*(A))$ and $B(r^*, g^*(A), x; b) = 0$; and (iii) if $A > A^*(b, x_z^q(v))$ (r, g, x) is uniquely defined, the budget constraint is binding, $r > r^*$ and $g < g^*(A)$. Denote the solution in case (iii) as $(r_z(b, A; v), g_z(b, A; v), x_z(b, A; v))$.

It follows from this that, if $A \leq A^*(b, x_z^n(v))$

$$T_z(v)(b,A) = u(w(1-r^*), g^*(A); A) + \frac{B(r^*), g^*(A), x_z^n(v); b)}{n} + \delta E v(x_z^n(v), A')$$

and

$$-\frac{\partial T_z(v)(b,A)}{\partial b} = \frac{1+\rho}{n}.$$

If $A \in (A^*(b, x_z^n(v)), A^*(b, x_z^q(v))]$, then

$$T_z(v)(b,A) = u(w(1-r^*), g^*(A); A) + \delta E v(pg^*(A) + (1+\rho)b - R(r^*), A')$$

and, since v is differentiable,

$$-\frac{\partial T_z(v)(b,A)}{\partial b} = -\delta E v'(pg^*(A) + (1+\rho)b - R(r^*), A')(1+\rho).$$

Notice for future reference that in this range, $x \in (x_z^n(v), x_z^q(v)]$ and hence

$$-\delta E v'(pg^*(A) + (1+\rho)b - R(r^*), A')(1+\rho) \in (\frac{1+\rho}{n}, \frac{1+\rho}{q}].$$

If $A > A^*(b, x_z^q(v))$ then

$$T_z(v)(b,A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r),g;A) + \frac{B(r,g,x;b)}{n} + \delta E v(x,A') \\ \\ B(r,g,x;b) \geq 0 \ \& \ x \in [z,\overline{x}] \end{array} \right\} \tag{A.4}$$

and

$$-\frac{\partial T_z(v)(b,A)}{\partial b} = \frac{1-r_z(b,A;v)}{n(1-r_z(b,A;v)(1+\varepsilon))}(1+\rho).$$

Since $r_z(b,A;v) > r^*$, in this range we have that

$$-\frac{\partial T_z(v)(b,A)}{\partial b} > \frac{(1+\rho)}{q}.$$

Combining all this and using the fact that $T_z(v)(b,\cdot)$ is continuous, we have that:

$$\begin{aligned} -n\delta E(\frac{\partial T_z(v)(b,A)}{\partial b}) \quad = \quad & G(A^*(b,x_z^n(v))) \\ & -n\int_{A^*(b,x_z^n(v))}^{A^*(b,x_z^q(v))} E[\frac{\partial v(pg^*(A)+(1+\rho)b-R(r^*),A')}{\partial b}]dG(A) \\ & +\int_{A^*(b,x_z^q(v))}^{\overline{A}} [\frac{1-r_z(b,A;v)}{1-r_z(b,A;v)(1+\varepsilon)}]dG(A). \end{aligned}$$

Applying this to the problem at hand, let $x \in [x_1^*, \overline{x}]$ and $f(x) = x - \frac{x_1^*-x_0^*}{1+\rho'}$. Then, to prove the claim for $k=1$, we need to show that

$$\begin{aligned} & G(A^*(x,x_{x_1^*}^n(v))) - n\int_{A^*(x,x_{x_1^*}^n(v))}^{A^*(x,x_{x_1^*}^q(v))} E[\tfrac{\partial v(pg^*(A)+(1+\rho)x-R(r^*),A')}{\partial b}]dG(A) \\ & + \int_{A^*(x,x_{x_1^*}^q(v))}^{\overline{A}} [\tfrac{1-r_{x_1^*}(x,A;v)}{1-r_{x_1^*}(x,A;v)(1+\varepsilon)}]dG(A) \\ > \ & G(A^*(f(x),x_{x_0^*}^n(v))) - n\int_{A^*(f(x),x_{x_0^*}^n(v))}^{A^*(f(x),x_{x_0^*}^q(v))} E[\tfrac{\partial v(pg^*(A)+(1+\rho)f(x)-R(r^*),A')}{\partial b}]dG(A) \\ & + \int_{A^*(f(x),x_{x_0^*}^q(v))}^{\overline{A}} [\tfrac{1-r_{x_0^*}(f(x),A;v)}{1-r_{x_0^*}(f(x),A;v)(1+\varepsilon)}]dG(A). \end{aligned}$$

It is straightforward to verify that the following four conditions are sufficient for this inequality to hold: (i) $A^*(x, x_{x_1^*}^n(v)) < A^*(f(x), x_{x_0^*}^n(v))$, (ii) $A^*(x, x_{x_1^*}^q(v)) < A^*(f(x), x_{x_0^*}^q(v))$, (iii) for all $A \in (A^*(f(x), x_{x_0^*}^n(v)), A^*(x, x_{x_1^*}^q(v))]$

$$-E\frac{\partial v(pg^*(A)+(1+\rho)x-R(r^*),A')}{\partial b} \geq -E\frac{\partial v(pg^*(A)+(1+\rho)f(x)-R(r^*),A')}{\partial b},$$

and (iv) for all $A \in (A^*(f(x), x_{x_0^*}^q(v)), \overline{A}]$

$$\frac{1-r_{x_0^*}(f(x),A;v)}{1-r_{x_0^*}(f(x),A;v)(1+\varepsilon)} \leq \frac{1-r_{x_1^*}(x,A;v)}{1-r_{x_1^*}(x,A;v)(1+\varepsilon)}.$$

We will now show that these four conditions are satisfied. We begin with condition (i). If it were not satisfied, then $A^*(x, x^n_{x_1^*}(v)) \geq A^*(f(x), x^n_{x_0^*}(v))$ which is equivalent to $x^n_{x_1^*}(v)-(1+\rho)x \geq x^n_{x_0^*}(v) - (1+\rho)f(x)$. Thus,

$$\begin{aligned} x^n_{x_0^*}(v) &\leq x^n_{x_1^*}(v) - (1+\rho)[x - f(x)] \\ &= x^n_{x_1^*}(v) - \frac{1+\rho}{1+\rho'}(x_1^* - x_0^*) < x^n_{x_1^*}(v) - (x_1^* - x_0^*) \end{aligned}$$

which implies that $x^n_{x_1^*}(v) - x^n_{x_0^*}(v) > x_1^* - x_0^*$. Given the definition of $x^n_{x_1^*}(v)$, this requires that $x^n_{x_1^*}(v) > x_1^*$. This in turn implies that

$$-\delta E \frac{\partial v(x^n_{x_1^*}(v), A')}{\partial b} = \frac{1}{n}$$

and hence that $x^n_{x_0^*}(v) = x^n_{x_1^*}(v)$ - a contradiction.

Condition (ii) can be established in the same way and condition (iii) follows directly from the assumption that $v(\cdot, A)$ is concave. This leaves condition (iv). From the first order conditions associated with problem (A.4) (see (10), (11) and (12)), we have that

$$\frac{1 - r_{x_1^*}(x, A; v)}{1 - r_{x_1^*}(x, A; v)(1+\varepsilon)} \geq -\delta n E \frac{\partial v(x_{x_1^*}(x, A; v), A')}{\partial b} \quad (= \text{ if } x_{x_1^*}(x, A; v) < \overline{x})$$

and that

$$\frac{1 - r_{x_0^*}(f(x), A; v)}{1 - r_{x_0^*}(f(x), A; v)(1+\varepsilon)} \geq -\delta n E \frac{\partial v(x_{x_0^*}(f(x), A; v), A')}{\partial b} \quad (= \text{ if } x_{x_0^*}(f(x), A; v) < \overline{x}).$$

Suppose that

$$\frac{1 - r_{x_0^*}(f(x), A; v)}{1 - r_{x_0^*}(f(x), A; v)(1+\varepsilon)} > \frac{1 - r_{x_1^*}(x, A; v)}{1 - r_{x_1^*}(x, A; v)(1+\varepsilon)}.$$

Then this implies that $r_{x_0^*}(f(x), A; v) > r_{x_1^*}(x, A; v)$, $g_{x_0^*}(f(x), A; v) < g_{x_1^*}(x, A; v)$, and $x_{x_0^*}(f(x), A; v) \geq x_{x_1^*}(x, A; v)$. Thus, we have that

$$\begin{aligned} pg_{x_0^*}(f(x), A; v) + (1+\rho)f(x) &= R(r_{x_0^*}(f(x), A; v)) + x_{x_0^*}(f(x), A; v) \\ &> R(r_{x_1^*}(x, A; v)) + x_{x_1^*}(x, A; v) = pg_{x_1^*}(x, A; v) + (1+\rho)x \end{aligned}$$

This means that $(1+\rho)[x - f(x)] = \frac{1+\rho}{1+\rho'}(x_1^* - x_0^*) < 0$, which is a contradiction.

Now assume the claim is true for $1, ..., k$ and consider it for $k+1$. We have that

$$-n\delta E(\frac{\partial T_{x_i^*}(v_k^i)(b, A)}{\partial b}) = G(A^*(b, x^n_{x_i^*}(v_k^i)))$$

$$-n\int_{A^*(b,x^n_{x_i^*}(v_k^i))}^{A^*(b,x^q_{x_i^*}(v_k^i))} E[\frac{\partial v_k^i(pg^*(A)+(1+\rho)b-R(r^*),A')}{\partial b}]dG(A)$$
$$+\int_{A^*(b,x^q_{x_i^*}(v_k^i))}^{\overline{A}} [\frac{1-r_{x_i^*}(b,A;v_k^i)}{1-r_{x_i^*}(b,A;v_k^i)(1+\varepsilon)}]dG(A).$$

Thus, we need to show that

$$G(A^*(x,x^n_{x_1^*}(v_k^1)))-n\int_{A^*(x,x^n_{x_1^*}(v_k^1))}^{A^*(x,x^q_{x_1^*}(v_k^1))} E[\tfrac{\partial v_k^1(pg^*(A)+(1+\rho)x-R(r^*),A')}{\partial b}]dG(A)$$
$$+\int_{A^*(x,x^q_{x_1^*}(v_k^1))}^{\overline{A}}[\tfrac{1-r_{x_1^*}(x,A;v_k^1)}{1-r_{x_1^*}(x,A;v_k^1)(1+\varepsilon)}]dG(A)$$
$$>G(A^*(f(x),x^n_{x_0^*}(v_k^0)))-n\int_{A^*(f(x),x^n_{x_0^*}(v_k^0))}^{A^*(f(x),x^q_{x_0^*}(v_k^0))} E[\tfrac{\partial v_k^0(pg^*(A)+(1+\rho)f(x)-R(r^*),A')}{\partial b}]dG(A)$$
$$+\int_{A^*(f(x),x^q_{x_0^*}(v_k^0))}^{\overline{A}}[\tfrac{1-r_{x_0^*}(f(x),A;v_k^0)}{1-r_{x_0^*}(f(x),A;v_k^0)(1+\varepsilon)}]dG(A)$$

Following the same approach as above, for this inequality to hold, the following four conditions are sufficient: (i) $A^*(x,x^n_{x_1^*}(v_k^1))<A^*(f(x),x^n_{x_0^*}(v_k^0))$, (ii) $A^*(x,x^q_{x_1^*}(v_k^1))<A^*(f(x),x^q_{x_0^*}(v_k^0))$, (iii) for all $A\in(A^*(f(x),x^n_{x_0^*}(v_k^0)),A^*(x,x^q_{x_1^*}(v_k^1))]$

$$-E\frac{\partial v_k^1(pg^*(A)+(1+\rho)x-R(r^*),A')}{\partial b}\geq -E\frac{\partial v_k^0(pg^*(A)+(1+\rho)f(x)-R(r^*),A')}{\partial b},$$

and (iv) for all $A\in(A^*(f(x),x^q_{x_0^*}(v_k^0)),\overline{A}]$

$$\frac{1-r_{x_0^*}(f(x),A;v_k^0)}{1-r_{x_0^*}(f(x),A;v_k^0)(1+\varepsilon)}\leq\frac{1-r_{x_1^*}(x,A;v_k^1)}{1-r_{x_1^*}(x,A;v_k^1)(1+\varepsilon)}.$$

We will again show that these four conditions are satisfied. We begin with condition (i). If it were not satisfied, then it must be the case that $A^*(x,x^n_{x_1^*}(v_k^1))\geq A^*(f(x),x^n_{x_0^*}(v_k^0))$ which is equivalent to $x^n_{x_1^*}(v_k^1)-(1+\rho)x\geq x^n_{x_0^*}(v_k^0)-(1+\rho)f(x)$. This implies that

$$x^n_{x_0^*}(v_k^0)\leq x^n_{x_1^*}(v_k^1)-\frac{1+\rho}{1+\rho'}[x_1^*-x_0^*]<x^n_{x_1^*}(v_k^1)-[x_1^*-x_0^*]. \tag{A.5}$$

Thus, we know by the induction step that

$$-\delta E\frac{\partial v_k^1(x^n_{x_1^*}(v_k^1),A')}{\partial b}>-\delta E\frac{\partial v_k^0(x^n_{x_0^*}(v_k^0),A')}{\partial b}\geq\frac{1}{n}.$$

This implies that $x^n_{x_1^*}(v_k^1)=x_1^*$ which in turn, together with (A.5), implies that $x^n_{x_0^*}(v_k^0)<x_0^*$ which is a contradiction.

We can use similar logic to conclude that condition (ii) is satisfied. Condition (iii) follows immediately from the induction step since we have that

$$pg^*(A)+(1+\rho)x-R(r^*)-[x_1^*-x_0^*]>pg^*(A)+(1+\rho)f(x)-R(r^*).$$

This leaves condition (iv). Suppose to the contrary that for some A

$$\frac{1-r_{x_0^*}(f(x),A;v_k^0)}{1-r_{x_0^*}(f(x),A;v_k^0)(1+\varepsilon)} > \frac{1-r_{x_1^*}(x,A;v_k^1)}{1-r_{x_1^*}(x,A;v_k^1)(1+\varepsilon)}$$

Again, from the first order conditions associated with problem (A.4) we have that

$$\frac{1-r_{x_1^*}(x,A;v_k^1)}{1-r_{x_1^*}(x,A;v_k^1)(1+\varepsilon)} \geq -\delta nE\frac{\partial v_k^1(x_{x_1^*}(x,A;v_k^1),A')}{\partial b} \quad (= \text{ if } x_{x_1^*}(x,A;v_k^1) < \overline{x})$$

and that

$$\frac{1-r_{x_0^*}(f(x),A;v_k^0)}{1-r_{x_0^*}(f(x),A;v_k^0)(1+\varepsilon)} \geq -\delta nE\frac{\partial v_k^0(x_{x_0^*}(f(x),A;v_k^0),A')}{\partial b} \quad (= \text{ if } x_{x_0^*}(f(x),A;v_k^0) < \overline{x}).$$

If $x_{x_0^*}(f(x),A;v_k^0) < \overline{x}$ these first order conditions imply that

$$-\delta nE\frac{\partial v_k^1(x_{x_1^*}(x,A;v_k^1),A')}{\partial b} < -\delta nE\frac{\partial v_k^0(x_{x_0^*}(f(x),A;v_k^0),A')}{\partial b}.$$

We know by the induction step that for any $x \geq x_1^*$, $-\delta nE(\frac{\partial v_k^1(x,A)}{\partial x}) > -\delta nE(\frac{\partial v_k^0(f(x),A)}{\partial x})$. Thus, it must be the case that

$$x_{x_0^*}(f(x),A;v_k^0) > f(x_{x_1^*}(x,A;v_k^1)) = x_{x_1^*}(x,A;v_k^1) - \frac{x_1^*-x_0^*}{1+\rho'}$$

In addition, we know that $r_{x_0^*}(f(x),A;v_k^0) > r_{x_1^*}(x,A;v_k^1)$ and that $g_{x_0^*}(f(x),A;v_k^0) < g_{x_1^*}(x,A;v_k^1)$. But this means that

$$\begin{aligned} pg_{x_0^*}(f(x),A;v_k^0) + (1+\rho)f(x) &= x_{x_0^*}(f(x),A;v_k^0) + R(r_{x_0^*}(f(x),A;v_k^0)) \\ &> x_{x_1^*}(x,A;v_k^1) + R(r_{x_1^*}(x,A;v_k^1)) - \frac{x_1^*-x_0^*}{1+\rho'} \\ &= pg_{x_1^*}(x,A;v_k^1) + (1+\rho)x - \frac{x_1^*-x_0^*}{1+\rho'}. \end{aligned}$$

This in turn implies that $(1+\rho)[x-f(x)] < \frac{x_1^*-x_0^*}{1+\rho'}$ which is a contradiction. If $x_{x_0^*}(f(x),A;v_k^0) = \overline{x}$, then it must be the case that $x_{x_1^*}(x,A;v_k^1) \leq x_{x_0^*}(f(x),A;v_k^0)$ and the same contradiction arises. ■

To complete the uniqueness proof, observe that for $i \in \{0,1\}$ the function $E(v^i(\cdot,A))$ is concave and differentiable. In addition, $\langle E(v_k^i(\cdot,A))\rangle$ is a sequence of concave and differentiable functions such that for all x $\lim_{k\to\infty} E(v_k^i(x,A)) = E(v^i(x,A))$. Thus, by Theorem 25.7 of Rockafellar (1970), we know that $\lim_{k\to\infty}\frac{dE(v_k^i(x,A))}{dx} = \frac{dE(v^i(x,A))}{dx}$. It follows that for any $x \in [x_1^*,\overline{x}]$

$$\begin{aligned} -\delta E(\frac{\partial v^1(x,A)}{\partial b}) &= \lim_{k\to\infty} -\delta E(\frac{\partial v_k^1(x,A)}{\partial b}) \\ &\geq \lim_{k\to\infty} -\delta E(\frac{\partial v_k^0(x-\frac{(x_1^*-x_0^*)}{1+\rho'},A)}{\partial b}) = -\delta E(\frac{\partial v^0(x-\frac{(x_1^*-x_0^*)}{1+\rho'},A)}{\partial b}). \end{aligned}$$

From equation (21) in the text, we know that

$$-\delta E(\frac{\partial v^1(x_1^*,A)}{\partial b}) = -\delta E(\frac{\partial v^0(x_0^*,A)}{\partial x}) = \frac{1}{q}.$$

Thus, it follows that

$$-\delta E(\frac{\partial v^0(x_1^* - \frac{(x_1^*-x_0^*)}{1+\rho'},A)}{\partial b}) \leq -\delta E(\frac{\partial v^1(x_1^*,A)}{\partial b}) = -\delta E(\frac{\partial v^0(x_0^*,A)}{\partial b}).$$

But this implies that $x_1^* - \frac{x_1^*-x_0^*}{1+\rho'} \leq x_0^*$, which contradicts the fact that $x_1^* > x_0^*$.

It follows that $x_0^* = x_1^*$. This, in turn, implies that $v^0 = v^1$ and hence that $\{r_\tau^0(b,A)$, $g_\tau^0(b,A)$, $x_\tau^0(b,A)$, $s_\tau^0(b,A)\}_{\tau=1}^T$ equals $\{r_\tau^1(b,A)$, $g_\tau^1(b,A)$, $x_\tau^1(b,A)$, $s_\tau^1(b,A)\}_{\tau=1}^T$. ■

6 Proof of Proposition 3

Given the discussion in the text, the only thing we need to show is that the equilibrium debt distribution converges to a unique invariant distribution. Let $\psi_t(x)$ denote the distribution function of the current level of debt at the beginning of period t. The distribution function $\psi_1(x)$ is exogenous and is determined by the economy's initial level of debt b_0. To describe the distribution of debt in periods $t \geq 2$, we must first describe the *transition function* implied by the equilibrium. First, define the function $\widehat{A} : [\underline{x}, \overline{x}] \times (x^*, \overline{x}] \rightarrow [\underline{A}, \overline{A}]$ as follows:

$$\widehat{A}(b,x) = \begin{cases} \underline{A} & \text{if } x < x(b,\underline{A}) \\ \min\{A \in [\underline{A},\overline{A}] : x(b,A) = x\} & \text{if } x \in [x(b,\underline{A}), x(b,\overline{A})] \\ \overline{A} & \text{if } x > x(b,\overline{A}) \end{cases} .$$

Intuitively, $\widehat{A}(b,x)$ is the smallest value of public goods under which the equilibrium debt level would be x given an initial level of debt b. Then, the transition function is given by

$$H(b,x) = \begin{cases} G(\widehat{A}(b,x)) & \text{if } x \in (x^*, \overline{x}] \\ G(A^*(b,x^*)) & \text{if } x = x^* \end{cases} .$$

Intuitively, $H(b,x)$ is the probability that in the next period the initial level of debt will be less than or equal to $x \in [x^*, \overline{x}]$ if the current level of debt is b. Using this notation, the distribution of debt at the beginning of any period $t \geq 2$ is defined inductively by

$$\psi_t(x) = \int_b H(b,x) d\psi_{t-1}(b).$$

The sequence of distributions $\langle \psi_t(x) \rangle$ *converges* to the distribution $\psi(x)$ if for all $x \in [x^*, \overline{x}]$, we have that $\lim_{t\to\infty} \psi_t(x) = \psi(x)$.[2] Moreover, $\psi^*(x)$ is an *invariant distribution* if

$$\psi^*(x) = \int_b H(b,x) d\psi^*(b).$$

We can now establish that the sequence of debt distributions $\langle \psi_t(x) \rangle$ converges to a unique invariant distribution $\psi^*(x)$.

It is easy to prove that the transition function $H(b,x)$ has the Feller Property and that it is monotonic in b (see Ch. 12.4 in Stokey, Lucas and Prescott (1989) for definitions). By Theorem 12.12 in Stokey, Lucas and Prescott (1989), therefore, the result follows if the following "mixing condition" is satisfied:

Mixing Condition: *There exists an* $\epsilon > 0$ *and* $m \geq 1$*, such that* $H^m(\overline{x}, x^*) \geq \epsilon$ *and* $1 - H^m(\underline{x}, x^*) \geq \epsilon$ *where the function* $H^m(b,x)$ *is defined inductively by* $H^1(b,x) = H(b,x)$ *and* $H^m(b,x) = \int_z H(z,x) dH^{m-1}(b,z)$.

Intuitively, this condition requires that if we start out with the highest level of debt $\overline{x}$, then we will end up at x^* with probability greater than ϵ after m periods, while if we start out with the lowest level of debt $\underline{x}$, we will end up above x^* with probability greater than ϵ in m periods. For any $b \in [\underline{x}, \overline{x}]$ and $A \in [\underline{A}, \overline{A}]$ define the sequence $\langle \phi_m(b,A) \rangle$ as follows: $\phi_0(b,A) = b$, $\phi_{m+1}(b,A) = x(\phi_m(b,A), A)$. Thus, $\phi_m(b,A)$ is the level of debt if the debt level were b at time 0 and the shock was A in periods 1 through m. Recall that, by assumption, there exists some positive constant $\xi > 0$, such that for any pair of realizations satisfying $A < A'$, the difference $G(A') - G(A)$ is at least as big as $\xi(A' - A)$. This implies that for any $b \in [\underline{x}, \overline{x}]$, $H^m(b, \phi_m(b, \underline{A} + \lambda)) - H^m(b, \phi_m(b, \underline{A})) \geq \xi^m \lambda^m$ for all λ such that $0 < \lambda < \overline{A} - \underline{A}$. Using this observation, we can prove:

Claim 1: For m sufficiently large, $H^m(\overline{x}, x^*) > 0$.

Proof: It suffices to show that for m sufficiently large $A^*(\phi_m(\overline{x}, \underline{A}), x^*) > \underline{A}$. Then, for any such m, by continuity there is a λ_m small enough such that $A^*(\phi_m(\overline{x}, \underline{A} + \lambda_m), x^*) > \underline{A}$. It then follows that

$$H^m(\overline{x}, x^*) = \int_z H(z, x^*) dH^{m-1}(\overline{x}, z) = \int_z G(A^*(z, x^*)) dH^{m-1}(\overline{x}, z)$$

[2] In the present environment, this definition is equivalent to the requirement that the sequence of probability measures associated with $\langle \psi_t(x) \rangle$ converges weakly to the probability measure associated with $\psi(x)$ (see Stokey, Lucas and Prescott (1989) Theorem 12.8).

$$\geq \int_{\phi_m(\overline{x},\underline{A})}^{\phi_m(\overline{x},\underline{A}+\lambda_m)} G(A^*(z,x^*))dH^{m-1}(\overline{x},z)$$
$$\geq G(A^*(\phi_m(\overline{x},\underline{A}+\lambda_m),x^*))\left[H^{m-1}(\overline{x},\phi_{m-1}(\overline{x},\underline{A}+\lambda_m)) - H^{m-1}(\overline{x},\phi_{m-1}(\overline{x},\underline{A}))\right]$$
$$\geq G(A^*(\phi_m(\overline{x},\underline{A}+\lambda_m),x^*))\left(\xi\lambda_m\right)^{m-1} > 0.$$

Suppose, to the contrary, that for all m we have that $A^*(\phi_m(\overline{x},\underline{A}),x^*) \leq \underline{A}$. Then, it must be the case that the sequence $\langle\phi_m(\overline{x},\underline{A})\rangle$ is decreasing. To see this note that since $r(b,A)$ is increasing in A we have that

$$\frac{1-r(\phi_k(\overline{x},\underline{A}),\underline{A})}{1-r(\phi_k(\overline{x},\underline{A}),\underline{A})(1+\varepsilon)} < \int_{\underline{A}}^{\overline{A}} (\frac{1-r(\phi_k(\overline{x},\underline{A}),A)}{1-r(\phi_k(\overline{x},\underline{A}),A)(1+\varepsilon)})dG(A).$$

But the first order condition for $x(b,A)$ (see (11) with appropriate value function) and the derivative of the value function (23) imply that:

$$\frac{1-r(\phi_{k-1}(\overline{x},\underline{A}),\underline{A})}{1-r(\phi_{k-1}(\overline{x},\underline{A}),\underline{A})(1+\varepsilon)} = \int_{\underline{A}}^{\overline{A}} (\frac{1-r(\phi_k(\overline{x},\underline{A}),A)}{1-r(\phi_k(\overline{x},\underline{A}),A)(1+\varepsilon)})dG(A). \tag{A.6}$$

Since $r(b,\underline{A})$ is increasing in b and A, this implies $\phi_{k-1}(\overline{x},\underline{A}) > \phi_k(\overline{x},\underline{A})$.

We can therefore assume without loss of generality that $\phi_m(\overline{x},\underline{A})$ converges to some finite $\beta \geq \underline{x}$. We now prove that this yields a contradiction. Taking the limit as $m \to \infty$, continuity of $r(\cdot,\underline{A})$ would imply $\lim_{k\to\infty} r(\phi_k(\overline{x},\underline{A}),A) = r(\phi_\infty(\overline{x},\underline{A}),A)$ for all A. Using condition (A.4):

$$\frac{1-r(\phi_\infty(\overline{x},\underline{A}),\underline{A})}{1-r(\phi_\infty(\overline{x},\underline{A}),\underline{A})(1+\varepsilon)} = \int_{\underline{A}}^{\overline{A}} (\frac{1-r(\phi_\infty(\overline{x},\underline{A}),A)}{1-r(\phi_\infty(\overline{x},\underline{A}),A)(1+\varepsilon)})dG(A)$$

which is impossible since $r(\phi_\infty(\overline{x},\underline{A}),A)$ is strictly increasing in A. We conclude therefore that for m sufficiently large $A^*(\phi_m(\overline{x},\underline{A}),x^*) > \underline{A}$, which yields the result. ■

Next, we can establish:

Claim 2: For all $m \geq 2$, $1 - H^m(\underline{x},x^*) \geq G(A^*(\underline{x},x^*))G(A^*(x^*,x^*))^{m-2}\left[1 - G(A^*(x^*,x^*))\right]$.

Proof: With probability $G(A^*(\underline{x},x^*))$ the level of debt chosen in period 1 is x^* when the initial level of debt is $\underline{x}$; so with probability $G(A^*(\underline{x},x^*))G(A^*(x^*,x^*))^{m-2}$ the level of debt is x^* for the first $m-1$ periods. Given this, the probability that the level of debt is larger than x^* in period m is at least $G(A^*(\underline{x},x^*))G(A^*(x^*,x^*))^{m-2}\left[1 - G(A^*(x^*,x^*))\right]$. ■

These two Claims imply that the Mixing Condition is satisfied if $q < n$. To see this, choose m sufficiently large so that $H^m(\overline{x},x^*) > 0$. This is always possible by Claim 1. Now let

$$\epsilon = \min\left\{G(A^*(\underline{x},x^*))G(A^*(x^*,x^*))^{m-2}\left[1 - G(A^*(x^*,x^*))\right]; H^m(\overline{x},x^*)\right\}$$

Assuming that $q < n$, we know from the definition of x^* that $A^*(x^*, x^*) \in (\underline{A}, \overline{A})$ (see equation (24)) and $A^*(\underline{x}, x^*) > A^*(x^*, x^*) > \underline{A}$. Thus, $\epsilon > 0$ and the condition is satisfied. ■

7 Proof of Proposition 4

This result follows from Step 7 of the existence part of the proof of Proposition 2. ■

8 Completion of the Proof of Proposition 1

As discussed in the text, Proposition 4 implies that legislative decision-making delivers the planner's solution when $q = n$. Thus, we just need to show that when $q = n$, the equilibrium debt level will reach $\underline{x}$ with probability one. Since $x^* = \underline{x}$ when $q = n$, Claim 1 of Proposition 3 implies that there exists a $\epsilon > 0$ and a m such that for any initial b, the probability that $x = \underline{x}$ in the next m periods is at least ϵ. Thus, the probability that x is never equal to $\underline{x}$ in the next $j \cdot m$ periods is not larger than $(1-\epsilon)^j$. Since $\lim_{j\to\infty}(1-\epsilon)^j = 0$, we conclude that the probability that x is never equal to $\underline{x}$ is zero. ■