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Inefficiency in Legislative Policy-Making: A Dynamic Analysis

Technical Appendix

Abstract

This appendix provides complete and detailed proofs of the results presented in the paper "Inefficiency in Legislative Policy-Making: A Dynamic Analysis."

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1 Proof of Proposition 1

To prove the result, we will need some additional notation. For any strictly concave function $v(x)$ consider the problem for all $\mu \in [0, \infty)$

$$\begin{aligned} &\max_{\{r,x\}} \mu[u(w(1-r), g) + \delta v(x)] + B(r, x; g) \\ &s.t. \quad B(r, x; g) \geq 0 \end{aligned} \tag{1}$$

Interpreting $v(x)$ as the expected payoff with x units of the public good, the problem is to maximize the aggregate utility of μ legislators under the assumption that any revenue that is not used for investment in the public good is used to finance pork in these legislators' districts. Under the assumption that v is strictly concave, there is a unique solution to this problem given by $r(g; \mu, v)$ and $x(g; \mu, v)$.

Note the following facts about this problem. First, for μ sufficiently small, the solution will involve a positive budget surplus (i.e., $B(r, x; g) > 0$). Second, for μ sufficiently large, the optimal tax rate will be such that all tax revenues are used to finance investment in the public good and hence $B(r, x; g) = 0$. Third, if it is the case that for some $\widetilde{\mu}$ it is optimal to select a tax rate-public good pair such that all revenues are used for investment (i.e., $B(r, x; g) = 0$), then this must also be optimal for all $\mu > \widetilde{\mu}$.

Now define $\mu(g; v)$ to be the size of the smallest group of legislators who would choose to devote all revenues to investment. Formally,

$$\mu(g; v) = \min\{\mu \in [0, \infty) : B(r(g; \mu, v), x(g; \mu, v); g) = 0\}.$$

Then all groups of legislators of size less than $\mu(g; v)$ would devote some revenues to pork and all larger groups would devote all revenues to investment. It should be noted that $\mu(g; v)$ exists and is unique for all g. We now have:

Lemma A.1: *Let* $\{r_\tau(g),\ s_\tau(g),\ x_\tau(g)\}_{\tau=1}^{T}$ *be an equilibrium with associated payoff function* $v_1(g)$. *(i) If* $\mu(g; v_1) \leq q$, *then*

$$(r_\tau(g), s_\tau(g), x_\tau(g)) = (r(g; n, v_1), 0, x(g; n, v_1)) \ \ \textit{for all } \tau = 1, ..., T.$$

(ii) If $\mu(g; v_1) > q$, *then*

$$(r_\tau(g), x_\tau(g)) = (r(g; q, v_1), x(g; q, v_1)) \ \ \textit{for all } \tau = 1, ..., T,$$

and

$$s_\tau(g) = \begin{cases} \frac{B(r(g;q,v_1),x(g;q,v_1);g)}{n} & \text{for all } \tau = 1, ..., T-1 \\ v_{T+1}(g) - u(w(1 - r(g;q,v_1)), g) - \delta v_1(x(g;q,v_1)) & \text{for } \tau = T \end{cases}.$$

Proof of Lemma A.1: We begin by considering the problem of the proposer in the final round T. From the discussion in the text, we know that $(r_T(g),\ s_T(g),\ x_T(g))$ must solve the round T proposer's problem

$$\max_{(r,s,x)} [u(w(1-r), g) + B(r, x; g) - (q-1)\, s + \delta v_1(x)]$$

subject to

$$u(w(1-r), g) + s + \delta v_1(x) \geq v_{T+1}(g),$$

$$B(r, x; g) \geq (q-1)s, \quad \text{and} \quad s \geq 0,$$

where

$$v_{T+1}(g) = \max_{(r,x)} \{u(w(1-r), g) + \frac{B(r, x; g)}{n} + \delta v_1(x) : B(r, x; g) \geq 0\}.$$

We now establish:

Claim A.1: *Let (r_T, s_T, x_T) solve the round T proposer's problem. Then (r_T, x_T) solves problem (1) with $\mu = q$ and $v = v_1$. In addition, $s_T = v_{T+1}(g) - \delta v_1(x_T) - u(w(1 - r_T), g)$.*

Proof of Claim A.1: It is easy to see that

$$s_T = v_{T+1}(g) - \delta v_1(x_T) - u(w(1 - r_T), g).$$

For if this were not the case it would follow from the definition of $v_{T+1}(g)$ that either the incentive constraint is violated or $s_T > 0$ and we could create a preferred proposal by just reducing s_T. It follows that we can write the proposer's payoff as

$$q\,[u(w(1 - r_T), g_T) + \delta v_1(x_T)] + B(r_T, x_T; g).$$

Now suppose that (r_T, x_T) does not solve problem (1) with $\mu = q$ and $v = v_1$. Let (r', x') solve problem (1) with $\mu = q$ and $v = v_1$, and let

$$s' = v_{T+1}(g) - \delta v_1(x') - u(w(1 - r'), g).$$

Then, the proposer's payoff under the proposal (r', s', x') is

$$q\,[u(w(1 - r'), g') + \delta v_1(x')] + B(r', x'; g').$$

By construction, the incentive constraint is satisfied and, by definition of $v_{T+1}(g)$, $s' \geq 0$. Note also that

$$\begin{aligned} B(r',x';g') - (q-1)s' &= (q-1)[u(w(1-r'),g') + \delta v_1(x')] + B(r',x';g') - (q-1)v_{T+1}(g) \\ &= q[u(w(1-r'),g') + \delta v_1(x')] + B(r',x';g') \\ &\quad -[u(w(1-r'),g') + \delta v_1(x') + (q-1)v_{T+1}(g)] \\ &\geq q[u(w(1-r'),g') + \delta v_1(x')] + B(r',x';g') - qv_{T+1}(g), \end{aligned}$$

where the last inequality follows by definition of $v_{T+1}(g)$. The difference on the right hand side must be non-negative. To see this, note that

$$v_{T+1}(g) = u(w(1-r^*),g) + \delta v_1(x^*) + B(r^*,x^*;g)/n$$

for some (r^*, x^*) such that $B(r^*, x^*; g) \geq 0$ and hence

$$\begin{aligned} q[u(w(1-r');g') + \delta v_1(x')] + B(r',x';g') &\geq q[u(w(1-r^*);g) + \delta v_1(x^*)] + B(r^*,x^*;g) \\ &\geq qv_{T+1}(g). \end{aligned}$$

Thus, (r', s', x') is feasible for the proposer's problem and yields a higher payoff than (r_T, s_T, x_T) - a contradiction. This completes the proof of Claim A.1. ■

It follows from Claim A.1 that if $\mu(g;v_1) \leq q$, then $(r_T(g),\ s_T(g),\ x_T(g))$ equals $(r(g;n,v_1), 0, x(g;n,v_1))$, while if $\mu(g;v_1) > q$, then $(r_T(g),\ s_T(g),\ x_T(g))$ equals

$$(r(g;q,v_1), v_{T+1}(g) - u(w(1-r(g;q,v_1)),g) - \delta v_1(x(g;q,v_1)), x(g;q,v_1)).$$

Now consider the round $T-1$ proposer's problem

$$\max_{(r,s,x)} [u(w(1-r),g) + B(r,x;g) - (q-1)\,s + \delta v_1(x)]$$

subject to

$$u(w(1-r),g) + s + \delta v_1(x) \geq v_T(g),$$

$$B(r,x;g) \geq (q-1)s, \quad \text{and} \quad s \geq 0.$$

If $\mu(g;v_1) \leq q$ then we know that

$$v_T(g) = u(w(1-r(g;n,v_1)),g) + \delta v_1(x(g;n,v_1)) = v_{T+1}(g)$$

so applying the exact same logic as above implies that the solution to the round $T-1$ proposer's problem is $(r(g;n,v_1),\ 0,\ x(g;n,v_1))$. Repeated application of the same logic implies that the solution to the proposer's problem is $(r(g;n,v_1),\ 0,\ x(g;n,v_1))$ in all earlier proposal rounds.

On the other hand, if $\mu(g;v_1) > q$ then we know that

$$v_T(g) = u(w(1-r(g;q,v_1)),g) + \delta v_1(x(g;q,v_1)) + \frac{B(r(g;q,v_1),x(g;q,v_1);g)}{n}.$$

So we need to show that the solution to the round $T-1$ proposer's problem with this level of reservation utility is

$$(r(g;q,v_1), \frac{B(r(g;q,v_1),x(g;q,v_1);g)}{n}, x(g;q,v_1))$$

Let $(r_{T-1}, s_{T-1}, x_{T-1})$ denote the solution. It is straightforward to show the desired result if $s_{T-1} > 0$, so we need only rule out the possibility that $s_{T-1} = 0$. If $s_{T-1} = 0$, it must be the case that $B(r_{T-1}, x_{T-1}; g) > 0$ and that (r_{T-1}, x_{T-1}) solves the problem

$$\max u(w(1-r),g) + \delta v_1(x) + B(r,x;g)$$

$$s.t. \;\; u(w(1-r),g) + \delta v_1(x) \geq v_T(g).$$

Now consider the proposal

$$(r', s', x') = (r(g;q,v_1), \frac{B(r(g;q,v_1),x(g;q,v_1);g)}{n}, x(g;q,v_1)).$$

The payoff to the proposer under the policy (r', s', x') is

$$q[u(w(1-r'),g) + \delta v_1(x')] + B(r',x';g) - (q-1)v_T(g). \tag{2}$$

But we know that $u(w(1-r_{T-1}),g) + \delta v_1(x_{T-1}) \geq v_T(g)$, and hence a lower bound of (2) is:

$$q[u(w(1-r'),g) + \delta v_1(x')] + B(r',x';g) - (q-1)[u(w(1-r_{T-1}),g) + \delta v_1(x_{T-1})].$$

The payoff to the proposer under the optimal policy $(r_{T-1}, 0, x_{T-1})$ is given by:

$$u(w(1-r_{T-1}),g) + \delta v_1(x_{T-1}) + B(r_{T-1}, x_{T-1}; g).$$

It must be the case that

$$\begin{aligned} & u(w(1-r_{T-1}),g) + \delta v_1(x_{T-1}) + B(r_{T-1}, x_{T-1}; g) \\ > \;\; & q[u(w(1-r'),g) + \delta v_1(x')] + B(r',x';g) - (q-1)[u(w(1-r_{T-1}),g) + \delta v_1(x_{T-1})], \end{aligned}$$

which implies that

$$\begin{aligned} & q[u(w(1-r_{T-1}),g)+\delta v_1(x_{T-1})]+B(r_{T-1},x_{T-1};g) \\ > \quad & q[u(w(1-r'),g)+\delta v_1(x')]+B(r',x';g). \end{aligned}$$

This contradicts the fact that $(r',x')=(r(g;q,v_1),x(g;q,v_1))$.

Repeated application of the same logic implies that the solution to the proposer's problem is

$$(r(g;q,v_1),\frac{B(r(g;q,v_1),x(g;q,v_1);g)}{n},x(g;q,v_1)).$$

in all earlier proposal rounds. This completes the proof of Lemma A.1. ■

Lemma A.1 tells us what equilibrium proposals must look like. The next step is to develop expressions for $(r(g;n,v_1),x(g;n,v_1))$ and $(r(g;q,v_1),x(g;q,v_1))$. If $\mu(g;v_1)\leq q$, then it must be the case that $B(r(g;n,v_1),x(g;n,v_1);g)=0$. It follows that we can write

$$(r(g;n,v_1),x(g;n,v_1))=(r(x^*(g),g),x^*(g)),$$

where $r(x,g)$ is the tax rate function from the analysis of the planner's problem, and

$$x^*(g)=\arg\max_x\{u(w(1-r(x,g)),g)+\delta v_1(x)\}.$$

If $\mu(g;v_1)>q$, then $B(r(g;q,v_1),x(g;q,v_1);g)>0$ and hence $(r(g;q,v_1),x(g;q,v_1))$ must solve the unconstrained problem

$$\max_{(r,x)} q[u(w(1-r),g)+\delta v_1(x)]+B(r,x;g). \tag{3}$$

Notice that the solutions to this problem are independent of g and thus we denote them by (r^*,x^*).

To complete the proof of the Proposition, it only remains to show that there exists a unique $g^*>0$ such that $\mu(g;v_1)\leq q$ for all $g\leq g^*$ and $\mu(g;v_1)>q$ for all $g>g^*$. We begin with the following useful observation.

Claim A.2: *For any strictly concave function $v(x)$, $g\geq 0$ and $\mu\in[0,\infty)$, let $(\widehat{r}(g;\mu,v),\widehat{x}(g;\mu,v))$ be the solution of the problem*

$$\max_{\{r,x\}} \mu[u(w(1-r),g)+\delta v(x)]+B(r,x;g) \tag{4}$$

and define $\widehat{\mu}(g;v)$ as

$$\widehat{\mu}(g;v)=\min\{\mu\in(0,\infty):B(\widehat{r}(g;\mu,v),\widehat{x}(g;\mu,v);g)=0\}.$$

Then, it is the case that $\widehat{\mu}(g;v)=\mu(g;v)$.

Proof of Claim A.2: Assume first that $\widehat{\mu}(g;v)<\mu(g;v)$, then for any $\mu\in(\widehat{\mu}(g;v),\mu(g;v))$ the unconstrained solution $(\widehat{r}(g;\mu,v),\widehat{x}(g;\mu,v))$ would violate the budget constraint:

$$B(\widehat{r}(g;\mu,v),\widehat{x}(g;\mu,v);g)<0$$

(note that $B(\widehat{r}(g;\mu,v),\widehat{x}(g;\mu,v);g)$ is strictly in decreasing in μ since $\widehat{r}(g;\mu,v)$ is strictly decreasing and $\widehat{x}(g;\mu,v)$ is non decreasing in μ). This implies that the constrained solution $(r(g;\mu,v),x(g;\mu,v))$ to problem (1) satisfies the budget constraint with equality: i.e.,

$$B(r(g;\mu,v),x(g;\mu,v);g)=0.$$

This implies that $\mu(g;v)\leq\mu$, a contradiction.

Assume now that $\widehat{\mu}(g;v)>\mu(g;v)$, then for any $\mu\in(\mu(g;v),\widehat{\mu}(g;v))$, since $\mu<\widehat{\mu}(g;v)$, it must be $B(\widehat{r}(g;\mu,v),\widehat{x}(g;\mu,v);g)>0$. In this case the solution of the unconstrained problem is the same as the constrained solution $r(g;\mu,v),x(g;\mu,v)$ (since the constraint is not binding): so $B(r(g;\mu,v),x(g;\mu,v);g)>0$. However, since $\mu>\mu(g;v)$ at the solution of the constrained problem, the constraint must be satisfied as equality, $B(r(g;\mu,v),x(g;\mu,v);g)=0$, a contradiction. This completes the proof of Claim A.2. ■

We can now show that $\mu(\cdot;v_1)$ is an increasing and continuous function. For monotonicity, let $g'>g$, $\mu'=\mu(g';v_1)$ and $\mu=\mu\left(g;v_1\right)$. We need to show that $\mu'>\mu$. Suppose, to the contrary, that $\mu'\leq\mu$. By Claim A.2, we know that $(r(g;\mu,v_1),x(g;\mu,v_1))$ solves the problem

$$\max_{(r,x)}\mu[u(w(1-r),g)+\delta v_1(x)]+B(r,x;g),$$

while $(r(g';\mu',v_1),x(g';\mu',v_1))$ solves the problem

$$\max_{(r,x)}\mu'[u(w(1-r),g')+\delta v_1(x)]+B(r,x;g').$$

It can easily verified that $r(g';\mu',v_1)\geq r(g;\mu,v_1)$ and $x(g';\mu',v_1)\leq x(g;\mu,v_1)$. Thus, since $g'>g$

$$B\left(r(g';\mu',v_1),x(g';\mu',v_1);g'\right)>B\left(r(g;\mu,v_1),x(g;\mu,v_1);g\right)=0$$

which contradicts the definition of μ'.

For continuity, let $g \geq 0$ and consider a sequence $g_n \to g$. Letting $\mu_n = \mu(g_n; v_1)$, we need to show that $\mu_n \to \mu(g; v_1)$. Note first that for any g there is an upperbound M such that $\mu(g_n; v_1) \in [0, M]$ (at least for n large enough): so we can assume, without loss of generality, that the limit $\mu^\infty = \lim_{n\to\infty} \mu_n$ exists. Since $B(r, x; g)$ is continuous in all its arguments and since $r(g; \mu, v_1)$ and $x(g; \mu, v_1)$ are continuous in g and μ by the *Theorem of the Maximum*, we have that

$$\lim_{n\to\infty} B\left(r(g_n; \mu_n, v_1), x(g_n; \mu_n, v_1); g_n\right) = B\left(r(g; \mu^\infty, v_1), x(g; \mu^\infty, v_1); g\right).$$

Moreover, since $B\left(r(g_n; \mu_n, v_1), v_1), x(g_n; \mu_n, v_1), v_1); g_n\right) = 0$ for all g_n we have that

$$B\left(r(g; \mu^\infty, v_1), x(g; \mu^\infty, v_1); g\right) = 0.$$

Clearly, it can not be that $\mu^\infty < \mu(g; v_1)$, because this would violate the definition of $\mu(g; v_1)$. Assume then that $\mu^\infty > \mu(g; v_1)$. In this case, we must have that $x(g; \mu^\infty, v_1) \geq x(g; \mu(g; v_1), v_1)$ and $r(g; \mu^\infty, v_1) < r(g; \mu(g; v_1), v_1)$, but this would imply

$$B\left(r(g; \mu^\infty, v_1), x(g; \mu^\infty, v_1); g\right) < B\left(r(g; \mu(g; v_1), v_1), x(g; \mu(g; v_1), v_1); g\right) = 0,$$

which is a contradiction.

The final step is to show that $\mu(0; v_1) < q$ while for g large enough $\mu(g; v_1) > q$. The latter is obvious, and the former is implied by Assumption 1. To see this, suppose to the contrary, that $\mu(0; v_1) \geq q$. Then it would follow that $\mu(g; v_1) > q$ for all $g > 0$. This would imply that for all $g > 0$

$$(r_1(g), s_1(g), x_1(g)) = (r^*, \frac{B(r^*, x^*; g)}{n}, x^*),$$

and hence that

$$v_1(g) = u(w(1 - r^*), g) + \frac{B(r^*, x^*; g)}{n} + \delta v_1(x^*).$$

This in turn implies that

$$v_1'(g) = A\alpha g^{\alpha-1} + \frac{p(1-d)}{n}$$

and hence that

$$x^* = \left(\frac{\delta q A\alpha}{p(1 - \frac{q}{n}\delta(1-d))}\right)^{\frac{1}{1-\alpha}}.$$

But then, since $\mu(0; v_1) \geq q$ it must be the case that

$$B(\widehat{r}(0; q, v_1), \widehat{x}(0; q, v_1); 0) = B(r^*, x^*; 0) \geq 0$$

or, equivalently,

$$\frac{R(r^*)}{p} \geq (\frac{\delta q A\alpha}{p(1-\frac{q}{n}\delta(1-d))})^{\frac{1}{1-\alpha}}$$

which violates Assumption 1 (since $r^* = (1-q/n)/(1+\varepsilon-q/n)$).

It now follows that there exists a unique $g^* > 0$ such that $\mu(g^*; v_1) = q$. Because $\mu(\cdot; v_1)$ is increasing, this g^* will have the property that for all $g \leq g^*$, $\mu(g; v_1) \leq q$ and for all $g > g^*$, $\mu(g; v_1) > q$. Thus, the proof of the Proposition is complete. *QED*

2 Proof of Proposition 5

The proof will proceed in two parts. First, we develop necessary and sufficient conditions for the existence of an equilibrium of each of the three types. Then we analyze when these conditions will be satisfied, relating them to $\underline{A}$ and $\overline{A}$.

Existence of a Type I equilibrium

From the analysis preceding Proposition 2 in this case we have that

$$x^* = (\frac{\delta q A\alpha}{p(1-\frac{q}{n}\delta(1-d))})^{\frac{1}{1-\alpha}}$$

and that

$$g^* = \frac{px^* - R(r^*)}{p(1-d)}.$$

It then follows that $x^* > g^*$ if and only if

$$\frac{R(r^*)}{pd} > [\frac{\delta q A\alpha}{p(1-\frac{q}{n}\delta(1-d))}]^{\frac{1}{1-\alpha}}. \tag{5}$$

This inequality is therefore a necessary condition for the existence of a Type I equilibrium.

In such an equilibrium, if $g > g^*$ then the legislature will choose the public good level x^* and tax rate r^* that period and every period thereafter. It must therefore be the case that the value function $v_1(g)$ satisfies:

$$v_1(g) = \begin{cases} \max_x \{u(w(1-r(x,g)),g) + \delta v_1(x)\} & g \leq g^* \\ u(w(1-r^*),g) + \frac{B(r^*,x^*;g)}{n} + \frac{\delta}{1-\delta}(u(w(1-r^*),x^*) + \frac{B(r^*,x^*;x^*)}{n}) & g > g^* \end{cases}. \tag{6}$$

The question is whether there exists a strictly concave value function v_1 that satisfies this relationship when inequality (5) is satisfied. If so, there will exist an associated equilibrium $\{(r_\tau(g), s_\tau(g), x_\tau(g))\}_{\tau=1}^{T}$

in which if $g \in [0, g^*]$

$$(r_\tau(g), s_\tau(g), x_\tau(g)) = (r(x^*(g), g), 0, x^*(g)) \text{ for all } \tau = 1, ..., T,$$

and if $g \in (g^*, \infty)$

$$(r_\tau(g), x_\tau(g)) = (r^*, x^*) \text{ for all } \tau = 1, ..., T,$$

and

$$s_\tau(g) = \begin{cases} \frac{B(r^*, x^*; g)}{n} \text{ for all } \tau = 1, ..., T-1 \\ v_{T+1}(g) - u(w(1-r^*), g) - \delta v_1(x^*) \text{ for } \tau = T \end{cases}.$$

The proof of the following Lemma establishes not only that there does exist such a function but also that it is unique.

Lemma A.2: *There exists a Type I equilibrium if and only if inequality (5) is satisfied. Moreover, there is a unique such equilibrium.*

Proof of Lemma A.2: Let $\overline{g} > x^*$ be an arbitrarily large but bounded scalar. We will first restrict the domain of public good levels to $[0, \overline{g}]$ and prove the existence of a unique $v_1(g)$ that satisfies (6) when this assumption is satisfied. Then we will extend the solution for $g > \overline{g}$. Define for $g \in [g^*, \overline{g}]$ the function

$$\overline{v}(g) = u(w(1-r^*), g) + \frac{B(r^*, x^*; g)}{n} + \frac{\delta}{1-\delta}(u(w(1-r^*), x^*) + \frac{B(r^*, x^*; x^*)}{n})$$

This function $\overline{v}(g)$ is continuous, bounded and strictly concave on $[g^*, \overline{g}]$. Then let F denote the set of continuous, bounded, weakly concave functions $v : \Re_+ \to \Re$ such that $v(g) = \overline{v}(g)$ for all $g \in [g^*, \overline{g}]$. This set is non empty, closed, bounded, and convex. Finally, define the functional Ψ on F as follows:

$$\Psi(v)(g) = \begin{cases} \max_x \{u(w(1 - r(x, g)), g) + \delta v(x)\} & g \in [0, g^*] \\ u(w(1-r^*), g) + \frac{B(r^*, x^*; g)}{n} + \delta \overline{v}(x^*) & g \in (g^*, \overline{g}] \end{cases}. \tag{7}$$

For a given expected continuation value function v at time $t+1$, $\Psi(v)$ provides the expected value function of a legislator at time t in an equilibrium with $x^* > g^*$.

We will now prove that there exists a unique $v_1 \in F$ such that $v_1 = \Psi(v_1)$. The first step is to show that Ψ maps F into itself; i.e., that $\Psi(v) \in F$. It is immediate that $\Psi(v)(g) = \overline{v}(g)$ for all $g \in [g^*, \overline{g}]$, and that $\Psi(v)$ is bounded on $[0, \overline{g}]$. However, we need to prove that $\Psi(v)$ is continuous and (strictly) concave.

Continuity. The function $\Psi(v)$ is continuous on $[0, g^*)$ by the *Theorem of the Maximum*, and on $(g^*, \overline{g}]$ by definition. We just need to show that it is continuous at $g = g^*$. Since $B(r^*, x^*; g^*) = 0$, we have that

$$\begin{aligned}\lim_{g \searrow g^*} \Psi(v)(g) &= u(w(1-r^*), g^*) + \frac{B(r^*, x^*; g^*)}{n} + \delta\overline{v}(x^*) \\ &= u(w(1-r^*), g^*) + \delta\overline{v}(x^*).\end{aligned}$$

Next note that $(r^*, x^*) = (r(x^*(g^*; v), g^*), x^*(g^*; v))$ where

$$x^*(g; v) = \arg\max_x \{u(w(1 - r(x, g)), g) + \delta v(x)\}.$$

To see this, suppose the converse. Then, since $B(r^*, x^*; g^*) = 0$, it must be that:

$$u(w(1 - r(x^*(g^*; v), g^*)), g^*) + \delta v(x^*(g^*; v)) > u(w(1-r^*), g^*) + \delta v(x^*),$$

which implies that

$$\begin{aligned}& q\left[u(w(1 - r(x^*(g^*; v), g^*)), g^*) + \delta v(x^*(g^*; v))\right] + B(r(x^*(g^*; v), g^*), x^*(g^*; v); g^*) \\ > \; & q\left[u(w(1-r^*), g^*) + \delta v(x^*)\right] + B(r^*, x^*; g^*).\end{aligned}$$

But this is a contradiction since (r^*, x^*) solves the problem

$$\max_{(r,x)} q[u(w(1-r), g) + \delta v(x)] + B(r, x; g).$$

To see the latter, note that given that $x^* > g^*$, we know that $v(x^*) = \overline{v}(x^*)$ and, by construction, $\delta q \overline{v}'(x^*) = p$. It follows, therefore, that

$$\begin{aligned}\lim_{g \nearrow g^*} \Psi(v)(g) &= u(w(1 - r(x^*(g^*; v), g^*)), g^*) + \delta v(x^*(g^*; v)) \\ &= u(w(1-r^*), g^*) + \delta\overline{v}(x^*) = \lim_{g \searrow g^*} \Psi(v)(g).\end{aligned}$$

Strict Concavity. We proceed in three steps.

Step 1. $\Psi(v)$ is strictly concave on $[0, g^*]$. In this case the budget constraint is binding and the value function is:

$$\Psi(v)(g) = \max_{(r,x)} \left\{ \begin{array}{c} u(w(1-r), g) + \frac{B(r,x;g)}{n} + \delta v(x) \\ B(r, x; g) \geq 0 \end{array} \right\}. \tag{8}$$

Take two points g_1 and g_2 with $0 \leq g_1 < g_2 \leq g^*$, and a scalar $\phi \in [0,1]$. Define r_i and x_i to be the optimal policies with public good level g_i $i = 1, 2$. Let $g_\phi = \phi g_1 + (1-\phi) g_2$, $r_\phi = \phi r_1 + (1-\phi) r_2$, and $x_\phi = \phi x_1 + (1-\phi) x_2$. Since $v(x)$ is concave, the function $u(w(1-r), g) + B(r, x; g)/n + \delta v(x)$ is concave in (r, x, g). Thus, we have that

$$\begin{aligned} \phi\Psi(v)(g_1) + (1-\phi)\Psi(v)(g_2) &= \phi[u(w(1-r_1), g_1) + \frac{B(r_1, x_1; g_1)}{n} + \delta v(x_1)] \\ &\quad +(1-\phi)[u(w(1-r_2), g_2) + \frac{B(r_2, x_2; g_2)}{n} + \delta v(x_2)] \\ &< u(w(1-r_\phi), g_\phi) + \frac{B(r_\phi, x_\phi; g_\phi)}{n} + \delta v(x_\phi). \end{aligned}$$

Since $B(r, x; g)$ is concave in (r, x, g), we have that $B(r_\phi, x_\phi; g_\phi) \geq 0$, so that

$$\begin{aligned} u(w(1-r_\phi), g_\phi) + \frac{B(r_\phi, x_\phi; g_\phi)}{n} + \delta v(x_\phi) &\leq \max_{(r,x)} \left\{ \begin{array}{c} u(w(1-r), g_\phi) + \frac{B(r,x;g_\phi)}{n} + \delta v(x) \\ B(r, x; g_\phi) \geq 0 \end{array} \right\} \\ &= \Psi(v)(g_\phi). \end{aligned}$$

Therefore $\phi\Psi(v)(g_1) + (1-\phi)\Psi(v)(g_2) < \Psi(v)(g_\phi)$ as required.

Step 2. $\Psi(v)$ is strictly concave on $(g^*, \overline{g}]$. This is immediate from the definition of $\Psi(v)(g)$.

Step 3. $\Psi(v)$ is strictly concave on $[0, \overline{g}]$. Let g_1 and g_2 be such that $0 \leq g_1 \leq g^* < g_2 \leq \overline{g}$. We have two possible cases. First it may be that $g_\phi \leq g^*$. For all $g \in [0, \overline{g}]$, let $(r'(g; v), x'(g; v))$ be the solution to the problem

$$\begin{array}{c} \max_{(r,x)} u(w(1-r), g) + \frac{B(r,x;g)}{n} + \delta v(x) \\ B(r, x; g) \geq 0 \end{array},$$

and let

$$\Xi(v)(g) = u(w(1-r'(g; v)), g) + \frac{B(r'(g; v), x'(g; v); g)}{n} + \delta v(x'(g; v)).$$

We have that $\Xi(v)(g) \geq \Psi(v)(g)$ for all $g \in [0, \overline{g}]$. Indeed, the two functionals are equivalent for $g \in [0, g^*]$ but, if $g > g^*$, $\Psi(v)(g)$ is less than $\Xi(v)(g)$. Therefore we have:

$$\begin{aligned} \phi\Psi(v)(g_1) + (1-\phi)\Psi(v)(g_2) &\leq \phi\Xi(v)(g_1) + (1-\phi)\Xi(v)(g_2) \\ &< u(w(1-r'_\phi), g_\phi) + \frac{B(r'_\phi, x'_\phi; g_\phi)}{n} + \delta v(x'_\phi) \end{aligned}$$

where $r'_\phi = \phi r'(g_1; v) + (1-\phi) r'(g_2; v)$ and $x'_\phi = \phi x'(g_1; v) + (1-\phi) x'(g_2; v)$. Moreover, $B(r'_\phi, x'_\phi; g_\phi) \geq 0$, implying:

$$\begin{aligned}
\Psi(v)(g_\phi) &= \max_{(r,x)} \left\{ \begin{array}{c} u(w(1-r), g_\phi) + B(r, x; g_\phi)/n + \delta v(x) \\ B(r, x; g_\phi) \geq 0 \end{array} \right\} \\
&\geq u(w(1-r'_\phi), g_\phi) + \frac{B(r'_\phi, x'_\phi; g_\phi)}{n} + \delta v\left(x'_\phi\right) \\
&> \phi \Psi(v)(g_1) + (1-\phi)\Psi(v)(g_2).
\end{aligned}$$

The second case arises when $g_\phi > g^*$. Let $\psi \in [0,1]$ be such that $g^* = \psi g_1 + (1-\psi) g_2$; by the previous step, we have that $\Psi(v)(g^*) > \psi \Psi(v)(g_1) + (1-\psi)\Psi(v)(g_2)$ (since obviously $g^* \in [0, g^*]$). Take now a scalar $\eta \in [0,1]$ such that $\eta g^* + (1-\eta) g_2 = g_\phi$. Since $\Psi(v)$ is strictly concave and continuous in $g \geq g^*$, it must be that $\Psi(v)(g_\phi) > \eta \Psi(v)(g^*) + (1-\eta)\Psi(v)(g_2)$. Therefore we have:

$$\begin{aligned}
\Psi(v)(g_\phi) &> \eta \Psi(v)(g^*) + (1-\eta)\Psi(v)(g_2) > \eta\psi \Psi(v)(g_1) + (1-\eta\psi)\Psi(v)(g_2) \\
&= \phi \Psi(v)(g_1) + (1-\phi)\Psi(v)(g_2)
\end{aligned}$$

where the second inequality follows from $\Psi(v)(g^*) > \psi \Psi(v)(g_1) + (1-\psi)\Psi(v)(g_2)$, and the last equality follows from the definitions of η and ψ: $\phi g_1 + (1-\phi) g_2 = g_\phi = \eta g^* + (1-\eta) g_2 = \eta\psi g_1 + (1-\eta\psi) g_2$, implying $\phi = \eta\psi$.

Given that $\Psi(v) \in F$, to prove existence and uniqueness of a fixpoint of Ψ in F, it is sufficient to prove that $\Psi(\cdot)$ is a contraction in F. Let $\omega_1, \omega_2 \in F$ be such that $\omega_1(g) \leq \omega_2(g)$ for all $g \in [0, \overline{g}]$. Define $x_{\omega_i}(g)$ as a solution of $\max_x \{u(w(1-r(x,g)), g) + \delta\omega_i(x)\}$ $\forall i = 1, 2$. For $g \in [0, g^*]$, we have:

$$\begin{aligned}
\Psi(\omega_2)(g) &= \max_x \{u(w(1-r(x,g)), g) + \delta\omega_2(x)\} \geq u(w(1-r(x_{\omega_1}(g), g)), g) + \delta\omega_2(x_{\omega_1}(g)) \\
&\geq u(w(1-r(x_{\omega_1}(g), g)), g) + \delta\omega_1(x_{\omega_1}(g)) \\
&= \Psi(\omega_1)(g)
\end{aligned}$$

and, by definition, $\Psi(\omega_2)(g) = \Psi(\omega_1)(g)$ for $g \in (g^*, \overline{g}]$. So $\Psi(\cdot)$ satisfies Blackwell's monotonicity condition (cf. Blackwell (1965)). Let a be a weakly positive scalar, then for any $g \in [0, g^*]$ and $v \in F$ we have:

$$\Psi(v+a)(g) = \max_x \{u(w(1-r(x,g)), g) + \delta v(x)\} + \delta a = \Psi(v)(g) + \delta a$$

and $\Psi(v+a)(g) = \Psi(v)(g)$ if $g \in (g^*, \overline{g}]$. Since $\delta \in (0,1)$, we conclude that Blackwell's discounting condition is satisfied as well (cf. Blackwell (1965)). It follows that Blackwell's sufficient conditions are satisfied and, by Theorem 5 in Blackwell (1965), $\Psi(\cdot)$ is a contraction with modulus δ. From all these properties, it follows that there exists a unique continuous, bounded, strictly concave value function v_1 that satisfies (6) on the domain $[0, \overline{g}]$.

To see that the equilibrium value function can be extended for $g > \overline{g}$, note that we can define $v(g) = \overline{v}(g)$ for $g > \overline{g}$. The resulting value function is continuous, concave and continues to be a fixpoint of (7). This completes the proof of Lemma A.2. ■

Existence of a Type II equilibrium

In this case, Proposition 3 tells us that the equilibrium converges monotonically to the planner's steady state. Thus, it must be the case that for all $g \leq g^*$, $v_1(g) = V(g)/n$ where $V(g)$ is the planner's value function. This means that x^* satisfies

$$\delta \frac{V'(x^*)}{n} = \frac{p}{q}.$$

In addition,

$$g^* = \frac{px^* - R(r^*)}{p(1-d)}.$$

It turns out that $x^* < g^*$ if and only if the marginal cost of public funds at the planner's steady state exceeds the ratio n/q; that is,

$$\left(\frac{1 - r(x^o, x^o)}{1 - r(x^o, x^o)(1+\varepsilon)}\right) > \frac{n}{q}. \tag{9}$$

To see this, note from the Euler equation for the planner's problem, that at the planner's steady state

$$\delta V'(x^o) = p\left(\frac{1 - r(x^o, x^o)}{1 - r(x^o, x^o)(1+\varepsilon)}\right).$$

Thus, if the condition is satisfied then it must be the case that $V'(x^o) > V'(x^*)$ which by the concavity of the planner's value function implies that $x^* > x^o$. But since the condition implies that $r^o = r(x^o, x^o) > r^*$, this means that

$$x^o = \frac{px^o - R(r^o)}{p(1-d)} < \frac{px^* - R(r^*)}{p(1-d)} = g^*$$

From Figure 3(b) it is clear that this implies that $x^o(g^*) = x^* < g^*$.

For $g > g^*$, we know that the legislature selects the public good level x^* which puts proposals back into the no-pork region in one period. Accordingly, it must be the case that

$$v_1(g) = \begin{cases} \frac{V(g)}{n} & g \le g^* \\ u(w(1-r^*), g) + \frac{1}{n} B(r^*, x^*; g) + \delta \frac{V(x^*)}{n} & g > g^* \end{cases}$$

It is now straightforward to show that this is indeed an equilibrium value function and is strictly concave. The associated equilibrium policy proposals $\{(r_\tau(g), s_\tau(g), x_\tau(g))\}_{\tau=1}^{T}$ are such that if $g \in [0, g^*]$

$$(r_\tau(g), s_\tau(g), x_\tau(g)) = (r(x^*(g), g), 0, x^*(g)) \text{ for all } \tau = 1, ..., T,$$

and if $g \in (g^*, \infty)$

$$(r_\tau(g), x_\tau(g)) = (r^*, x^*) \text{ for all } \tau = 1, ..., T,$$

and

$$s_\tau(g) = \begin{cases} \frac{B(r^*, x^*; g)}{n} \text{ for all } \tau = 1, ..., T-1 \\ v_{T+1}(g) - u(w(1-r^*), g) - \delta v_1(x^*) \text{ for } \tau = T \end{cases}.$$

Thus we have:

Lemma A.3: *There exists a Type II equilibrium if and only if inequality (9) is satisfied. Moreover, there is a unique such equilibrium.*

Existence of a Type III equilibrium

In this case, we know that

$$x^* = g^* = \frac{R(r^*)}{pd}.$$

Further, we know that it must be the case that

$$\delta[A\alpha x^{*\alpha-1} + \frac{p(1-d)}{n}] \le \frac{p}{q} \le \delta[A\alpha x^{*\alpha-1} + (\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})]. \tag{10}$$

In such an equilibrium, if $g > x^*$ then the legislature will choose the public good level x^* and tax rate r^* that period and every period thereafter. It must therefore be the case that the value function $v_1(g)$ satisfies

$$v_1(g) = \begin{cases} \max_x \{u(w(1-r(x,g)), g) + \delta v_1(x)\} & g \le x^* \\ u(w(1-r^*), g) + \frac{B(r^*, x^*; g)}{n} + \frac{\delta}{1-\delta} u(w(1-r^*), x^*) & g > x^* \end{cases}. \tag{11}$$

The question is whether there exists a strictly concave value function which satisfies this relationship when inequality (10) is satisfied. If so, there will exist an associated equilibrium $\{(r_\tau(g), s_\tau(g), x_\tau(g))\}_{\tau=1}^{T}$ in which if $g \in [0, x^*]$

$$(r_\tau(g), s_\tau(g), x_\tau(g)) = (r(x^*(g), g), 0, x^*(g)) \text{ for all } \tau = 1, ..., T,$$

and if $g \in (x^*, \infty)$

$$(r_\tau(g), x_\tau(g)) = (r^*, x^*) \text{ for all } \tau = 1, ..., T,$$

and

$$s_\tau(g) = \begin{cases} \frac{B(r^*, x^*; g)}{n} \text{ for all } \tau = 1, ..., T-1 \\ v_{T+1}(g) - u(w(1-r^*), g) - \delta v_1(x^*) \text{ for } \tau = T \end{cases}$$

The following Lemma shows that the answer is yes.

Lemma A.4: *There exists a Type III equilibrium if and only if inequality (10) is satisfied. Moreover, there is a unique such equilibrium.*

Proof of Lemma A.4: The proof is similar to the proof of Lemma A.2. Let $\overline{g} > x^*$ be an arbitrarily large but bounded scalar. We will first restrict the range of public good levels to $[0, \overline{g}]$ and prove the existence of a unique $v_1(g)$ that satisfies (11) when this assumption is satisfied, then we will extend the solution for $g > \overline{g}$. Define for $g \in [x^*, \overline{g}]$ the function

$$\widetilde{v}(g) = u(w(1-r^*), g) + \frac{B(r^*, x^*; g)}{n} + \frac{\delta}{1-\delta} u(w(1-r^*), x^*).$$

Then let $\widetilde{F}$ denote the set of continuous, bounded, weakly concave functions $v : \Re_+ \to \Re$ such that $v(g) = \widetilde{v}(g)$ for all $g \in [x^*, \overline{g}]$. Finally, define the functional $\widetilde{\Psi}$ on $\widetilde{F}$ as follows:

$$\widetilde{\Psi}(v)(g) = \begin{cases} \max_x \{u(w(1 - r(x, g)), g) + \delta v(x)\} & g \in [0, x^*] \\ u(w(1-r^*), g) + \frac{1}{n} B(r^*, x^*; g) + \delta \widetilde{v}(x^*) & g \in (x^*, \overline{g}] \end{cases}. \quad (12)$$

It can be shown that $\widetilde{\Psi}(v) \in \widetilde{F}$ and, further, that $\widetilde{\Psi}(v)$ is strictly concave. It can also be shown that $\widetilde{\Psi}(v)$ is a contraction mapping which implies that there exists a unique function v_1 such that $v_1 = \widetilde{\Psi}(v_1)$. This function is strictly concave and satisfies (11) on the range $[0, \overline{g}]$. As in the case with $x^* > g^*$, the value function can be extended in $(\overline{g}, \infty)$ by defining $v(g) = \widetilde{v}(g)$ for $g > \overline{g}$. The resulting value function is continuous, concave and continues to be a fixpoint of (12). This completes the proof of Lemma A.4. ■

When are the conditions satisfied?

Define $\overline{A}$ to be the value of A that would be such as to make the discounted marginal benefit of the public good in the minimum winning coalition range equal to p/q at the public good level $R(r^*)/pd$; that is,

$$\delta[\overline{A}\alpha(\frac{R(r^*)}{pd})^{\alpha-1} + \frac{p(1-d)}{n}] = \frac{p}{q}. \tag{13}$$

Similarly, let $\underline{A}$ be the value of A that would be such as to make the discounted marginal benefit of the public good in the unanimity range equal to p/q at the public good level $R(r^*)/pd$; that is,

$$\delta[\underline{A}\alpha(\frac{R(r^*)}{pd})^{\alpha-1} + (\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})] = \frac{p}{q}. \tag{14}$$

Notice that $\underline{A}$ must be less than $\overline{A}$ since, holding constant public good preferences, the value of an additional unit is higher in the unanimity range.

Now we have the following convenient result.

Lemma A.5: *(i) Condition (5) is satisfied if and only if $A \in (0, \overline{A})$. (ii) Condition (9) is satisfied if and only if $A > \underline{A}$. (iii) Condition (10) is satisfied if and only if $A \in [\underline{A}, \overline{A}]$.*

Proof of Lemma A.5: (i) Let

$$x^*(A) = (\frac{\delta q A\alpha}{p(1-\frac{q}{n}\delta(1-d))})^{\frac{1}{1-\alpha}}$$

Then, we know that condition (5) is satisfied if and only if $x^*(A) < \frac{R(r^*)}{pd}$ or equivalently if only if $A \in (0, \widehat{A})$ where

$$x^*(\widehat{A}) = \frac{R(r^*)}{pd}.$$

But note that

$$\begin{aligned} x^*(\widehat{A}) &= \frac{R(r^*)}{pd} \Leftrightarrow (\frac{\delta q\widehat{A}\alpha}{p(1-\frac{q}{n}\delta(1-d))})^{\frac{1}{1-\alpha}} = \frac{R(r^*)}{pd} \\ &\Leftrightarrow \delta[\widehat{A}\alpha(\frac{R(r^*)}{pd})^{\alpha-1} + \frac{p(1-d)}{n}] = \frac{p}{q} \end{aligned}$$

which implies that $\widehat{A} = \overline{A}$.

(ii) Condition (9) is that

$$(\frac{1-r(x^o,x^o)}{1-r(x^o,x^o)(1+\varepsilon)}) > \frac{n}{q}.$$

Let $x^o(A)$ denote the planner's steady state public good level with public good preference parameter A and $r^o(A) = r(x^o(A), x^o(A))$ the associated tax rate. Clearly, $x^o(A)$ and $r^o(A)$ are increasing in A. Letting $\widetilde{A}$ be such that

$$(\frac{1-r^o(A)}{1-r^o(A)(1+\varepsilon)})=\frac{n}{q},$$

it is clear that condition (9) is satisfied if and only if $A>\widetilde{A}$. But note that $r^o(\widetilde{A})=r^*$ and that

$$\delta n\widetilde{A}\alpha x^o(\widetilde{A})^{\alpha-1}=p[1-\delta(1-d)]\left[\frac{1-r^*}{1-r^*(1+\varepsilon)}\right]$$

or, equivalently,

$$\delta[\underline{A}\alpha x^o(\widetilde{A})^{\alpha-1}+(\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})]=\frac{p}{q}.$$

Furthermore, $B(x^o(\widetilde{A}),r^*;x^o(\widetilde{A}))=0$ which implies that

$$x^o(\widetilde{A})=\frac{R(r^*)}{pd}$$

and hence that $\widetilde{A}=\underline{A}$ as required.

(iii) This is immediate.

This completes the proof of Lemma A.5. ■

Proposition 5 now follows by combining Lemmas A.2 - A.5. *QED*

3 Proof of Lemma 1

We know from Proposition 2 that the equilibrium steady state is

$$(r^*,(\frac{\delta qA\alpha}{p(1-\frac{q}{n}\delta(1-d))})^{\frac{1}{1-\alpha}}).$$

The planner's steady state satisfies:

$$\delta[nA\alpha(x^o)^{\alpha-1}+(\frac{1-r^o}{1-r^o(1+\varepsilon)})p(1-d)]=\frac{1-r^o}{1-r^o(1+\varepsilon)}\cdot p$$

and

$$R(r^o)=pdx^o,$$

which means that

$$\delta[A\alpha(\frac{R(r^o)}{pd})^{\alpha-1}+(\frac{1-r^o}{1-r^o(1+\varepsilon)})\frac{p(1-d)}{n}]=\frac{1-r^o}{1-r^o(1+\varepsilon)}\cdot\frac{p}{n}.$$

The equilibrium steady state satisfies

$$\delta[A\alpha x^{\alpha-1}+\frac{p(1-d)}{n}]=\frac{p}{q}.$$

We know from (14) that

$$\delta[\underline{A}\alpha(\frac{R(r^*)}{pd})^{\alpha-1}+(\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})]=\frac{p}{q}=\frac{1-r^*}{1-r^*(1+\varepsilon)}\cdot\frac{p}{n}.$$

Thus, if $A<\underline{A}$ we have that

$$\delta[A\alpha(\frac{R(r^*)}{pd})^{\alpha-1}+(\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})]<\frac{1-r^*}{1-r^*(1+\varepsilon)}\cdot\frac{p}{n}.$$

This implies that $r^*>r^o$ - the equilibrium tax rate is higher than the planner's tax rate. On the other hand, if $A\in(\underline{A},\overline{A}]$ we have that

$$\delta[A\alpha(\frac{R(r^*)}{pd})^{\alpha-1}+(\frac{1-r^*}{1-r^*(1+\varepsilon)})(\frac{p(1-d)}{n})]>\frac{1-r^*}{1-r^*(1+\varepsilon)}\cdot\frac{p}{n},$$

which implies that $r^*<r^o$.

What about the claim concerning the level of public goods? The result is obvious if $r^*<r^o$, so suppose that $r^*>r^o$. The level of public goods at the equilibrium steady state satisfies

$$\delta[A\alpha x^{\alpha-1}+\frac{p(1-d)}{n}]=\frac{1-r^*}{1-r^*(1+\varepsilon)}\cdot\frac{p}{n}>\frac{1-r^o}{1-r^o(1+\varepsilon)}\cdot\frac{p}{n}.$$

Thus, it is clear that the level of public goods is below that at the planner's steady state, because the marginal cost is higher and the marginal benefit is lower. *QED*

4 Proof of Lemma 2

It suffices to show that $r^*<r^o$. But we know from the proof of Lemma 1 that this follows if $A\in(\underline{A},\overline{A}]$. *QED*

5 Proof of Proposition 7

Solving equations (13) and (14) for $\underline{A}$ and $\overline{A}$ yields

$$\underline{A}=\frac{p(1-\delta(1-d))(\frac{R(r^*)}{pd})^{1-\alpha}}{q\delta\alpha}$$

and

$$\overline{A}=\frac{p(1-\frac{q}{n}\delta(1-d))(\frac{R(r^*)}{pd})^{1-\alpha}}{q\delta\alpha}.$$

Moreover, we have that

$$R(r^*) = nr^*(1-r^*)^\varepsilon w^{\varepsilon+1}\varepsilon^\varepsilon = \frac{(n-q)\varepsilon^{2\varepsilon}w^{\varepsilon+1}}{(1+\varepsilon-q/n)^{\varepsilon+1}}.$$

From these expressions it is clear that $\underline{A}$ and $\overline{A}$ are decreasing in δ and q and increasing in p and w. For the claims about the impact of increasing the elasticity of labor supply ε, define the function $w(\varepsilon)$ from the equality

$$nw^{\varepsilon+1}\varepsilon^\varepsilon = K,$$

for some constant K. Then let $\widetilde{R}(\varepsilon)$ be the function that equals $R(r^*)$ when the elasticity is ε and the wage is $w(\varepsilon)$; that is,

$$\widetilde{R}(\varepsilon) = \frac{(n-q)\varepsilon^{2\varepsilon}w(\varepsilon)^{\varepsilon+1}}{(1+\varepsilon-q/n)^{\varepsilon+1}} = \frac{(1-q/n)\varepsilon^\varepsilon}{(1+\varepsilon-q/n)^{\varepsilon+1}}.$$

We need to show that $\widetilde{R}(\varepsilon)$ is decreasing in ε. Taking logs, we have that

$$\begin{aligned}
\ln \widetilde{R}(\varepsilon) &= \ln(1-q/n)\varepsilon^\varepsilon - \ln(1+\varepsilon-q/n)^{\varepsilon+1} \\
&= \ln(1-q/n) + \varepsilon\ln\varepsilon - (\varepsilon+1)\ln(1+\varepsilon-q/n)
\end{aligned}$$

Thus,

$$\frac{d\ln\widetilde{R}(\varepsilon)}{d\varepsilon} = \ln\varepsilon - \ln(1+\varepsilon-q/n) + 1 - \left(\frac{\varepsilon+1}{\varepsilon+1-q/n}\right) < 0$$

which implies the result. *QED*