

Appendix

Proof of Lemma 4. First, we show that the sequence $\{\theta_k\}$ is bounded below. By contradiction: for every N finite there exists $k > -\infty$ such that $N \in [\theta_{k-1}, \theta_k)$ for some $\theta_{k-1} > -\infty$. By definition of PBEC and $\bar{\theta}_{k-1} = \mathbb{E}[\theta | \theta_{k-1} \leq \theta < \theta_k]$ this means that

$$\begin{aligned} \mathbb{E}[\theta | \theta_{k-1} \leq \theta < \theta_k] &= 2\theta_k - q - \mathbb{E}[\theta | \theta_k \leq \theta < \theta_{k+1}] \\ &> \mathbb{E}[\theta | \theta < \theta_k] \end{aligned}$$

because θ has full support on the real line. Rearranging

$$\begin{aligned} \theta_k - \mathbb{E}[\theta | \theta < \theta_k] &= f(\theta_k | H) \\ &> q + \mathbb{E}[\theta | \theta_k \leq \theta < \theta_{k+1}] - \theta_k \\ &> q \\ &> 0. \end{aligned}$$

Since this must be true for every N and every $\theta_{k-1} \leq N$ unbounded below, this contradicts Lemma 3.

Second, we show that the sequence $\{\theta_k\}$ is bounded above. Using $\bar{\theta}_k = 2\theta_k - q - \bar{\theta}_{k-1} \geq \theta_k$, $\mathbb{E}[\theta | \theta > \theta_k] \geq \bar{\theta}_k$, and rearranging,

$$\begin{aligned} \mathbb{E}[\theta | \theta > \theta_k] - \theta_k &= g(\theta_k | H) \geq \bar{\theta}_k - \theta_k \\ &= \theta_k - \bar{\theta}_{k-1} - q \geq 0 \end{aligned}$$

where the last inequality follows from $\bar{\theta}_k \geq \theta_k$. By contradiction, suppose θ_k grows unbounded with k . Then by Lemma 3 $\lim_{k \rightarrow \infty} g(\theta_k | H) = 0$, so for every $\varepsilon > 0$ there

exists K_ε such that $g(\theta_k|H) < \varepsilon$ for all $k > K_\varepsilon$, and therefore

$$\varepsilon > \theta_k - \bar{\theta}_{k-1} - q \geq 0$$

which in turn implies

$$\lim_{k \rightarrow \infty} (\theta_k - \bar{\theta}_{k-1}) = q. \quad (7.1)$$

It follows:

$$\lim_{k \rightarrow \infty} [\theta_k - \bar{\theta}_{k-1} - (\theta_{k+1} - \bar{\theta}_k)] = 0.$$

So for every $\delta > 0$ there is K_δ such that for all $k > K_\delta$

$$\theta_{k+1} - \theta_k < \bar{\theta}_k - \bar{\theta}_{k-1} + \delta = 2(\theta_k - \bar{\theta}_{k-1}) + \delta - q$$

where in the last equality we used (3.9). If we take $k > \max\langle K_\delta, K_\varepsilon \rangle$:

$$\theta_{k+1} - \theta_k < 2(q + \varepsilon) + \delta - q = q + 2\varepsilon + \delta.$$

This fact, along with $\theta_{k+1} - \theta_k > q$ from Lemma 2, finally proves

$$\lim_{k \rightarrow \infty} (\theta_k - \theta_{k-1}) = q. \quad (7.2)$$

So we have established that we can always satisfy

$$\theta_{k-1} + q - \varepsilon < \bar{\theta}_{k-1} < \theta_k < \theta_{k-1} + q + \delta.$$

It follows:

$$\begin{aligned} \theta_{k-1} + q - \varepsilon &< \bar{\theta}_{k-1} \\ &= \mathbb{E}[\theta | \theta_{k-1} \leq \theta < \theta_k] \\ &< \mathbb{E}[\theta | \theta_{k-1} \leq \theta < \theta_{k-1} + q + \delta] \\ &= \frac{\int_{\theta_{k-1}}^{\theta_{k-1} + q + \delta} \theta e^{-H \frac{\theta^2}{2}} d\theta}{\int_{\theta_{k-1}}^{\theta_{k-1} + q + \delta} e^{-H \frac{\theta^2}{2}} d\theta} \end{aligned}$$

But notice that for $\varepsilon = \delta = 0$ the leftmost term is strictly larger than the rightmost term, so this inequality must be violated for some ε, δ small enough and $k > \max \langle K_\delta, K_\varepsilon \rangle$, the desired contradiction.

Dynamic Analysis when (6.2) fails. When condition (6.2) fails, we can construct equilibria with partial communication, supported by the threat of grim trigger to “babbling for ever” (flow loss \mathcal{L}_D) after P detects a lie by CB “outside the equilibrium” partition (message space).

First, we show that equilibria are again partitional also in this dynamic setting. Given any equilibrium, whether partitional or not, the associated payoff when observing θ and announcing $\bar{\theta}$ is

$$\begin{aligned} \mathcal{L}(\bar{\theta}|\theta) + \frac{\beta}{1-\beta} \mathbb{E} [\mathcal{L}(\bar{\theta}|\theta)] &= \frac{s^2 H^2 \lambda}{\lambda + s^2} (\theta - \bar{\theta})^2 - 2bsH (\theta - \bar{\theta}) + \mathcal{L}_T \\ &\quad + \frac{\beta}{1-\beta} \mathbb{E} \left[\frac{s^2 H^2 \lambda}{\lambda + s^2} (\theta - \bar{\theta})^2 - 2bsH (\theta - \bar{\theta}) + \mathcal{L}_T \right] \\ &= \frac{s^2 H^2 \lambda}{\lambda + s^2} (\theta - \bar{\theta})^2 - 2bsH (\theta - \bar{\theta}) + \frac{s^2 H^2 \lambda}{\lambda + s^2} \frac{\beta}{1-\beta} \mathbb{E} [(\theta - \bar{\theta})^2] + \frac{\mathcal{L}_T}{1-\beta} \end{aligned}$$

where $\mathbb{E} [\mathcal{L}(\bar{\theta}|\theta)]$ is the expected flow loss from equilibrium play, and $\mathbb{E} [(\theta - \bar{\theta})^2]$ is the expected variance of θ conditioning on equilibrium communication. The incentive constraint, comparing equilibrium play to a deviation to another of the equilibrium messages followed by babbling for ever, is

$$\begin{aligned} \mathcal{L}(\bar{\theta}|\theta) + \frac{\beta}{1-\beta} \mathbb{E} [\mathcal{L}(\bar{\theta}|\theta)] &\leq \mathcal{L}(\bar{\theta}'|\theta) + \frac{\beta}{1-\beta} \mathcal{L}_B \\ &\leq \frac{s^2 H^2 \lambda}{\lambda + s^2} (\theta - \bar{\theta})^2 - 2bsH (\theta - \bar{\theta}) + \frac{\beta}{1-\beta} \frac{s^2 H^2 \lambda}{\lambda + s^2} \mathbb{E} [(\theta - \bar{\theta})^2] + \frac{\mathcal{L}_T}{1-\beta} \\ &\leq \frac{s^2 H^2 \lambda}{\lambda + s^2} (\theta - \bar{\theta}')^2 - 2sbH (\theta - \bar{\theta}') + \mathcal{L}_T + \frac{\beta}{1-\beta} \left(\frac{s^2 H \lambda}{s^2 + \lambda} + \mathcal{L}_T \right) \end{aligned}$$

rearranging terms

$$\begin{aligned}
& \frac{s^2 H^2 \lambda}{\lambda + s^2} \left[\bar{\theta}^2 - \bar{\theta}'^2 - 2\theta (\bar{\theta} - \bar{\theta}') \right] + 2bsH (\bar{\theta} - \bar{\theta}') + \\
& + \frac{s^2 H^2 \lambda}{\lambda + s^2} \frac{\beta}{1 - \beta} \mathbb{E} \left[(\theta - \bar{\theta})^2 \right] \leq \frac{\beta}{1 - \beta} \frac{s^2 H \lambda}{s^2 + \lambda} \\
& (\bar{\theta} - \bar{\theta}') (\bar{\theta} + \bar{\theta}' - 2\theta + q) \leq \frac{\beta}{1 - \beta} \left\{ \frac{1}{H} - \mathbb{E} \left[(\theta - \bar{\theta})^2 \right] \right\}.
\end{aligned}$$

The payoff from a deviation (the LHS) must fall short of the damage (the RHS), which is the decrease in the variance of the signal θ from the prior $1/H = \mathbb{E}[\theta^2]$ to the posterior after communication $\mathbb{E}[(\theta - \bar{\theta})^2] \leq 1/H$. This latter inequality is the same condition as in the static game (Cf. Equation 3.4), except that the zero on the RHS is replaced by a positive magnitude.

Let

$$\hat{q} \equiv q - \frac{\beta}{1 - \beta} \frac{1}{\bar{\theta} - \bar{\theta}'} \left(\frac{1}{H} - \mathbb{E} \left[(\theta - \bar{\theta})^2 \right] \right).$$

When $\bar{\theta}' < \bar{\theta}$, namely, when comparing the equilibrium message with a downward lie, this is equivalent to reduce the normalized bias, as $\hat{q} < q$, and the inequality becomes

$$\bar{\theta}' \leq 2\theta - \bar{\theta} - \hat{q}$$

which is less binding than the analogous condition in the static case $\bar{\theta}' \leq 2\theta - \bar{\theta} - q$.

That is, lower equilibrium messages $\bar{\theta}'$ can be closer to the message $\bar{\theta}$ prescribed for type θ . So CB is more credible because the lie is followed by a punishment. When $\bar{\theta}' > \bar{\theta}$, when comparing the equilibrium message with an upward lie, this is equivalent to increase the normalized bias, as $\hat{q} > q$, and the inequality becomes

$$\bar{\theta}' \geq 2\theta - \bar{\theta} - \hat{q}$$

which is less binding than the analogous condition in the static case $\bar{\theta}' \geq 2\theta - \bar{\theta} - q$. In either direction, the equilibrium takes the form of a partition of the state space, where the interval's bounds solve the above two conditions with equality, and messages are as usual the conditional means of θ inside those intervals:

$$\begin{aligned}\theta_k &= \frac{\bar{\theta}_{k-1} + \bar{\theta}_k + \hat{q}}{2} \\ \bar{\theta}_k &= \sqrt{\frac{H}{2\pi}} \int_{\theta_{k-1}}^{\theta_k} \theta e^{-\frac{H\theta^2}{2}} d\theta.\end{aligned}$$

This double recursion allows us to construct all equilibria, similarly to the static case, and to verify the comparative statics effects of changes in H . For given parameter values, the best equilibrium's partition is finer than in the static case. Given the tail properties of the normal distribution, there always exists a two-message equilibrium.