Technical appendix to

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Appendix

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This mathematical appendix works out the details of the analysis in the text, including some parameter restrictions required for the model to make sense. It has four sections:

A1 Optimal production plans for risk-neutral firms.

A2 Locational equilibrium, including the properties of the *QQ* curve.

A3 Labor market equilibrium, including the properties of the *LL* curve.

A4 Comparative statics of location and wages in general equilibrium.

A1 The firm's problem

As discussed in the text, we consider risk-neutral firms who face two consecutive realizations of demand, and must decide how much to produce and how much to sell in each period. The timing is as follows:

1) All firms decide how much to produce
2) Demand level for period 1 is realized, and firms decide how much to sell. Any output not sold can be held until period 2.
3) Flexible firms produce again; nonflexible firms do not
4) Demand level for period 2 is realized, and firms decide how much to sell. Any output not sold is thrown out.

We consider the nonflexible firm's problem first. We begin with the problem facing the firm after it has already produced, and then work out optimal production given the solution. After producing some level of output and observing a_1, output costs are sunk, so the objective is simply to maximize expected revenue, subject to the constraint that total sales not exceed output. Both revenue and the shadow value of second period output depend on the realization of a_2. As a result, the constrained maximization problem is to choose $\{s_1, s_2^H, s_2^L\}$ to maximize

$$L = s_1(a_1 - bs_1) + \rho s_2^H(a_H - bs_2^H) + (1-\rho)s_2^L(a_L - bs_2^L) + \lambda_H(q - s_1 - s_2^H) + \lambda_L(q - s_1 - s_2^L)$$

where ρ is the probability that $a_i = a_H$. This is a complementary slackness problem, for which the first order conditions and associated solutions are

$$\frac{\partial L}{\partial s_1} = a_1 - 2bs_1 - \lambda_H - \lambda_L = 0 \quad \rightarrow \quad s_1 = \frac{a_1}{2b} - \frac{(\lambda_H + \lambda_L)}{2b}$$

$$\frac{\partial L}{\partial s_2^H} = \rho a_H - 2\rho b s_2^H - \lambda_H = 0 \quad \rightarrow \quad s_2^H = \frac{a_H}{2b} - \frac{\lambda_H}{2\rho b}$$

$$\frac{\partial L}{\partial s_2^L} = (1-\rho) a_L - 2(1-\rho) b s_2^L - \lambda_L = 0 \quad \rightarrow \quad s_2^L = \frac{a_L}{2b} - \frac{\lambda_L}{2(1-\rho) b}$$

There are four possible configurations for the Lagrange multipliers:

A. $\lambda_H > 0, \lambda_L > 0$

B. $\lambda_H > 0, \lambda_L = 0$

C. $\lambda_H = 0, \lambda_L > 0$

D. $\lambda_H = 0, \lambda_L = 0$.

The logic of the model means we can dismiss cases C and D. Case A is the simplest, and corresponds to the solutions in the text, which are

$$s_1 = \frac{q}{2} + \frac{a_1 - \bar{a}}{4b}, \tag{A1}$$

$$s_2 = \frac{q}{2} - \frac{a_1 - \bar{a}}{4b}. \tag{A2}$$

We can also calculate the value of the Lagrange multipliers in case A,

$$\lambda^H = \rho \left[\frac{2a_H + a_1 - \bar{a}}{2} - bq \right], \tag{A3}$$

$$\lambda^L = (1-\rho) \left[\frac{2a_L + a_1 - \bar{a}}{2} - bq \right] \tag{A4}$$

Optimal output q maximizes expected revenue minus actual costs. In case A, where second period marginal revenue is always positive, substitution of (A1) and (A2) into the definition of revenue gives, after a bit of manipulation,

$$Expected\ revenue_A = const + \bar{a}q - \frac{bq^2}{2}$$

Plugging this into the definition of profit and maximizing immediately yields the result

$$q^N = \frac{\bar{a} - w}{b} \tag{A5}$$

We can substitute this into the solutions (A3) and (A4) for the Lagrange multipliers to find their values at the optimum:

$$\lambda^H = \rho\left[\frac{2a_H + a_1 - 3\overline{a} + 2w}{2}\right] \tag{A6}$$

$$\lambda^L = (1-\rho)\left[\frac{2a_L + a_1 - 3\overline{a} + 2w}{2}\right] \tag{A7}$$

We can use these expressions to see for what range of parameters case A is relevant. A sufficient condition is that λ^L is always positive, which occurs if

$$w > \frac{3}{2}(\overline{a} - a_L) \tag{A8}$$

If this condition is not satisfied, then case B is applies. In the paper and for most of the rest of this appendix we proceed on the assumption that (A8) holds in equilibrium. We return to the analysis of case B at the end of this section.

Plugging (A5) into (A1) and (A2) gives the value of sales at the optimum:

$$s_1^N = \frac{\overline{a} - w}{2b} + \frac{a_1 - \overline{a}}{4b} = \frac{\overline{a} + a_1 - 2w}{4b} \tag{A9}$$

$$s_2^N = \frac{\overline{a} - w}{2b} - \frac{a_1 - \overline{a}}{4b} = \frac{3\overline{a} - a_1 - 2w}{4b} \tag{A10}$$

We know that the revenue-maximizing value of sales in period 2 when demand is low is $\frac{a_L}{2b}$. For (A10) to make sense, s_2^N must be less than this upper bound when first period demand is low, or

$$\frac{3\overline{a} - a_L - 2w}{4b} \le \frac{a_L}{2b} \tag{A11}$$

The implied restriction on w is exactly the lower bound on w derived in (A8).

Moving to the solution for flexible firms, again we work backward. After first period demand is realized but before second period demand is realized, the firm chooses first-period sales and second-period output and conditional sales to maximize expected profit subject to the constraint that sales are no greater than output. The Lagrangean is

$$L = s_1(a_1 - bs_1) + \rho s_2^H(a_H - bs_2^H) + (1-\rho)s_2^L(a_L - bs_2^L) - wq_2$$
$$+\lambda_1(q_1 - s_1) + \lambda_H(q_1 + q_2 - s_1 - s_2^H) + \lambda_L(q_1 + q_2 - s_1 - s_2^L) \tag{A12}$$

As for the nonflexible firm, this is a complementary slackness problem, for which the first order conditions and associated solutions are

$$\frac{\partial L}{\partial s_1} = a_1 - 2bs_1 - \lambda_H - \lambda_L - \lambda_1 = 0 \quad \rightarrow \quad s_1 = \frac{a_1}{2b} - \frac{(\lambda_H + \lambda_L + \lambda_1)}{2b}$$

$$\frac{\partial L}{\partial s_2^H} = \rho a_H - 2\rho b s_2^H - \lambda_H = 0 \quad \rightarrow \quad s_2^H = \frac{a_H}{2b} - \frac{\lambda_H}{2\rho b}$$

$$\frac{\partial L}{\partial s_2^L} = (1-\rho)a_L - 2(1-\rho)bs_2^L - \lambda_L = 0 \quad \rightarrow \quad s_2^L = \frac{a_L}{2b} - \frac{\lambda_L}{2(1-\rho)b}$$

$$\frac{\partial L}{\partial q_2} = -w + \lambda_H + \lambda_L = 0 \quad \rightarrow \quad \lambda_H + \lambda_L = w$$

For the moment, we assume that the first period output constraint is always slack, so that λ_1 =0. As before, there are four possible configurations for the Lagrange multipliers λ_H and λ_L. When both are positive the solution is

$$s_1^F = \frac{a_1 - w}{2b} \tag{A13}$$

$$s_2^F = \frac{\overline{a} - w}{2b} \tag{A14}$$

$$q_2^F = \frac{\overline{a} - w}{2b} - (q_1 - s_1) \geq 0 \tag{A15}$$

The Lagrange multipliers at the optimum are

$$\lambda_H = \rho(a_H - \overline{a}) + \rho w \tag{A16}$$

$$\lambda_L = (1-\rho)(a_L - \overline{a}) + (1-\rho)w \tag{A17}$$

Obviously λ_H is always positive, but λ_L will be positive only if

$$\overline{a} - a_L < w \tag{A18}$$

The inequality (A8) is sufficient for (A18) but not necessary: there is a range of wages for which the second-period output constraint is slack for flexible firms but not for inflexible firms. For the moment we will focus on the case where (A18) is satisfied, returning to the $\lambda_L = 0$ case at the end of this section.

We now come to the last element of the problem for flexible firms, the choice of q_1. Expected first period revenue is highest if the multiplier λ_1 in (A12) is always zero (as tentatively assumed), and since unsold first-period output can be sold in the second period, total profits will be maximized if λ_I is always zero. This can be accomplished by producing at least enough in period 1 to sell (A13) if demand is high, which is

$$q_1^F = \frac{a_H - w}{2b} \tag{A19}$$

Another consideration is given by the constraint in (A15) that second period-output can't be negative. To check if (A15) is satisfied when first period demand is low, substitute (A19) and (A13) into (A15), setting $a_I = a_L$. The result is

$$\bar{a} - w \geq a_H - a_L \tag{A21}$$

This inequality states that average demand must be sufficiently large relative to the variance in demand, which we assume to hold in equilibrium.

Finally, we note that there is some indeterminacy in the solution for optimal first period output. Equation (A19) gives the minimum output level to guarantee that first-period sales are ex-post optimal, but since sales continue in the second period, the firm may choose to produce more in period 1 and less in period 2. To resolve this indeterminacy, we assume very small storage costs, which will lead the firm to produce q_1^F given by (A19) without affecting anything else about the problem.

The solutions discussed above for both flexible and non-flexible firms have focused on the case where wages are high enough so that the Lagrange multipliers on the output constraints are always positive at the optimum, conditions which were given by the inequalities (A8) and (A18) for non-flexible and flexible firms respectively. For lower wages, this will not be the case, and both flexible and non-flexible firms may end up discarding some output in the second period when demand is low. In the rest of this section we discuss the equilibrium in these cases.

For non-flexible firms, using the first-order conditions preceding equation (A1) and setting $\lambda_L = 0$ gives

$$\lambda_H = \rho\left[a_H - 2b\left(q - s_1\right)\right] \tag{A22}$$

$$s_1 = \frac{1}{1+\rho}\left[\rho q + \frac{a_1 - \rho a_H}{2b}\right] \tag{A23}$$

$$s_2^H = \frac{1}{1+\rho}\left[q - \frac{a_1 - \rho a_H}{2b}\right] \tag{A24}$$

$$s_2^L = \frac{a_L}{2b} \tag{A25}$$

To calculate optimal output when $\lambda_L = 0$, we plug the expressions for optimal sales into the definition of profit and then maximize with respect to q. The resulting expression for revenue is a quadratic in output q,

$$R = \alpha + \frac{\rho(\bar{a} + a_H)q - b\rho q^2}{1+\rho}$$

where

$$k_H = \frac{a_H - \rho a_H}{(1+\rho)2b}, \quad k_L = \frac{a_L - \rho a_H}{(1+\rho)2b}, \quad \bar{k} = \rho k_H + (1-\rho)k_L = \frac{1-\rho}{1+\rho}\frac{a_L}{2b}$$

and

$$\alpha = \rho a_H k_H + (1-\rho)a_L k_L - b\rho k_H^2 - b(1-\rho)k_L^2 - \rho^2 k_H(a_H + bk_H) - \rho(1-\rho)k_L(a_H + bk_L) + (1-\rho)\frac{a_L^2}{4b}$$

Setting marginal revenue equal to marginal cost w gives optimal output in the nonflexible location when $\lambda_L = 0$ as

$$q^N = \frac{\bar{a} + a_H}{2b} - \frac{(1+\rho)w}{2\rho b} \tag{A26}$$

Turning now to the flexible firm, we set $\lambda_L = 0$ in the first order conditions following (A12) and find

$$s_1 = \frac{a_1 - w}{2b} \tag{A27}$$

$$q_1 = \frac{a_H - w}{2b} \tag{A28}$$

$$s_2^H = \frac{\rho a_H - w}{2\rho b} \tag{A29}$$

$$s_2^L = \frac{a_L}{2b} \tag{A30}$$

$$q_2 = \frac{a_1 - a_H}{2b} + \frac{\rho a_H - w}{2\rho b} \geq 0 \tag{A31}$$

For q_2 given by (A31) to satisfy the nonnegativity constraint when $a_1 = a_L$ requires $\rho a_L - w > 0$, which is slightly stronger than the already-assumed condition $\bar{a} > w$.

A2 Locational equilibrium

The choice of production location involves a tradeoff between the benefits of flexibility and higher wage costs. To derive the cutoff point, we calculate expected profits in each location as a function of the variance of output, with outputs chosen optimally as outlined above. By assumption, producers in *C* are flexible, while producers in *A* are not. As in the previous section, we first focus on the case where wages are high enough in both locations so that the Lagrange multipliers in the firms maximization problems are always positive, which is the case discussed in the text. We discuss the other case at the end of the section.

Expected profits for producers in *A* are expected revenues minus costs. Costs are found by multiplying wages in *A* by optimal output as given by (4):

$$costs_A = \frac{\bar{a}w_A - w_A^2}{b} \tag{A32}$$

Expected revenue is computed by substituting optimal sales choices into the definition of revenue, and using the fact that

$$V(a) = \rho a_H^2 + (1-\rho)a_L^2 - \bar{a}^2 . \tag{A33}$$

The result is that expected revenue in *A* is

$$revenue_A = \frac{4\bar{a}^2 - 4w_A^2 + V(a)}{8b} . \tag{A34}$$

Similarly, expected costs and revenue in *C* are calculated as

$$costs_C = \frac{\bar{a}w_C - w_C^2}{b} \tag{A35}$$

$$revenue_C = \frac{4\bar{a}^2 - 4w_C^2 + 2V(a)}{8b} \tag{A36}$$

In both (A34) and (A36), revenue is increasing in the variance of demand, which is a consequence of the convexity of the profit function. Both types of producers respond to period 1 shocks, but producers in *C* respond more, so expected revenues in *C* increase faster as a function of $V(a)$ than they do in *A*. It is clear by inspection that if wages are equal in the two locations then profits will be higher in *C*, and it is also clear that, holding $V(a)$ constant, a big enough wage premium in *C* will cause profits there to fall below profits in *A*. Finally, for a given wage premium in *C*, there is a level of $V(a)$ which will equalize profits.

To find the critical value of $V(a)$, we substitute equation (7) into (A34) and (A36), substitute (A32) and (A34)-(A36) into the definition of profits, and set profits in the two locations equal. Solving for i gives the result for i_L in the text, equation (8). The total derivative of i_L is

$$di_L = \frac{8}{\sigma^2}\left[(\bar{a} - w_C)dw_C - (\bar{a} - w_A)dw_A\right] \tag{A37}$$

and the partial derivatives are

$$\frac{\partial i_L}{\partial w_C} = \frac{8(\bar{a} - w_C)}{\sigma^2} \tag{A38}$$

$$\frac{\partial i_L}{\partial w_A} = -\frac{8(\bar{a} - w_A)}{\sigma^2} \tag{A39}$$

As long as $\bar{a} > w_C$ and $\bar{a} > w_A$, which we have already assumed (and which simply means that demand is high enough for the model to make sense), we get the expected result that i_L is increasing in w_C and decreasing in w_A. The same conditions are sufficient to guarantee that $i_L > 0$ when $w_C > w_A$. To see this, re-write (8) as

$$i_L \frac{\sigma^2}{4(w_C - w_A)} = (\bar{a} - w_C) + (\bar{a} - w_A) > 0 \tag{8′}$$

The inequality can only be satisfied if the two terms on the left hand side of the equality are of the same sign, which is what is illustrated in the Figures. But it is not possible to solve for i_L as a function of the wage differential $\hat{w} = w_C - w_A$, since (8) is a quadratic in w_C and w_A separately. With the restrictions that the wage differential cannot be negative in equilibrium and that $i_L \in [0,1]$, we can depict equation (8) as a surface in i_L- w_C-w_A space, as in figure A1. The surface is a quadratic, increasing at a decreasing rate in w_C and decreasing at a decreasing rate in w_A. The *QQ* curve of Figures 1 and 2 is traced out by the equilibrium movements of w_C and w_A as i_U changes, which are derived in the next section of this appendix, where we show that $\partial \hat{w} / \partial i_U > 0$. In general all we know is that the relationship is monotonically increasing in w_C-w_A along the equilibrium path, so the shape of *QQ* in Figures 1 and 2 is otherwise free hand.

Turning to the low-wage case where output constraints are sometimes slack, we proceed in the same way as we did above. For non-flexible firms in *A*, expected profit can be computed by substituting optimal output (A26) into the expressions for revenue and costs. For flexible firms in *C*, the expression for expected profit derived from using equations (A27)-(A31) in the definitions of revenue and costs. For both locations the resulting expressions for profits are hard to analyze except in the case where $\rho = 0.5$, so for the rest of this section we assume this. When $\rho = 0.5$, $a_H = \bar{a} + \sigma$, $a_L = \bar{a} - \sigma$, and $V(a) = \sigma^2$. Using this, the expressions for profits in the two locations for a given firm with demand variance σ^2 are

$$\pi^N = \frac{1}{b}\left[\bar{a}^2 - \frac{9}{4}w_A^2 - \frac{1}{2}w_A\sigma + \frac{1.25}{3}\sigma^2\right] \tag{A40}$$

$$\pi^F = \frac{1}{4b}\left[2\bar{a}^2 + 3w_C^2 - 4w_C\bar{a} - 2w_C\sigma + 2\sigma^2\right] \tag{A41}$$

Equations (A40) and (A41) are increasing in σ as long as $\sigma > 0.6w_A$, which we assume to hold. Subtracting (A40) from (A41) gives the profit differential as a function of relative wages and demand variance:

$$\pi^F - \pi^N = \frac{1}{b}\left[\left(\bar{a} + \frac{\sigma}{2}\right)(w_A - w_C) + \frac{9}{4}w_A^2 + \frac{3}{4}w_C^2 + \frac{5}{60}\sigma^2\right] \tag{42}$$

Since this expression involves both σ and σ^2 separately, we can't proceed in quite the same way that we did when deriving the *QQ* surface. But the economics of the problem are identical, since the profit differential is increasing in w_A and in σ, and decreasing in w_C, just as in the case where $\lambda_L > 0$. Figure A2 illustrates the equilibrium location of firms for a given wage differential: for firms with a standard deviation of demand less than the level where the profit curves intersect, profits are higher in *A*, while the remaining firms earn higher profits by locating in *C*.

A3 Labor Market equilibrium

For given i_U and i_L, there are i_U - i_L firms that locate in *C*, with the remaining 1 - (i_U - i_L) firms located in *A*. Each firm in each location has average annual labor demand of $2q^*$, where

$$q^* = \frac{\bar{a} - w}{2b}, \tag{A43}$$

so total labor demand in *C* and *A* respectively is

$$(i_U - i_L)\frac{\overline{a} - w_C}{b} \tag{A44}$$

$$(1 + i_L - i_U)\frac{\overline{a} - w_A}{b} \tag{A45}$$

Setting labor supply equal to labor demand in each region and solving for wages gives equations (9) and (11) in the text, and subtracting (11) from (9) gives (12). The vertical intercept of (12) at $i_L = 0$ is

$$\hat{w} = b\left(\frac{L_A}{1 - i_U} - \frac{L_C}{i_U}\right), \tag{A46}$$

which is positive due to the parameter restriction given by equation (10). The economically relevant range of the function is $[0,i_U)$, and $\hat{w}$ asymptotically approaches minus infinity as $i_L \to i_U$ from below.

The point where the *LL* curve crosses the horizontal axis is found be setting $\hat{w} = 0$ to find

$$i_L = i_U - \frac{L_C}{L_A + L_C}$$

This is strictly less than i_U, and as noted in the text this be greater than zero as long as (13) is satisfied.

From equation (12), the slope of $\hat{w}$ when graphed against i_L is

$$\frac{\partial \hat{w}}{\partial i_L} = -b\left(\frac{L_A}{(1 + i_L - i_U)^2} + \frac{L_C}{(i_U - i_L)^2}\right) \tag{A47}$$

which is strictly negative. The second derivative is

$$\frac{\partial^2 \hat{w}}{\partial i_L^2} = 2b\left(\frac{L_A}{(1 + i_L - i_U)^3} - \frac{L_C}{(i_U - i_L)^3}\right) \tag{A48}$$

which may be of either sign for small values of i_L but is strictly negative as $i_L \to i_U$ from below. As i_U approaches 1, (A48) must be strictly positive at $i_L = 0$, and the shape of the *LL* curve shown in Figures 1 and 2 illustrates this case.

A4 Comparative statics

Our core comparative static experiment is an increase in the range of products for which flexible production is feasible, and we focus on the case where output constraints always bind. We model this through an increase in i_U, which has two effects. The first is straightforward, which is a horizontal shift of Δi_U in the LL curve (to prove this, totally differentiate (12), set $\hat{w} = 0$, and solve for $di_L/di_U = 1$). The second part of the story is that an increase in i_U changes the QQ curve, which makes analysis somewhat tricky.

We start with the simplest case. Here we suppose that there is a one-to-one negative relationship between selling season and demand variance, so that (7) is valid for all $i \in [0,1]$. In this case an increase in i_U simply extends the upper boundary of the QQ relationship, without changing its shape, which is the case analyzed graphically in Figure 2. This means that we can use calculus to analyze the comparative statics. We have three equations (9), (11) and (8) in the three unknowns w_A, w_C, and i_L. Excluding parameters, the exogenous variables that can change the equilibrium are L_A, L_C, and i_U.

Totally differentiating the three equations gives

$$dw_C = \frac{-b}{i_U - i_L} dL_C + \frac{bL_C}{(i_U - i_L)^2} di_U - \frac{bL_C}{(i_U - i_L)^2} di_L \tag{9'}$$

$$dw_A = \frac{-b}{1 + i_L - i_U} dL_A - \frac{bL_A}{(1 + i_L - i_U)^2} di_U + \frac{bL_A}{(1 + i_L - i_U)^2} di_L \tag{11'}$$

$$di_L = \frac{8(\bar{a} - w_C)}{\sigma^2} dw_C - \frac{8(\bar{a} - w_A)}{\sigma^2} dw_A \tag{8'}$$

Defining $\theta = i_U - i_L$, $\beta_C \equiv \frac{8}{\sigma^2}(\bar{a} - w_C)$, $\beta_A \equiv \frac{8}{\sigma^2}(\bar{a} - w_A)$ and bringing the endogenous variables to the left hand side, gives

$$dw_C + \frac{bL_C}{\theta^2} di_L = \frac{-b}{\theta} dL_C + \frac{bL_C}{\theta^2} di_U \tag{9'}$$

$$dw_A - \frac{bL_A}{(1-\theta)^2} di_L = -\frac{b}{1-\theta} dL_A - \frac{bL_A}{(1-\theta)^2} di_U \tag{11'}$$

$$di_L - \beta_C dw_C + \beta_A dw_A = 0 \tag{8'}$$

Writing this system out in matrix notation,

$$\begin{bmatrix} 1 & 0 & \dfrac{bL_C}{\theta^2} \\ 0 & 1 & \dfrac{-bL_A}{(1-\theta)^2} \\ -\beta_C & \beta_A & 1 \end{bmatrix} \begin{bmatrix} dw_c \\ dw_A \\ di_L \end{bmatrix} = \begin{bmatrix} \dfrac{-b}{\theta} & 0 & \dfrac{bL_C}{\theta^2} \\ 0 & \dfrac{-b}{1-\theta} & \dfrac{-bL_A}{(1-\theta)^2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dL_C \\ dL_A \\ di_U \end{bmatrix} \tag{A49}$$

Using the shorthand **A**dy=**B**dx to represent the system (A49) above, we have

$$\det(\mathbf{A}) = 1 + \frac{\beta_C bL_C}{\theta^2} + \frac{\beta_A bL_A}{(1-\theta)^2} > 0 \tag{A50}$$

Solving the system (A49) for $dy=\mathbf{A}^{-1}\mathbf{B}dx$ gives

$$\begin{bmatrix} dw_C \\ dw_A \\ di_L \end{bmatrix} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} 1+\dfrac{\beta_A bL_A}{(1-\theta)^2} & \dfrac{\beta_A bL_c}{\theta^2} & \dfrac{-bL_c}{\theta^2} \\ \dfrac{\beta_C bL_A}{(1-\theta)^2} & 1+\dfrac{\beta_C bL_c}{\theta^2} & \dfrac{bL_A}{(1-\theta)^2} \\ \beta_C & -\beta_A & 1 \end{bmatrix} \begin{bmatrix} \dfrac{-b}{\theta} & 0 & \dfrac{bL_c}{\theta^2} \\ 0 & \dfrac{-b}{1-\theta} & \dfrac{-bL_A}{(1-\theta)^2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dL_c \\ dL_A \\ di_U \end{bmatrix} \tag{A51}$$

or equivalently,

$$\begin{bmatrix} dw_C \\ dw_A \\ di_L \end{bmatrix} = \frac{b}{\det(\mathbf{A})} \begin{bmatrix} \dfrac{-1}{\theta}\left(1+\dfrac{\beta_A bL_A}{(1-\theta)^2}\right) & \dfrac{-\beta_A bL_C}{(1-\theta)\theta^2} & \dfrac{L_C}{\theta^2} \\ -\dfrac{\beta_C bL_A}{\theta(1-\theta)^2} & \dfrac{-1}{1-\theta}\left(1+\dfrac{\beta_C bL_c}{\theta^2}\right) & \dfrac{-L_A}{(1-\theta)^2} \\ \dfrac{-\beta_C}{\theta} & \dfrac{\beta_A}{1-\theta} & \dfrac{\beta_C L_c}{\theta^2}+\dfrac{\beta_A L_A}{(1-\theta)^2} \end{bmatrix} \begin{bmatrix} dL_c \\ dL_A \\ di_U \end{bmatrix} \tag{A52}$$

Scrutinizing the solution establishes

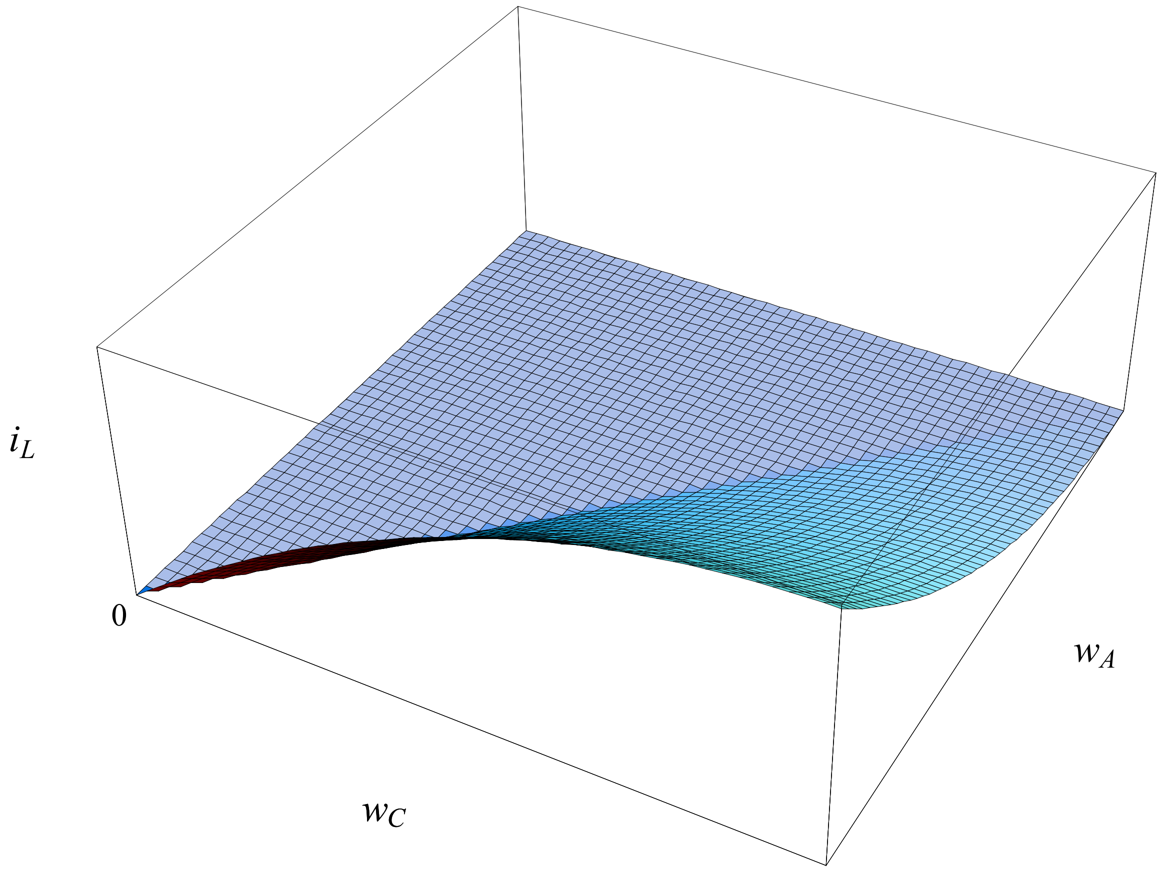
$$\begin{bmatrix} \dfrac{dw_C}{dL_C} & \dfrac{dw_C}{dL_A} & \dfrac{dw_C}{di_U} \\ \dfrac{dw_A}{dL_C} & \dfrac{dw_A}{dL_A} & \dfrac{dw_A}{di_U} \\ \dfrac{di_L}{dL_C} & \dfrac{di_L}{dL_A} & \dfrac{di_L}{di_U} \end{bmatrix} = \begin{bmatrix} - & - & + \\ - & - & - \\ - & + & + \end{bmatrix} \tag{A53}$$

These signs are all as expected. It is also the case that $di_L/di_U < 1$ (by inspection). All of this corresponds to the results from the graphical analysis. In particular, as we claimed at the end of section A2 above, the third column of (A53) establishes that $\partial\hat{w}/\partial i_U > 0$.

The model is not susceptible to analysis using calculus when selling season and demand variance do not have a one-to-one relationship, but the substantive conclusions are the same. At any given relative wage $\hat{w}$ there is a mass of potentially flexible firms that wants to produce in *C*, and this mass is weakly decreasing in $\hat{w}$. This is because there is only so much labor in *C*, and the equilibrium $\hat{w}$ prices some of the potentially flexible firms out of the market, so that they produce in *A* instead. Now increase the mass of potentially flexible firms by the amount Δi_u. Of these new entrants into the market for *C* labor, some fraction γ will want to produce in *C* at the old equilibrium wage, since their revenue increase from flexible production exceeds the increased wage costs associated with *C*. This means there is excess demand for *C* labor at the old equilibrium $\hat{w}$, so w_C rises, choking off some of the increased demand for *C* labor. As a result, less than $\gamma\Delta i_u$ move to *C* on net. Since some firms have left the market for *A* labor, wages there must fall. Therefore, $\hat{w}$ rises. If $\gamma = 0$, there is no effect: all of the newly-flexible firms are content to stay in low-wage *A* even though flexible production is now feasible, since it is not profitable. If $\gamma = 1$, the algebraic results will be the same as found in equations (A52).

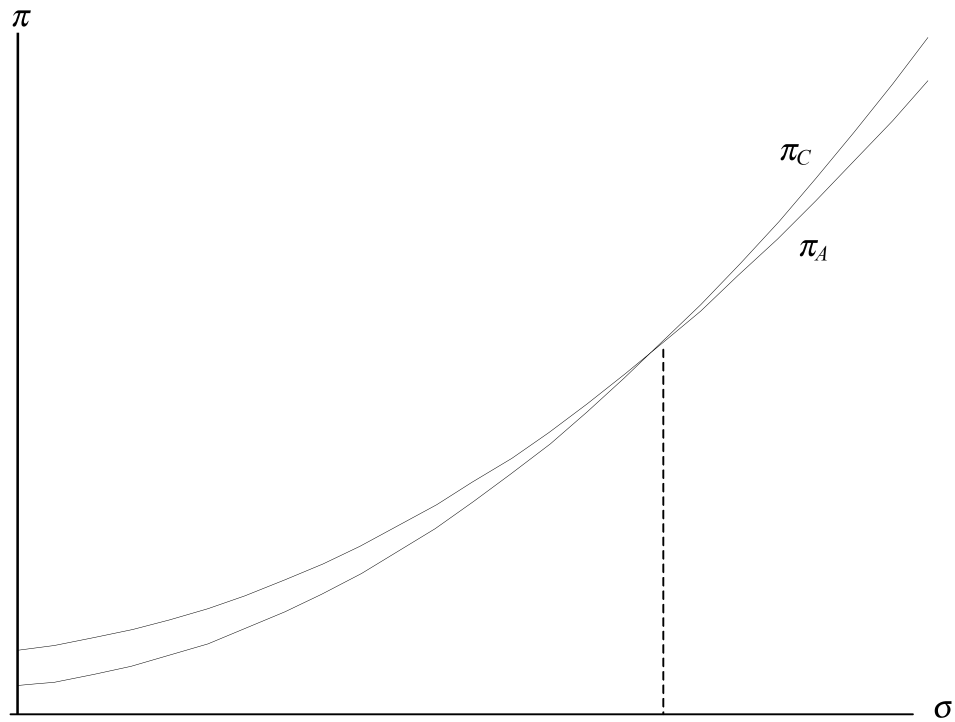
Lastly, our comparative statics results have been derived for the case where wages are high enough that (A8) is satisfied, so that output constraints always bind. The relationship between profits, wages, and demand variance when (A8) is not satisfied is given by (A42), with the labor market equilibrium conditions unchanged. Since we can't derive a closed-form expression for i_L when (A8) is not satisfied, we can't derive comparative statics results analogous to those in (A52). However, we can show that similar comparative statics results occur in numerical examples, which is not surprising since the economics of the model are fundamentally no different when (A8) is not satisfied than when it is.

Figure A1 - QQ surface
(the origin is in the bottom left corner of the box)



Notes to Figure A1: This graph depicts equation (8) in the text. The origin is in the bottom left corner of the box. The range of the vertical axis is (0,1), while the range of the other two axes is $(0, \overline{a})$. The flat shaded portion of the figure is where $w_C < w_A$, which can not occur in equilibrium.

Figure A2 - Profits and variance



Notes to Figure A2: The equations for the π_C and π_A curves are given by (A41) and (A40) respectively. The figure is drawn for $w_C > w_A$.