Web Appendix for "Tricks for Hicks: The EASI Demand System"*

I. General Cost Functions for Implicit Marshallian Demands

Instead of Stone index related constructions, we could more generally define implicit utility by

$$y = \frac{x - G(\mathbf{w}, \mathbf{p}, \mathbf{z})}{1 + s(\mathbf{p}, \mathbf{z})} \tag{I.1}$$

for any deflator function G. Setting $y = u$ will then require

$$C(\mathbf{p}, u, \mathbf{z}) = [1 + s(\mathbf{p}, \mathbf{z})]\, u + G(\nabla_{\mathbf{p}} C(\mathbf{p}, u, \mathbf{z}), \mathbf{p}, \mathbf{z}) \tag{I.2}$$

and the resulting implicit Marshallian demand functions are

$$\mathbf{w} = \omega\left[\mathbf{p}, \mathbf{z}, \frac{x - G(\mathbf{w}, \mathbf{p}, \mathbf{z})}{(1 + s(\mathbf{p}, \mathbf{z})}\right]. \tag{I.3}$$

where $\omega(\mathbf{p}, \mathbf{z}, u) = \nabla_{\mathbf{p}} C(\mathbf{p}, u, \mathbf{z})$. At this level of generality, $u = y$ is possible for almost any log cost function $C(\mathbf{p}, u, \mathbf{z})$ even with $s(\mathbf{p}, \mathbf{z}) = 0$ as the following Theorem shows.

THEOREM 5: *Assume $C(\mathbf{p}, u, \mathbf{z})$ is regular. A sufficient condition for existence of a function $G(\mathbf{w}, \mathbf{p}, \mathbf{z})$ such that $u = x - G(\mathbf{w}, \mathbf{p}, \mathbf{z})$ is the existence of some scalar valued function of $\nabla_{\mathbf{p}} C(\mathbf{p}, u, \mathbf{z})$ and $\mathbf{p}$ that is strictly monotonic in u.*

PROOF OF THEOREM 5:

Let R be a scalar valued function of $\nabla_{\mathbf{p}} C(\mathbf{p}, u, \mathbf{z})$ and $\mathbf{p}$ that is strictly monotonic in u. Since $R(\mathbf{p}, \mathbf{z}, \omega(\mathbf{p}, \mathbf{z}, u))$ is strictly monotonic in u, it can be inverted to obtain $u = S(\mathbf{p}, \mathbf{z}, R)$, and we may define a function G by $G(\mathbf{w}, \mathbf{p}, \mathbf{z}) = C(\mathbf{p}, S(\mathbf{p}, \mathbf{z}, R(\mathbf{w}, \mathbf{p}, \mathbf{z}))\mathbf{z},) - S(\mathbf{p}, \mathbf{z}, R(\mathbf{w}, \mathbf{p}, \mathbf{z}))$.

Theorem 5 provides a very weak sufficient condition for existence of implicit Marshallian demands $\mathbf{w} = \omega[\mathbf{p}, \mathbf{z}, x - G(\mathbf{w}, \mathbf{p}, \mathbf{z})]$ for some function G. For example, this condition is satisfied if any good or combination of goods has a budget share that is strictly increasing or strictly decreasing in total expenditures (and hence in utility). Moreover, this condition is sufficient but not necessary, e.g., if preferences are homothetic so $C(\mathbf{p}, u, \mathbf{z}) = u + t(\mathbf{p}, \mathbf{z})$ for some function t, then even though the sufficient condition need not hold now, we can define $G(\mathbf{w}, \mathbf{p}, \mathbf{z}) = -t(\mathbf{p}, \mathbf{z})$ to obtain implicit marshallian demands. So the only kinds of situations where G might fail to exist is when all budget shares are independent of u for some, but not all, ranges of values of u.

To give an example of Theorem 5, suppose that the Hicksian budget share function for good 1, $w_1 = \omega_1(\mathbf{p}, \mathbf{z}, u)$, is invertible in u, so $u = \omega_1^{-1}(\mathbf{p}, \mathbf{z}, w_1)$. Then a deflator G that makes $u = y$ is $G(\mathbf{w}, \mathbf{p}, \mathbf{z}) = C(\mathbf{p}, \omega_1^{-1}(\mathbf{p}, \mathbf{z}, w_1), \mathbf{z}) - \omega_1^{-1}(\mathbf{p}, \mathbf{z}, w_1)$.

The function G satisfying equation (I.2) is not unique, e.g., it can be freely translated by an ordinal transformation of u, and in the previous example, different invertible budget shares could be used to obtain different expressions for G. For applications, the main point of implicit Marshallian demands is not existence or uniqueness, but rather convenience for demand estimation, which is why this paper focuses on examples where G has a simple closed form, namely, a translation of the log Stone index.

*Please note: there are additional appendix materials contained in the full article (http://www.aeaweb.org/articles.php?doi=10.1257/aer.99.3.827).

II. Generalized EASI Cost Functions

A generalization of the EASI class of cost functions (B5) is

$$x = C(\mathbf{p}, u, \mathbf{z}, \varepsilon) = u + \mathbf{c}(\mathbf{p}, \mathbf{z})'\mathbf{m}(u, \mathbf{z}, \varepsilon) + T(\mathbf{p}, \mathbf{z}) + S(\mathbf{p}, \mathbf{z})\, u \tag{II.1}$$

where $\mathbf{c}(\mathbf{p}, \mathbf{z})$ is a J-vector valued function. The Hicksian budget shares for this class are

$$\mathbf{w} = \omega(\mathbf{p}, u, \mathbf{z}, \varepsilon) = \nabla_{\mathbf{p}}\mathbf{c}(\mathbf{p}, \mathbf{z})'\mathbf{m}(u, \mathbf{z}, \varepsilon) + \nabla_{\mathbf{p}} T(\mathbf{p}, \mathbf{z}) + \nabla_{\mathbf{p}} S(\mathbf{p}, \mathbf{z})\, u$$

Solving this expression for $\mathbf{m}(u, \mathbf{z}, \varepsilon)$ and substituting the result into (II.1) gives

$$x = u + \mathbf{c}(\mathbf{p}, \mathbf{z})'[\nabla_{\mathbf{p}}\mathbf{c}(\mathbf{p}, \mathbf{z})']^{-1}[\mathbf{w} - \nabla_{\mathbf{p}} T(\mathbf{p}, \mathbf{z}) - \nabla_{\mathbf{p}} S(\mathbf{p}, \mathbf{z})\, u] + T(\mathbf{p}, \mathbf{z}) + S(\mathbf{p}, \mathbf{z})\, u$$

Now solve this expression for u and call the result y to get

$$y = \frac{x - T(\mathbf{p}, \mathbf{z}) - \mathbf{c}(\mathbf{p}, \mathbf{z})'[\nabla_{\mathbf{p}}\mathbf{c}(\mathbf{p}, \mathbf{z})']^{-1}[\mathbf{w} - \nabla_{\mathbf{p}} T(\mathbf{p}, \mathbf{z})]}{(1 + S(\mathbf{p}, \mathbf{z}) - \mathbf{c}(\mathbf{p}, \mathbf{z})'[\nabla_{\mathbf{p}}\mathbf{c}(\mathbf{p}, \mathbf{z})']^{-1}\nabla_{\mathbf{p}} S(\mathbf{p}, \mathbf{z})}$$

which we can write as

$$y = \frac{x - \overline{\mathbf{c}}(\mathbf{p}, \mathbf{z})'\mathbf{w} - \overline{t}(\mathbf{p}, \mathbf{z})}{1 + \overline{s}(\mathbf{p}, \mathbf{z})}$$

for appropriately defined functions $\overline{t}$, $\overline{s}$, and J-vector $\overline{\mathbf{c}}$, and with this definition of y we obtain implicit Marshallian demand

$$\mathbf{w} = \nabla_{\mathbf{p}}\mathbf{c}(\mathbf{p}, \mathbf{z})'\mathbf{m}(y) + \nabla_{\mathbf{p}} T(\mathbf{p}, \mathbf{z}) + \nabla_{\mathbf{p}} S(\mathbf{p}, \mathbf{z})\, y$$

This is a generalization of EASI demands where y is an affine transform of $x - \overline{\mathbf{c}}(\mathbf{p}, \mathbf{z})'\mathbf{w}$ instead of an affine transform of $x - \mathbf{p}'\mathbf{w}$. This generalization is useful for the closure under unit scaling property discussed later, and could be used to introduce additional interactions among y, $\mathbf{p}$, and $\mathbf{z}$ if required.

III. Coherency, Invertibility, and the Distribution of Errors

Coherency of a structural model is defined as the property that, for each value of the exogenous variables and errors, there exists a unique corresponding value of the endogenous variables. Invertibility is the property that a unique value of errors is associated with each value of the endogenous and exogenous variables. Coherency is essentially the condition that the model is fully specified, in the sense that the joint distribution of the endogenous variables is completely determined by the joint distribution of the errors and exogenous variables. Invertibility is the condition that permits uniquely recovering estimates of the errors given estimates of the model parameters. Since our model takes the errors to be random utility parameters, we will require invertibility to do policy analyses that depend upon these random utility parameters.

In the demand system context (see, e.g., van Soest, Kapteyn, and Kooreman (1993); Brown and Matzkin (1998); and Beckert and Blundell (2004)), coherency requires that a unique value of budget shares $\mathbf{w}$ be associated with each possible value of $\mathbf{p}, x, \mathbf{z}, \varepsilon$, and invertibility requires that a unique value of errors ε be associated with each possible value of $\mathbf{p}, x, \mathbf{z}, \mathbf{w}$.

THEOREM 7: *Assume a log cost function in the EASI class (B4) that is regular. Assume* $\tilde{\mathbf{m}} = \mathbf{m}(u, \mathbf{z}, \varepsilon)$ *is invertible in* ε, *so we may write* $\varepsilon = m^{-1}(u, \mathbf{z}, \tilde{\mathbf{m}})$. *Then the resulting budget share demand functions are coherent and invertible.*

PROOF OF THEOREM 7:

Invertibility of this class is established by observing that by equation (3)

$$\varepsilon = \mathbf{m}^{-1}(y, \mathbf{z}, \mathbf{w} - [\nabla_{\mathbf{p}} T(\mathbf{p})] - [\nabla_{\mathbf{p}} S(\mathbf{p})]\, y)$$

where y is given by equation (4). This uniquely defines ε as a function of $\mathbf{p}, x, \mathbf{z}, \mathbf{w}$. Coherency follows from $C(\mathbf{p}, u, \mathbf{z}, \varepsilon)$ having all of the properties of a regular cost function, which ensures existence of Marshallian demands (treating ε as preference parameters) and hence of a unique value of $\mathbf{w}$ associated with each possible value of $\mathbf{p}, x, \mathbf{z}, \varepsilon$, even though we do not have a closed form analytic expression for it.

Our functional form for empirical work, the cost function (5), satisfies the conditions of Theorem 7 assuming regularity holds on the support of $\mathbf{p}, x, \mathbf{z}, \varepsilon$ (which, e.g., is satisfied given Theorem 6). For that model $\mathbf{m}$ is invertible with, by equation (6), $\mathbf{m}^{-1}(u, \mathbf{z}, m) = \mathbf{m} - (\sum_{r=0}^{5} \mathbf{b}_r u^r) - \mathbf{Cz} - \mathbf{Dz}u$. More generally, invertibility is satisfied if $\mathbf{m}(y, \mathbf{z}, \varepsilon) = \mathbf{Bn}(y, \mathbf{z}) + \varepsilon$ for any parameter matrix $\mathbf{B}$ and functions $\mathbf{n}$.

For consistency of our estimator it was assumed that the distribution of ε is mean independent of $\mathbf{p}, x, \mathbf{z}$, and our 3SLS estimator is efficient (relative to GMM) if ε is homoskedastic. A sufficient condition for these to hold is that ε be distributed independently of $\mathbf{p}, x, \mathbf{z}$. Having ε independent of $\mathbf{p}$ is also appropriate for interpreting ε as preference heterogeneity parameters. Having ε be distributed independently of $\mathbf{p}, x, \mathbf{z}$ requires that the support of ε not depend on $\mathbf{p}, x, \mathbf{z}$. Theorem 7 does not impose this condition.

THEOREM 8: *Assume a log cost function in the EASI class* (2). *Then cost function regularity, the constraint that budget shares lie between zero and one, coherency, invertibility, and ε distributed independently of* $\mathbf{p}, x, \mathbf{z}$ *can all hold simultaneously.*

PROOF OF THEOREM 8:

Consider equation (2) cost functions with $\mathbf{m}(u, \mathbf{z}, \varepsilon) = \mathbf{Bn}(y, \mathbf{z}) + \varepsilon$ for some matrix of parameters $\mathbf{B}$ and vector of functions $\mathbf{n}(y, \mathbf{z})$. Our empirical model (5) is an example of this form. Then

$$\mathbf{w} = \mathbf{Bn}(y, \mathbf{z}) + \nabla_{\mathbf{p}} T(\mathbf{p}, \mathbf{z}) + \nabla_{\mathbf{p}} S(\mathbf{p}, \mathbf{z})\, y + \varepsilon. \tag{III.1}$$

Consider bounding each element of $\mathbf{Bn}(y, \mathbf{z}) + \nabla_{\mathbf{p}} T(\mathbf{p}, \mathbf{z}) + \nabla_{\mathbf{p}} S(\mathbf{p}, \mathbf{z}) y$ to lie inside some interval $[\tau, 1 - \tau]$ for every value of y in some positive interval $[\lambda_0, \lambda_1]$ and for every value of $\mathbf{p}, \mathbf{z}$ on their support, where $0 < \tau < ½$. The interval $[\lambda_0, \lambda_1]$ can be made arbitrarily large by restricting the range of values $\mathbf{n}(y, \mathbf{z})$ and $\nabla_{\mathbf{p}} S(\mathbf{p}, \mathbf{z})$ can take on. Let ε be drawn from any distribution that satisfies $\varepsilon' \mathbf{1}_J = 0$ and has the absolute value of each element of ε be less than τ. Let x be drawn from any distribution that has support sufficiently bounded to make $\tilde{y}$ defined by

$$\tilde{y} = \frac{x - \mathbf{p}'\tilde{\mathbf{w}} - T(\mathbf{p}, \mathbf{z}) + \mathbf{p}'[\nabla_{\mathbf{p}} T(\mathbf{p}, \mathbf{z})]}{1 + S(\mathbf{p}, \mathbf{z}) - \mathbf{p}'[\nabla_{\mathbf{p}} S(\mathbf{p}, \mathbf{z})]} \tag{III.2}$$

lie in $[\lambda_0, \lambda_1]$ for every value of of $\mathbf{p}, \mathbf{z}$ on their support and for every value of $\tilde{\mathbf{w}}$ on the unit simplex. Note that, as in the second part of Theorem 6, the units x is measured in can be adjusted to move the location of the support of x and hence of $\tilde{y}$. It then follows that y given by equation (4) will lie in $[\lambda_0, \lambda_1]$, and then by construction $\mathbf{w}$ defined by equation (III.1) will lie in the unit simplex.

Given the cost function (B4) with $\mathbf{m}(u, \mathbf{z}, \varepsilon) = \mathbf{Bn}(u, \mathbf{z}) + \varepsilon$ and the assumptions already made above, the remaining restrictions required for global regularity in Theorem 6 are $\mathbf{1}_J' \mathbf{Bn}(u, \mathbf{z}) = 1$, $\exp T(\mathbf{p}, \mathbf{z})$ and $\exp S(\mathbf{p}, \mathbf{z})$ homogeneous of degree zero in $\exp(\mathbf{p})$, $T(\mathbf{p}, \mathbf{z})$, and $S(\mathbf{p}, \mathbf{z})$ concave and differentiable in $\mathbf{p}$, and $1 + \inf[\mathbf{p}'\mathbf{B}\nabla_u \mathbf{n}(u, \mathbf{z})] > \inf[S(\mathbf{p}, \mathbf{z})]$. Given that the support of u lies inside an interval $[\lambda_0, \lambda_1]$ for any value ε can take on, we obtain global regularity for any ε on its support as long as these conditions (which do not depend directly on ε) hold for every $\mathbf{p}, \mathbf{z}$ on their support and every u in the interval $[\lambda_0, \lambda_1]$. We then also obtain coherency and invertibility, because the conditions of Theorem 7 all hold for this example.

None of the conditions imposed in the construction of the cost function and the distribution of ε in Theorem 8 conflict with assumptions imposed for consistency of the 3SLS estimates of our exact, symmetry restricted empirically estimated EASI demand functions. For example, the proof of Theorem 8 allows each ε to have conditional or unconditional mean zero, because it permits ε to be drawn from any distribution that satisfies $\varepsilon'\mathbf{1}_J = 0$ and has the absolute value of each element of ε be less than τ. However, having ε be distributed independently of $\mathbf{p}, x, \mathbf{z}$ means that for any value of $\mathbf{p}, x, \mathbf{z}$, the range of possible values of $\mathbf{w}$ is limited by the support of ε. For example, if the support of ε is such that each element of ε is less than 0.1 in absolute value, and if for a given $\mathbf{p}, x, \mathbf{z}$ we have each element of $\mathbf{Bn}(y, \mathbf{z}) + \nabla_{\mathbf{p}} T(\mathbf{p}, \mathbf{z}) + \nabla_{\mathbf{p}} S(\mathbf{p}, \mathbf{z}) y$ lying between 0.2 and 0.8 for any value of ε (which, along with $\mathbf{p}, x, \mathbf{z}$ determines y), then each element of $\mathbf{w}$ for consumers having this observed value of $\mathbf{p}, x, \mathbf{z}$ must lie between 0.1 and 0.9 (though consumers with different values of $\mathbf{p}, x, \mathbf{z}$ could have smaller or larger budget shares). This restriction on the conditional support of $\mathbf{w}$ is a general feature of virtually all nontrivial demand systems that map demand errors into unobserved preference heterogeneity parameters, such as Brown and Matzkin (1998) and Matzkin (2005).

Having ε distributed independently of $\mathbf{p}, x, \mathbf{z}$ implies conditional homoskedasticity of ε. Brown and Walker (1989) and Lewbel (2001) show that additive errors in Marshallian budget share demand systems must be heteroskedastic except under special circumstances (like homotheticity). Their results do not apply to the EASI model, because the EASI model has additive errors in implicit Marshallian (and hence in the Hicksian) budget share demand functions, rather than in the Marshallian budget share demand functions. Still, one might expect heteroskedastic errors, e.g., empirical demands may have the property noted by Hildenbrand (1994) of variance increasing in x. This need not conflict with the EASI model, or with regularity, coherence, or invertibility since all these conditions can be satisfied if $\mathbf{m}(y, \mathbf{z}, \varepsilon) = \mathbf{Bn}(y, \mathbf{z}) + \mathbf{N}(y, \mathbf{z}, \varepsilon)$ where $\mathbf{N}$ is mean zero and invertible in ε, which would then generate implicit Marshallian budget errors given by $\mathbf{N}$ instead of ε.

IV. The Rank of EASI Demand Systems

For clarity, in this section the heterogeneity parameters $\mathbf{z}, \varepsilon$ are held fixed, and so without loss of generality are omitted from the equations here. A demand system is defined to be exactly aggregable if it has Marshallian demands of the form $\mathbf{w} = \mathbf{G}(\mathbf{p})\,\mathbf{f}(x)$ for some matrix valued function $\mathbf{G}(\mathbf{p})$ and some vector valued function $\mathbf{f}(x)$. Gorman (1981) proved that for any exactly aggregable demand system that is derived from utility maximization, the maximum possible rank of the matrix $\mathbf{G}(\mathbf{p})$ is three. Generalizing Gorman, Lewbel (1991) defined the rank of any demand system to be dimension of the space spanned by its Engel curves. This definition coincides with Gorman's rank of $\mathbf{G}(\mathbf{p})$ for the special case of exactly aggregable demands. Lewbel (1991) shows that rank of the demand system corresponding to any cost function $C(\mathbf{p}, u)$ equals the smallest integer ℓ such that there exist functions $g, \theta_1, \ldots, \theta_\ell$, where $C(\mathbf{p}, u) = g[\theta_1(\mathbf{p}), \ldots, \theta_\ell(\mathbf{p}), u]$. The maximum possible rank of any demand system is the number of goods in the system minus one.

It follows by inspection of equations (2) and (5) that EASI demands are not constrained to have rank less than three, and can in fact have any rank, even when the EASI demands are linear in y. For example, for the reference type household the EASI demands we use for our empirical work has the cost function $C(\mathbf{p}, u) = \mathbf{p}'[\sum_{r=0}^{5} \mathbf{b}_r u^r] + ½\mathbf{p}'\mathbf{A}_0\mathbf{p} + ½\mathbf{p}'\mathbf{Bp}u$, which to minimize ℓ can be written as $C(\mathbf{p}, u) = (\mathbf{p}'\mathbf{b}_0 + ½\mathbf{p}'\mathbf{A}_0\mathbf{p}) + (\mathbf{p}'\mathbf{b}_1 + ½\mathbf{p}'\mathbf{Bp})u + \sum_{r=2}^{5} (\mathbf{p}'\mathbf{b}_r) u^r$, and so has rank $\ell = 6$.

V. Shape Invariance and Equivalence Scales

Shape-invariance is a property of demand functions that is relevant for the construction of equivalence scales, is convenient for semiparametric demand modelling, and has been found to at least approximately hold empirically in some data sets. See, e.g., Blundell and Lewbel (1991); and Blundell, Duncan, and Pendakur (1998); Pendakur (1999); and Blundell, Chen, and Kristensen (2003). Shape-invariance is satisfied if and only if Marshallian budget shares are identical across household types except for equation specific vertical translations and a horizontal translation that is common across equations.

In our notation, shape-invariance is satisfied if and only if the log-cost function may be written as $C(\mathbf{p}, u, \mathbf{z}, \varepsilon) = f[\mathbf{p}, h(u, \mathbf{z}, \varepsilon)] + G(\mathbf{p}, \mathbf{z}, \varepsilon)$. In this case, Hicks demands are given by $\omega(\mathbf{p}, u, \mathbf{z}, \varepsilon) = \nabla_{\mathbf{p}} f[\mathbf{p}, h(u, \mathbf{z}, \varepsilon)] + \nabla_{\mathbf{p}} G(\mathbf{p}, \mathbf{z}, \varepsilon)$. We may then invert C with respect to utility to obtain indirect utility $h(\cdot, u, \mathbf{z}, \varepsilon) = f^{-1}[\cdot, x - G(\mathbf{p}, \mathbf{z}, \varepsilon)]$ and substituting this expression into Hicks demands yields $\mathbf{w} = \nabla_{\mathbf{p}} f[\mathbf{p}, x - G(\mathbf{p}, \mathbf{z}, \varepsilon)] + \nabla_{\mathbf{p}} G(\mathbf{p}, \mathbf{z}, \varepsilon)$. This structure for demands may be very complex over x, but the characteristics $\mathbf{z}$ and ε enter demands in a very simple way: they translate expenditure shares vertically by the J-vector $\nabla_{\mathbf{p}} G(\mathbf{p}, \mathbf{z}, \varepsilon)$ and horizontally by the scalar $G(\mathbf{p}, \mathbf{z}, \varepsilon)$.

Define an equivalence scale E as the ratio of costs of a household with characteristics $\mathbf{z}, \varepsilon$ and a household with reference characteristics $\mathbf{0}_L, \mathbf{0}_J$, so that $\ln E(\mathbf{p}, u, \mathbf{z}, \varepsilon) = C(\mathbf{p}, u, \mathbf{z}, \varepsilon) - C(\mathbf{p}, u, \mathbf{0}_L, \mathbf{0}_J)$. If demands are shape invariant and the untestable restriction that $h(u, \mathbf{z}, \varepsilon) = \bar{h}(u)$ holds then $C(\mathbf{p}, u, \mathbf{z}, \varepsilon) = f[\mathbf{p}, \bar{h}(u)] + G(\mathbf{p}, \mathbf{z}, \varepsilon)$ so

$$\ln E(\mathbf{p}, u, \mathbf{z}, \varepsilon) = G(\mathbf{p}, \mathbf{z}, \varepsilon) \tag{V.1}$$

and therefore the equivalence scale depends only on prices and characteristics, but not on the utility level u. This property for E is called equivalence scale exactness (ESE) or independence-of-base (IB) by Blackorby and Donaldson (1993) and Lewbel (1989), respectively. Shape invariance is a necessary condition for IB/ESE.

EASI models can be shape invariant and can satisfy IB/ESE. The cost function (B4) satisfies shape-invariance if and only if S is independent of $\mathbf{z}$ and the vector-function $\mathbf{m}$ is additively separable into a vector-function $\mathbf{m}_1$ that depends on utility only and a vector-function $\mathbf{m}_2$ that depends on characteristics $\mathbf{z}, \varepsilon$ only. In this case, we have

$$\begin{aligned} C(\mathbf{p}, u, \mathbf{z}, \varepsilon) &= u + \mathbf{p}'\mathbf{m}_1(h(u, \mathbf{z}, \varepsilon)) + \mathbf{p}'\mathbf{m}_2(\mathbf{z}, \varepsilon) + T(\mathbf{p}, \mathbf{z}) + S(\mathbf{p})\, h(u, \mathbf{z}, \varepsilon) \\ &= [u + \mathbf{p}'\mathbf{m}_1(h(u, \mathbf{z}, \varepsilon)) + S(\mathbf{p})\, h(u, \mathbf{z}, \varepsilon)] + [\mathbf{p}'\mathbf{m}_2(\mathbf{z}, \varepsilon) + T(\mathbf{p}, \mathbf{z})]. \end{aligned}$$

This specification also satisfies IB/ESE if it satisfies the additional, untestable restriction that $h(u, \mathbf{z}, \varepsilon) = \overline{h}(u)$.

Shape-invariance is easily imposed on our empirically estimated model given by (5). In particular, that model has shape invariant demands if $\mathbf{D} = \mathbf{0}$. However, with $\mathbf{D} = \mathbf{0}$ the log equivalence-scale is given by $\ln E(\mathbf{p}, u, \mathbf{z}, \varepsilon) = \mathbf{p}'(\mathbf{Cz} + \varepsilon) + ½ \sum_{l=1}^{L} z_l \mathbf{p}'\mathbf{A}_l \mathbf{p}$, which takes on a fixed value at the base price vector, so $\ln E(\mathbf{0}_J, u, \mathbf{z}, \varepsilon) = 0$. To relax this implausible restriction, one could add a term linear in $\mathbf{z}$ to the log-cost function with $\mathbf{D} = \mathbf{0}$, so that

$$C(\mathbf{p}, u, \mathbf{z}, \varepsilon) = u + \mathbf{d}'\mathbf{z} + \mathbf{p}'\left[\sum_{r=0}^{5} \mathbf{b}_r u^r + \mathbf{Cz} + \varepsilon\right] + \frac{1}{2}\sum_{l=0}^{L} z_l\, \mathbf{p}'\mathbf{A}_l\, \mathbf{p} + \frac{1}{2}\mathbf{p}'\mathbf{Bp}u, \tag{V.2}$$

where $\mathbf{d}$ is a T-vector of parameters. The equivalence scale is then given by $\ln E(\mathbf{p}, u, \mathbf{z}, \varepsilon) = \mathbf{d}'\mathbf{z} + \mathbf{p}'(\mathbf{Cz} + \varepsilon) + ½\sum_{l=1}^{L} z_l\, \mathbf{p}'\mathbf{A}_l\, \mathbf{p}$, which takes on the value $\mathbf{d}'\mathbf{z}$ when evaluated at the base price vector. In this version of the model, implicit utility y includes the constant term $\mathbf{d}'\mathbf{z}$ and is given by

$$y = \frac{x - \mathbf{p}'\mathbf{w} - \mathbf{d}'\mathbf{z} + \sum_{l=0}^{L} z_l\, \mathbf{p}'\mathbf{A}_l\, \mathbf{p}/2}{1 - \mathbf{p}'\mathbf{Bp}/2}.$$

Other than this change in y, the demand functions are same as before, so the parameter vector $\mathbf{d}$ enters the model only through y. Estimates of the model in this form are available on request from the authors.

VI. Closure under Unit Scaling

A desirable feature of demand models is that they be closed under unit scaling, that is, that a change in the units that goods are measured in (or equivalently, a change in the base year or region where prices are normalized to equal one) only changes the values of the parameters or functions that define the model, leaving predicted values and estimated elasticities unchanged. See, e.g., Pollak and Wales (1980), especially footnote 15. The AID system is closed under unit scaling. The Quadratic AID of Banks, Blundell, and Lewbel (1997) is closed if the constant scalar parameter a_0 in that model is estimated, though in practice that parameter is usually fixed at some convenient value.

The parametric models proposed in this paper are not closed under unit scaling. To close them, we could replace $\mathbf{p}$ with $\mathbf{p} + \mathbf{k}$ everywhere that $\mathbf{p}$ appears in equations (2), (3), and (4), or in our empirically estimated model defined by equations (5), (8), and (9), where $\mathbf{k}$ is an additional J-vector of parameters to be estimated, with the free normalization $\mathbf{k}'\mathbf{1}_J = 0$. With the addition of $\mathbf{z}$ and ε terms, these are examples of equation (II.1) with $\overline{\mathbf{c}}(\mathbf{p}) = \mathbf{c}(\mathbf{p}) = \mathbf{p} + \mathbf{k}$.

To check possible sensitivity of our empirical results to unit scaling, we tried to reestimate our empirical model including the additional parameter vector $\mathbf{k}$, but in every attempt $\mathbf{k}$ was either completely insignificant or the model failed to converge. Also, reestimating the model (without $\mathbf{k}$) after changing the base year and region had little effect on the estimates, leaving Engel curve shapes unchanged and altering elasticities by a few percent at most. Like the Quadratic AID model with a_0 fixed, the parametric EASI models without $\mathbf{k}$ are approximately, though not exactly, closed under unit scaling. Formally, the objective function used for parameter estimation is relatively flat in directions corresponding to changes in $\mathbf{k}$.

To see why the parametric EASI models as estimated are almost closed, consider the general cost function

(VI.1) $$C(\mathbf{p},u,\mathbf{z},\varepsilon) = u + \mathbf{p}'\mathbf{m}(u,\mathbf{z},\varepsilon) + \mathbf{p}'[\mathbf{A}_1 + \mathbf{A}_2 h(u,\mathbf{z},\varepsilon)]\,\mathbf{p}/2$$

where h and the J-vector $\mathbf{m}$ are nonparametric functions. This cost function has demands that are closed under unit scaling (up to possible inequality constraints on the nonparametric functions) because

$$\mathrm{C}(\mathbf{p}+\mathbf{k},u,\mathbf{z},\varepsilon) = u + (\mathbf{p}+\mathbf{k})'\mathbf{m}(u,\mathbf{z},\varepsilon) + (\mathbf{p}+\mathbf{k})'[\mathbf{A}_1 + \mathbf{A}_2 h(u,\mathbf{z},\varepsilon)](\mathbf{p}+\mathbf{k})/2$$

$$u^* + \mathbf{p}'\mathbf{m}^*(u^*,\mathbf{z},\varepsilon) + \mathbf{p}'[\mathbf{A}_1 + \mathbf{A}_2 h^*(u^*,\mathbf{z},\varepsilon)]\,\mathbf{p}/2$$

where $u^* = u + \mathbf{k}'\mathbf{m}(u,\mathbf{z},\varepsilon) + \mathbf{k}'[\mathbf{A}_1 + \mathbf{A}_2 h(u,\mathbf{z},\varepsilon)]\,\mathbf{k}/2$, $\mathbf{m}^*(u^*,\mathbf{z},\varepsilon) = \mathbf{m}(u,\mathbf{z},\varepsilon) + [\mathbf{A}_1 + \mathbf{A}_2 h(u,\mathbf{z},\varepsilon)]\,\mathbf{k}$, and $h^*(u^*,\mathbf{z},\varepsilon) = h(u,\mathbf{z},\varepsilon)$, so by suitably redefining the functions $\mathbf{m}^*$ and h^*, adding $\mathbf{k}$ to the log price vector is equivalent to ordinally transforming u, which leaves the resulting demand functions unchanged. This paper's parametric EASI models are special cases of the cost function (31), so they fail to be closed under unit scaling only because $\mathbf{m}^*$ and h^* need not be contained in the same family of functional forms that are assumed for $\mathbf{m}$ and h. However, our flexible choice of these functions, particularly of $\mathbf{m}$, means that $\mathbf{m}^*$ and h^* can be closely approximated by suitable choice of parameterization of $\mathbf{m}$ and h, which explains the empirical finding that the numerical effects of violation of closure under unit scaling are very small.

VII. Marshallian Elasticity Calculations

Cost functions in the class of equation (21) have y given equation (8), so Marshallian demand functions $\mathbf{w}(\mathbf{p},x,\mathbf{z},\varepsilon)$ for these EASI models are implicitly given by

$$\mathbf{w}(\mathbf{p},x,\mathbf{z},\varepsilon) = \omega\left[\mathbf{p},\ \frac{x - \mathbf{p}'\mathbf{w}(\mathbf{p},x,\mathbf{z},\varepsilon) - \mathbf{p}'\mathbf{A}_1\mathbf{p}/2}{1 + \mathbf{p}'\mathbf{A}_2\mathbf{p}/2},\ \mathbf{z},\ \varepsilon\right]$$

Taking the total derivative of this expression with respect to any variable v gives

$$\nabla_v \mathbf{w}(\mathbf{p},x,\mathbf{z},\varepsilon) = \nabla_v \omega(\mathbf{p},y,\mathbf{z},\varepsilon) + [\nabla_y \omega(\mathbf{p},y,\mathbf{z},\varepsilon)]\left[\nabla_v\left(\frac{x - \mathbf{p}'\mathbf{w} - \mathbf{p}'\mathbf{A}_1\mathbf{p}/2}{1 + \mathbf{p}'\mathbf{A}_2\mathbf{p}/2}\right) - \left(\frac{x - \mathbf{p}'\nabla_v \mathbf{w}(\mathbf{p},x,\mathbf{z},\varepsilon)}{1 + \mathbf{p}'\mathbf{A}_2\mathbf{p}/2}\right)\right]$$

and solving for the Marshallian semielasticity $\nabla_v \mathbf{w}(\mathbf{p},x,\mathbf{z},\varepsilon)$ yields

(VII.1) $$\nabla_v \mathbf{w}(\mathbf{p},x,\mathbf{z},\varepsilon) = \left[I_J - \frac{[\nabla_y \omega(\mathbf{p},y,\mathbf{z},\varepsilon)]\,\mathbf{p}'}{1 + \mathbf{p}'\mathbf{A}_2\mathbf{p}/2}\right]^{-1}$$

$$\left[\nabla_v \omega(\mathbf{p},y,\mathbf{z},\varepsilon) + [\nabla_y \omega(\mathbf{p},y,\mathbf{z},\varepsilon)]\left[\nabla_v\left(\frac{x - \mathbf{p}'\mathbf{w} - \frac{\mathbf{p}'\mathbf{A}_1\mathbf{p}}{2}}{1 + \mathbf{p}'\mathbf{A}_2\mathbf{p}/2}\right) - \frac{x}{1 + \frac{\mathbf{p}'\mathbf{A}_2\mathbf{p}}{2}}\right]\right]$$

where I_J is the J by J identity matrix. In particular, taking v to be x above shows that, after algebraic simplification, the Marshallian semielasticity with respect to nominal expenditures x is

$$\nabla_x \mathbf{w}(\mathbf{p},x,\mathbf{z},\varepsilon) = \left(I_J - \frac{[\nabla_y \omega(\mathbf{p},y,\mathbf{z},\varepsilon)]\,\mathbf{p}'}{1 + \mathbf{p}'\mathbf{A}_2\mathbf{p}/2}\right)^{-1}\left(\frac{(1-x)\nabla_y \omega(\mathbf{p},y,\mathbf{z},\varepsilon)}{1 + \mathbf{p}'\mathbf{A}_2\mathbf{p}/2}\right)$$

where $\nabla_y \omega(\mathbf{p}, y, \mathbf{z}, \varepsilon)$ is given by equation (13). Equation (32) could also be evaluated taking v to be $\mathbf{p}$ to obtain Marshallian price elasticities, but it is simpler to recover them from the Hicksian $\mathbf{p}$ elasticities (12) and the above Marshallian x elasticities using the Slutsky matrix

$$\nabla_{\mathbf{p}'} \mathbf{w}(\mathbf{p}, x, \mathbf{z}, \varepsilon) = \nabla_{\mathbf{p}'} \omega(\mathbf{p}, u, \mathbf{z}, \varepsilon) - [\nabla_x \mathbf{w}(\mathbf{p}, x, \mathbf{z}, \varepsilon)]\, \omega(\mathbf{p}, y, \mathbf{z}, \varepsilon)'.$$

Finally, again using equation (32), the Marshallian semielasticity with respect to $\mathbf{z}$ is

$$\nabla_{\mathbf{z}} \mathbf{w}(\mathbf{p}, x, \mathbf{z}, \varepsilon) = \left(I_J - \frac{[\nabla_y \omega(\mathbf{p}, y, \mathbf{z}, \varepsilon)]\, \mathbf{p}'}{1 + \mathbf{p}'\mathbf{A}_2 \mathbf{p}/2} \right)^{-1} \left(\nabla_{\mathbf{z}} \omega(\mathbf{p}, y, \mathbf{z}, \varepsilon) - \frac{x \nabla_y \omega(\mathbf{p}, y, \mathbf{z}, \varepsilon)}{1 + \mathbf{p}'\mathbf{A}_2 \mathbf{p}/2} \right)$$

where $\nabla_y \omega(\mathbf{p}, y, \mathbf{z}, \varepsilon)$ and $\nabla_{\mathbf{z}} \omega(\mathbf{p}, y, \mathbf{z}, \varepsilon)$ are given by equations (13) and (14).

Some of the above elasticity expressions depend on ε either directly or through $\mathbf{w}$. Given estimated model parameters, ε for each consumer can be estimated as the model residuals (the difference between fitted and observed $\mathbf{w}$). We may therefore estimate mean elasticities in the population by calculating the estimated elasticities for each individual in the sample, plugging in their observed $\mathbf{w}$ or estimated ε where needed, and averaging the result. Other features of the population distribution of elasticities such as its median or variance can likewise be estimated from the corresponding empirical distribution.

If Marshallian demands are of direct interest, evaluated at points other than those observed in the sample, they can be obtained numerically by, e.g., numerically solving $x = C(\mathbf{p}, u, \mathbf{z}, \varepsilon)$ for u and substituting the result into the Hicksian demand functions. However, as the above equations show, this will often not be necessary for evaluating the effects on demand or welfare of price, expenditure, or demographic changes.