Entropy and the value of information for investors

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Web Appendix

1 Further proofs

1.1 Proof of Theorem 2

By Theorem 1 and a fixed prior p, the only possible index is $I(\alpha, p) = H(p) - \sum_s p_\alpha(s) H(q^s_\alpha)$. Therefore, it suffices to construct an example to show that this index orders two information structures in different ways for two different priors. The example follows.

Let $K = \{1, 2, 3\}$. Let $p_1 = (1/2, 1/2, 0)$ and $p_2 = (1/3, 1/3, 1/3)$ and an agent with $u(x) = \ln(x)$.

Let information structures α_1 and α_2 be described by these two-signal three-state matrices:

$$\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.5 & 0.5 \end{bmatrix},\ \alpha_2 = \begin{bmatrix} 0.3 & 0.7 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly, the expected utility for the agent with logarithmic utility under α_1 is larger than that for α_2 when priors are p_1 as the former gives her full information while the latter does not. It thus follows that:

$$I(\alpha_1, p_1) = H(p_1) - \sum_s p_{\alpha_1}(s) H(q^s_{\alpha_1}) > H(p_1) - \sum_s p_{\alpha_2}(s) H(q^s_{\alpha_2}) = I(\alpha_2, p_1).$$

What is the expected entropy of the posteriors generated by α_1 and α_2 under p_2? First, the utility for a ln agent of prior p_2 is $\ln(1/3)$. Then for α_1 the expected utility is:

$$\left(\frac{2}{3}\right) \ln\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right) \ln\left(\frac{1}{3}\right) = -0.63651.$$

Therefore,

$$H(p_2) - \sum_s p_{\alpha_1}(s) H(q^s_{\alpha_1}) = \frac{(2/3)\ln 2}{\ln 2} = \frac{0.462\,10}{\ln 2} = 2/3.$$

As for α_2, the (conditional on p_2) probability of either signal is 13/30 and 17/30. After she observes each signal, her posteriors are $(3/13, 0, 10/13)$ and $(7/17, 10/17, 0)$, respectively. Thus, her expected ln utility from α_2 is:

$$\left(\frac{13}{30}\right)\left(\left(\frac{3}{13}\right)\ln\left(\frac{3}{13}\right) + \left(\frac{10}{13}\right)\ln\left(\frac{10}{13}\right)\right)$$
$$+\left(\frac{17}{30}\right)\left(\left(\frac{7}{17}\right)\ln\left(\frac{7}{17}\right) + \left(\frac{10}{17}\right)\ln\left(\frac{10}{17}\right)\right) = -0.618\ .$$

Noting that $\ln(1/3)$ is the expected utility from the prior, we can derive:

$$H(p_2) - \sum_s p_{\alpha_2}(s) H(q^s_{\alpha_2}) = \frac{0.480\,61}{\ln 2}.$$

That is,

$$I(\alpha_1, p_2) = H(p_2) - \sum_s p_{\alpha_1}(s) H(q^s_{\alpha_1}) < H(p_2) - \sum_s p_{\alpha_2}(s) H(q^s_{\alpha_2}) = I(\alpha_2, p_2).$$

Hence, whereas for prior p_1 information structure α_1 is more informative than α_2, the opposite is true for prior p_2.$\blacksquare$

1.2 Proof of Theorem 4

Given a class $\mathcal{U} \subseteq U_0$ of utility functions, we say that an investment b individually satisfies NINI if for every $u \in \mathcal{U}$ and $w \in \mathbb{R}_+$ such that b is feasible,

$$\sum_k p(k) u(w + b_k) \leq u(w).$$

Thus, b individually satisfies NINI when, under no information, the agent does not prefer b to opting out. We denote as $\tilde{B}$ the set of investments that individually satisfy the NINI property. Since 0_K satisfies NINI, the NINI investment set is a nonempty investment set.

Lemma 1.1 *Given $\mathcal{U}$, $\mathcal{B}$ satisfies NINI if and only if $\mathcal{B}$ is the class of investment sets contained in $\tilde{B}$.*

Proof. $\mathcal{B}$ satisfies NINI if and only if it contains all the investment sets B such that for every $w > 0$ and $u \in \mathcal{U}$,

$$V(\underline{\alpha}, u, w, B) = 0.$$

That is, if B is such that for every $w > 0$, $u \in \mathcal{U}$, it is true that

$$\sup_{b \in B,\ b \text{ feasible}} \sum_k p(k) u(w + b_k) = 0.$$

An equivalent way to write the previous statement is: for every $w > 0, u \in \mathcal{U}$ and $b \in B$ feasible, then we have:

$$\sum_k p(k) u(w + b_k) \leq u(w),$$

which is finally equivalent to $B \subseteq \tilde{B}$. ■

Therefore, we assume from this point on that $\tilde{B}$ is the NINI investment set corresponding to a set of utility functions $\mathcal{U}$, and that $\mathcal{B}$ is the class of investment sets contained in $\tilde{B}$.

We say that a set $A \subseteq \mathbb{R}^K$ is *comprehensive* if, for every feasible b' and for every feasible $b \in A$ such that $b'_k \leq b_k$ for every k, we also have $b' \in A$.

Lemma 1.2 *$\tilde{B}$ is comprehensive.*

Proof. Assume that $b \in \tilde{B}$ and that b' is such that $b'_k \leq b_k$ for every k. Then, b is feasible at wealth w whenever b' is; and for every $u \in \mathcal{U}, w \in \mathbb{R}_+$,

$$\sum_k p(k) u(w + b'_k) \leq \sum_k p(k) u(w + b_k) \leq u(w).$$

Hence, $b' \in \tilde{B}$. ■

We observe that if $\tilde{B}$ is not investment-prone, neither is any element of $\mathcal{B}$, a subset of $\tilde{B}$. In this case, SCAI becomes trivially equivalent to $\mathcal{U} = \emptyset$. In contrast, the following proposition characterizes SCAI when $\tilde{B}$ is investment-prone.

Proposition 1.3 *If $\tilde{B}$ is investment-prone, then $\mathcal{U}$ satisfies SCAI if and only if $\mathcal{U} = \mathcal{U}^*$.*

Proof. We divide the proof into a series of lemmata.

Lemma 1.4 *Let $u \in \mathcal{U}_0$. If $u(0) > -\infty$, then for every w and for every B that is investment-prone and feasible, there exists an* always-uncertain α *such that*

$$V(u, w, \alpha, B) > 0.$$

Proof. Fix u, w, and the set B that is investment-prone and feasible.

For $1 > \varepsilon > 0$, let α^ε be defined by $S_{\alpha^\varepsilon} = K$, $\alpha_k^\varepsilon(s) = 1 - \varepsilon$ if $k = s$, and $\alpha_k(s) = \frac{\varepsilon}{K-1}$ otherwise. It can easily be verified that α^ε is *always uncertain* for every $\varepsilon > 0$, and that as $\varepsilon \to 0$, $q_{\alpha^\varepsilon}^k(k) \to 1$ for every s.

Since B is investment-prone, there exist k^* and $b^* \in B$ such that $b_{k^*}^* > 0$. We now have

$$\begin{aligned} v(u, w, B, q_{\alpha^\varepsilon}^{k^*}) &= \sup_{b \in B} \sum_k q_{\alpha^\varepsilon}^{k^*}(k) u(w + b_k) \\ &\geq \sum_k q_{\alpha^\varepsilon}^{k^*}(k) u(w + b_k^*) \\ &\geq q_{\alpha^\varepsilon}^{k^*}(k^*) u(w + b_{k^*}^*) + (1 - q_{\alpha^\varepsilon}^{k^*}(k^*)) u(0). \end{aligned}$$

Hence,

$$\lim_{\varepsilon \to 0} v(u, w, B, q_{\alpha^\varepsilon}^{k^*}) = u(w + b_{k^*}^*) > u(w),$$

which shows that for ε small enough, $v(u, w, B, q_{\alpha^\varepsilon}^{k^*}) > 0$ and therefore $V(u, w, \alpha^\varepsilon, B) > 0$. ■

Lemma 1.5 *Let $u \in \mathcal{U}_0$ and assume that $\tilde{B}$ is investment-prone. If $u(0) = -\infty$, then there exist w and an investment-prone set B that is feasible at w such that*

$$V(u, w, \alpha, B) = 0 \textit{ for every always-uncertain } \alpha.$$

Proof. Since $\tilde{B}$ is investment-prone, for every $k \in K$ there exists b'^k such that $b'^k_k > 0$. Let $b^+ = \min_k b'^k_k > 0$, and $b^- = \min(\min_{k \neq j} b'^k_j, -1) < 0$. The investment b^k given by $b_k^k = b^+$ and $b_j^k = b^-$ for every $j \neq k$ is such that for every j, $b_j^k \leq b'^k_j$. Since $\tilde{B}$ is comprehensive from Lemma 1.2, it follows that $b^k \in \tilde{B}$. Let B be the investment-prone set $B = \{0_K\} \cup \{b^k, k \in K\}$. B

is feasible at wealth-level $w = -b^-$. Let α be *always uncertain* and assume $u(0) = -\infty$. For every $s \in S_\alpha$ and for every $b^k \in B$, the expected utility from investing in b^k conditional on s is

$$\begin{aligned}\sum_{k'} q^s_\alpha(k')u(w + b^k_{k'}) &= q^s_\alpha(k)u(w + b^+) + (1 - q^s_\alpha(k))u(w + b^-)\\ &= q^s_\alpha(k)u(w + b^+) + (1 - q^s_\alpha(k))u(0)\\ &= -\infty.\end{aligned}$$

Thus, for every $s \in S_\alpha$,

$$v(u, w, B, q^s_\alpha) = u(w),$$

which implies that:

$$V(u, w, \alpha, B) = 0.$$

■

Lemmata 1.4 and 1.5 provide the proof of Proposition 1.3. ■

Lemma 1.6 *If $\mathcal{U} = \mathcal{U}^*$, then the only class $\tilde{B}$ that satisfies NINI is the class $\mathcal{B}^*$ of investment sets not subject to arbitrage.*

Proof. We need to show that the NINI investment set $\tilde{B}$ coincides with the set B^* of assets not subject to arbitrage.

For any $b \in B^*$, and for any $u \in \mathcal{U}^*$ and w such that b is feasible, u is concave and increasing. This implies:

$$\sum_k p(k)u(w + b_k) \leq u(w + \sum_k p(k)b_k) \leq u(w),$$

and hence, $b \in \tilde{B}$.

Now consider $b \in \tilde{B}$. Note that u given by $u(z) = \ln(z)$ for $z > 0$ is in $\mathcal{U}^*$. Hence, $b \in \tilde{B}$ implies that for every w large enough,

$$\sum_k p(k)\ln(w + b_k) \leq \ln(w),$$

which is equivalent to

$$\sum_k p(k)\ln(1 + \frac{b_k}{w}) \leq 0.$$

Hence, for every $\varepsilon > 0$ small enough,

$$\sum_k p(k) \ln(1 + \varepsilon b_k) \leq 0.$$

A first-order Taylor expansion shows that this implies

$$\sum_k p(k) b_k \leq 0,$$

and hence, $b \in B^*$. ■

To wrap up the proof of Theorem 4, assume first that $\mathcal{U}$ and $\mathcal{B}$ satisfy NINI and SCAI. Then from Lemma 1.1, $\mathcal{B}$ is the class of investment sets contained in $\tilde{B}$. If $\tilde{B}$ is not investment-prone, then $\mathcal{U} = \emptyset$, in which case $\tilde{B} = \mathbb{R}^K$, a contradiction. Hence, $\tilde{B}$ is investment-prone, and from Proposition 1.3, $\mathcal{U} = \mathcal{U}^*$. Finally, it follows from Lemma 1.6 that $\mathcal{B} = \mathcal{B}^*$.

We now show that $\mathcal{U}^*$ and $\mathcal{B}^*$ satisfy NINI and SCAI. With the assumption that $\mathcal{B} = \mathcal{B}^*$, $\mathcal{U}^*$ satisfies SCAI from Proposition 1.3. With $\mathcal{U} = \mathcal{U}^*$, $\mathcal{B}^*$ satisfies NINI from Lemma 1.6.■