The Environment and Directed Technical Change

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Web Appendix

APPENDIX B: OMITTED PROOFS AND FURTHER DETAILS (NOT FOR PUBLICATION)

B1. Allocation of scientists in laissez-faire equilibrium when the inputs are complementary ($\varepsilon < 1$)

Under Assumption 1 and if $\varepsilon < 1$, there is a unique equilibrium in laissez-faire where innovation first occurs in the clean sector, then occurs in both sectors, and asymptotically the share of scientists devoted to the clean sector is given by $s_c = \eta_d / \left(\eta_c + \eta_d\right)$; the long-run growth rate of dirty input production in this case is $\gamma \widetilde{\eta}$, where $\widetilde{\eta} \equiv \eta_c \eta_d / \left(\eta_c + \eta_d\right)$.

This proposition is proved using the following lemma:

When $\varepsilon < 1$, long-run equilibrium innovation will be in both sectors, so that the equilibrium share of scientists in the clean sector converges to $s_c = \eta_d/(\eta_c + \eta_d)$.

Suppose that at time t innovation occurred in both sectors so that $\Pi_{ct}/\Pi_{dt} = 1$. Then from (18), we have

$$\frac{\Pi_{ct+1}}{\Pi_{dt+1}} = \left(\frac{1+\gamma\,\eta_c s_{ct+1}}{1+\gamma\,\eta_d s_{dt+1}}\right)^{-\varphi-1} \left(\frac{1+\gamma\,\eta_c s_{ct}}{1+\gamma\,\eta_d s_{dt}}\right).$$

Innovation will therefore occur in both sectors at time $t+1$ whenever the equilibrium allocation of scientists (s_{ct+1}, s_{dt+1}) at time $t+1$ is such that

$$\text{(B.1)} \qquad \frac{1+\gamma\,\eta_c s_{ct+1}}{1+\gamma\,\eta_d s_{dt+1}} = \left(\frac{1+\gamma\,\eta_c s_{ct}}{1+\gamma\,\eta_d s_{dt}}\right)^{\frac{1}{\varphi+1}}.$$

This equation defines $s_{ct+1}(= 1-s_{dt+1})$ as a function of $s_{ct}(= 1-s_{dt})$. We next claim that this equation has an interior solution $s_{ct+1} \in (0,1)$ when $s_{ct} \in (0,1)$ (i.e., when s_{ct} is itself interior). First, note that when $\varphi > 0$ (that is, $\varepsilon < 1$), the function $z(x) = x^{1/(\varphi+1)} - x$ is strictly decreasing for $x < 1$ and strictly increasing for $x > 1$. Therefore, $x = 1$ is the unique positive solution to $z(x) = 0$. Second, note also that the function

$$X(s_{ct}) = \frac{1+\gamma\,\eta_c s_{ct}}{1+\gamma\,\eta_d s_{dt}} = \frac{1+\gamma\,\eta_c s_{ct}}{1+\gamma\,\eta_d (1-s_{ct})},$$

is a one-to-one mapping from (0, 1) onto $(\left(1+\gamma\,\eta_d\right)^{-1}, 1+\gamma\,\eta_c)$. Finally, it can be verified that whenever $X \in (\left(1+\gamma\,\eta_d\right)^{-1}, 1+\gamma\,\eta_c)$, we also have $X^{1/(\varphi+1)} \in (\left(1+\gamma\,\eta_d\right)^{-1}, 1+$

$\gamma\,\eta_c)$. This, together with (B.1), implies that if $s_{ct} \in (0, 1)$, then $s_{ct+1} = X^{-1}(X(s_{ct})^{1/(\varphi+1)}) \in (0, 1)$, proving the claim at the beginning of this paragraph.

From Appendix A, when $\varphi > 0$, the equilibrium allocation of scientists is unique at each t. Thus as $t \to \infty$, this allocation must converge to the unique fixed point of the function $Z(s) = X^{-1} \circ (X(s))^{\frac{1}{\varphi+1}}$, which is

$$s_c = \frac{\eta_d}{\eta_c + \eta_d}.$$

This completes the proof of the lemma.

Now given the characterization of the equilibrium allocations of scientists in Appendix A, under Assumption 1 the equilibrium involves $s_{dt} = 0$ and $s_{ct} = 1$, i.e., innovation occurs initially in the clean sector only. >From (11), A_{ct}/A_{dt} will grow at a rate $\gamma\,\eta_c$, and in finite time, it will exceed the threshold $\left(1+\gamma\,\eta_c\right)^{-(\varphi+1)/\varphi}\left(\eta_c/\eta_d\right)^{1/\varphi}$. Lemma B.B1 implies that when this ratio is in the interval $\left(\left(1+\gamma\,\eta_c\right)^{-(\varphi+1)/\varphi}\left(\eta_c/\eta_d\right)^{1/\varphi}, \left(\eta_c/\eta_d\right)^{1/\varphi}\left(1+\gamma\,\eta_d\right)^{(\varphi+1)/\varphi}\left(\eta_c/\eta_d\right)^{1/\varphi}\right)$, equilibrium innovation occurs in both sectors, i.e., $s_{dt} > 0$ and $s_{ct} > 0$, and from this point onwards, innovation will occur in both sectors and the share of scientists devoted to the clean sector converges to $\eta_d/(\eta_d + \eta_c)$. This completes the proof of Proposition B.B1.

B2. Speed of disaster in laissez-faire

>From the expressions in (19), dirty input production is given by:

$$Y_{dt} = \left(A_{ct}^{\varphi} + A_{dt}^{\varphi}\right)^{-\frac{\alpha+\varphi}{\varphi}} A_{ct}^{\alpha+\varphi} A_{dt} = \frac{A_{dt}}{\left(1+\left(\frac{A_{dt}}{A_{ct}}\right)^{\varphi}\right)^{\frac{\alpha+\varphi}{\varphi}}}.$$

When the two inputs are gross substitutes ($\varepsilon < 1$), we have $\varphi = \varphi^{su} < 0$, whereas when they are complements ($\varepsilon > 1$), we have $\varphi = \varphi^{co} > 0$. Since all innovations occur in the dirty sector in the substitutability case, but not in the complementarity case, if we start with the same levels of technologies in both cases, at any time $t > 0$ we have $A_{dt}^{su} > A_{dt}^{co}$ and $A_{ct}^{su} < A_{ct}^{co}$, where A_{kt}^{su} and A_{kt}^{co} denote the average productivities in sector k at time t respectively in the substitutability and in the complementarity case, starting from the same initial productivities $A_{k0}^{su} = A_{k0}^{co}$.

Assumption 1 implies that

$$\left(\frac{A_{dt}^{su}}{A_{ct}^{su}}\right)^{\varphi^{su}} < \frac{\eta_d}{\eta_c} \leq \left(\frac{A_{dt}^{co}}{A_{ct}^{co}}\right)^{\varphi^{co}}$$

so that

$$Y_{dt}^{su} = \frac{A_{dt}^{su}}{\left(1+\left(\frac{A_{dt}^{Su}}{A_{ct}^{su}}\right)^{\varphi^{su}}\right)^{\frac{\alpha}{\varphi^{su}}+1}} > \frac{A_{ct}^{su}}{\left(1+\left(\frac{A_{ct}^{co}}{A_{dt}^{co}}\right)^{\varphi^{co}}\right)}\left(1+\left(\frac{A_{dt}^{Su}}{A_{ct}^{su}}\right)^{\varphi^{su}}\right)^{-\frac{\alpha}{\varphi^{su}}}$$

$$> \frac{A_{ct}^{su}}{\left(1+\left(\frac{A_{ct}^{co}}{A_{dt}^{co}}\right)^{\varphi^{co}}\right)} > \frac{A_{ct}^{su}}{\left(1+\left(\frac{A_{ct}^{co}}{A_{dt}^{co}}\right)^{\varphi^{c0}}\right)^{\frac{\alpha}{\varphi^{c0}}+1}} > Y_{dt.}^{co}$$

Repeating the same argument for $t+1$, $t+2$,..., we have that $Y_{dt}^{su} > Y_{dt}^{co}$ for all t. This establishes that, under Assumption 1, there will be a greater amount of dirty input production for each t when $\varepsilon > 1$ than when $\varepsilon < 1$, implying that an environmental disaster will occur sooner when the two sectors are gross substitutes.

B3. *Proof of Proposition 4*

Using the fact that the term $\Omega(S_t)$ premultiplies all A's, equation (19) is now be replaced by:

$$Y_{dt} = \Omega(S_t)^{\frac{1}{1-\alpha}}\left(A_{ct}^{\varphi} + A_{dt}^{\varphi}\right)^{-\frac{\alpha+\varphi}{\varphi}} A_{ct}^{\alpha+\varphi} A_{dt}, \text{ and } Y_t = \Omega(S_t)^{\frac{1}{1-\alpha}}\left(A_{ct}^{\varphi} + A_{dt}^{\varphi}\right)^{-\frac{1}{\varphi}} A_{ct} A_{dt}.$$

In particular, as in Section II, under laissez-faire, all innovation is directed towards the dirty sector, A_{dt} grows to infinity. Then, an environmental disaster can only be avoided if Y_{dt} and thus $\Omega(S_t)^{1/(1-\alpha)} A_{dt}$ remain bounded. Since A_{dt} is growing exponentially, this is only possible if S_t converges to 0. Now, suppose that $\Omega(S_t)^{1/(1-\alpha)} A_{dt}$ converges to a finite value as time t goes to infinity. Then there exists $\eta > 0$ such that for any T there exists $\upsilon > T$ such that $\Omega(S_\upsilon)^{1/(1-\alpha)} A_d > \eta/\xi$. But for $\upsilon > T$ sufficiently high, we also have
$\left|Y_{d\upsilon} - \Omega(S_\upsilon)^{1/(1-\alpha)} A_{d\upsilon}\right| < \eta/(3\xi)$ since, asymptotically, $Y_{dt} \simeq \Omega(S_t)^{1/(1-\alpha)} A_{dt}$, and $(1+\delta) S_\upsilon < \eta/3$ as S_t converges to 0. But then (12) gives $S_{\upsilon+1} = 0$, which corresponds to an environmental disaster. Consequently, to avoid a disaster under laissez-faire, it must be the case that $\Omega(S_t)^{1/(1-\alpha)} A_{dt}$ converges to 0. But this implies that Y_t converges to 0 as well, and so does C_t.

B4. *Proof of Proposition 6*

First we need to derive the optimal production of inputs given technologies and the tax implemented. Using (A.9) and (A.10), the shadow values of clean and dirty inputs satisfy

$$\widehat{p}_{ct}^{1-\varepsilon} + \left(\widehat{p}_{dt}(1+\tau_t)\right)^{1-\varepsilon} = 1. \quad \text{(B.2)}$$

This, together with (A.15), yields

$$\widehat{p}_{dt} = \frac{A_{ct}^{1-\alpha}}{\left(A_{ct}^{\varphi}\left(1+\tau_t\right)^{1-\varepsilon} + A_{dt}^{\varphi}\right)^{\frac{1}{1-\varepsilon}}} \text{ and } \widehat{p}_{ct} = \frac{A_{dt}^{1-\alpha}}{\left(A_{ct}^{\varphi}\left(1+\tau_t\right)^{1-\varepsilon} + A_{dt}^{\varphi}\right)^{\frac{1}{1-\varepsilon}}}. \tag{B.3}$$

Using (7), (A.12), (A.16) and (B.3), we obtain

$$Y_{ct} = \left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}} \frac{\left(1+\tau_t\right)^{\varepsilon} A_{ct} A_{dt}^{\alpha+\varphi}}{\left(A_{dt}^{\varphi} + \left(1+\tau_t\right)^{1-\varepsilon} A_{ct}^{\varphi}\right)^{\frac{\alpha}{\varphi}} \left(A_{ct}^{\varphi} + \left(1+\tau_t\right)^{\varepsilon} A_{dt}^{\varphi}\right)}, \text{ and} \tag{B.4}$$

$$Y_{dt} = \left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}} \frac{A_{ct}^{\alpha+\varphi} A_{dt}}{\left(A_{dt}^{\varphi} + \left(1+\tau_t\right)^{1-\varepsilon} A_{ct}^{\varphi}\right)^{\frac{\alpha}{\varphi}} \left(A_{ct}^{\varphi} + \left(1+\tau_t\right)^{\varepsilon} A_{dt}^{\varphi}\right)}. \tag{B.5}$$

Equation (B.5) implies that the production of dirty input is decreasing in τ_t. Moreover, clearly as $\tau_t \to \infty$, we have $Y_{dt} \to 0$.

We next characterize the behavior of this tax rate and the research subsidy, q_t. Recall that to avoid an environmental disaster, the optimal policy must always ensure that Y_{dt} remained bounded, in particular, $Y_{dt} \leq (1+\delta)\,\overline{S}/\xi$.

Assume $\varepsilon > 1$. The proof consists of six parts: (1) We show that, for a discount rate ρ sufficiently low, the optimal allocation cannot feature a bounded Y_{ct}, thus Y_{ct} must be unbounded as t goes to infinity. (2) We show that this implies that A_{ct} must tend towards infinity. (3) We show that if the optimal allocation involves Y_{ct} unbounded (i.e $\limsup Y_{ct} = \infty$), then it must be the case that at the optimum $Y_{ct} \to \infty$ as t goes to infinity. (4) We prove that the economy switches towards clean research, that is, $s_{ct} \to 1$. (5) We prove that the switch in research to clean technologies occurs in finite time, that is, there exists $\tilde{T}$ such that $s_{ct} = 1$ for all $t \geq \tilde{T}$. (6) We then derive the implied behavior of τ_t and q_t.

Part 1: To obtain a contradiction, suppose that the optimal allocation features Y_{ct} remaining bounded as t goes to infinity. If Y_{dt} was unbounded, then there would be an environmental disaster, but then the allocation could not be optimal in view of the assumption that $\lim_{S\downarrow 0} u(C,S) = -\infty$ (equation (2)). Thus Y_{dt} must also remain bounded as t goes to infinity. But if both Y_{ct} and Y_{dt} remain bounded, so will Y_t and C_t. We use the superscript ns (ns for "no switch") to denote the variables under this allocation.

Consider an alternative (feasible) allocation, featuring all research being directed to clean technologies after some date $\hat{t}$, i.e., $s_{ct} = 1$ for all $t > \hat{t}$ and no production of dirty input (by taking an infinite carbon tax τ_t). This in turn implies that S_t reaches $\overline{S}$ in finite time because of regeneration at the rate δ in (12). Moreover, (B.4) implies that $Y_t/A_{ct} \to$ constant and thus $C_t/A_{ct} \to$ constant. Let us use superscript a to denote all variables under this alternative allocation. Then there exists a consumption level $\overline{C} < \infty$, and a date $T < \infty$ such that for $t \geq T$, $C_t^{ns} < \overline{C}$, $C_t^{a} > \overline{C} + \theta$ (where $\theta > 0$) and $S_t^{a} = \overline{S}$. Now using the fact that u is strictly increasing in C and S, for all $t \geq T$ we have

$$u\left(C_t^a, S_t^a\right) - u\left(C_t^{ns}, S_t^{ns}\right) \geq u\left(C_t^a, \overline{S}\right) - u\left(\overline{C}, \overline{S}\right) > 0$$

which is positive and strictly increasing over time. Then the welfare difference between the alternative and the no-switch allocations can be written as

$$\begin{aligned} W^a - W^{ns} &= \sum_{t=0}^{T-1} \frac{1}{(1+\rho)^t}\left(u\left(C_t^a, S_t^a\right) - u\left(C_t^{ns}, S_t^{ns}\right)\right) + \sum_{t=T}^{\infty} \frac{1}{(1+\rho)^t}\left(u\left(C_t^a, S_t^a\right) - u\left(C_t^{ns}, S_t^{ns}\right)\right) \\ &\geq \sum_{t=0}^{T-1} \frac{1}{(1+\rho)^t}\left(u\left(C_t^a, S_t^a\right) - u\left(C_t^{ns}, S_t^{ns}\right)\right) + \frac{1}{(1+\rho)^T} \sum_{t=T}^{\infty} \frac{1}{(1+\rho)^{t-T}}\left(u\left(C_t^a, \overline{S}\right) - u\left(\overline{C}, \overline{S}\right)\right). \end{aligned}$$

Since the utility function is continuous in C, and C_t^{ns} is finite for all $t < T$ (for all ρ), then as ρ decreases the first term remains bounded above by a constant, while the second term tends to infinity. This establishes that $W^a - W^{ns} > 0$ for ρ sufficiently small, yielding a contradiction and establishing that we must have Y_{ct} unbounded when t goes to infinity.

Part 2: Now (B.4) directly implies that

$$A_{ct} \geq g\left(Y_{ct}\right) = \left(\frac{\alpha}{\psi}\right)^{\frac{-\alpha}{1-\alpha}} Y_{ct}\left(1 + \left(\frac{Y_{ct}}{M}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right)^{\frac{\alpha}{\varphi}}$$

where M is an upper-bound on Y_{dt}. g is an increasing function and $\limsup Y_{ct} = \infty$, so $\limsup A_{ct} = \infty$ and as A_{ct} is weakly increasing, $\lim A_{ct} = \infty$.

Part 3: Now suppose by contradiction that $\liminf Y_{ct} \neq \infty$, then by definition if must be the case that $\exists M'$ such that $\forall T$, $\exists t > T$ with $Y_{ct} < M'$. Let us consider such an M' and note that we can always choose it to be higher than the upper bound on Y_{dt}. Then we can define a subsequence t_n with $t_n \geq n$ and $Y_{ct_n} < M'$ for all n. Since $Y_{dt} < M'$ as well, we have that for all n: $C_{t_n} < Y_{t_n} < 2^{\varepsilon/(\varepsilon-1)}M'$. Moreover, since $\lim_{t\to\infty} A_{ct} = \infty$, there exists an integer υ such that for any $t > \upsilon$, $A_{ct} > (\alpha/\psi)^{-\alpha/(1-\alpha)}\, 2^{\varepsilon/(\varepsilon-1)}M'/\left(1-\alpha\right)$. Consequently, for $n \geq \upsilon$ we have: $C_{t_n} < Y_{t_n} < 2^{\varepsilon/(\varepsilon-1)}M'$ and $A_{ct_n} > (\alpha/\psi)^{-\alpha/(1-\alpha)}\, 2^{\varepsilon/(\varepsilon-1)}M'/\left(1-\alpha\right)$.

Consider now the alternative policy that mimics the initial policy, except that in all periods t_n for $n \geq \upsilon$ the social planner chooses the carbon tax $\tau_{t_n}^a$ to be sufficiently large (the superscript a designates "alternative") that $Y_{dt_n}^a = 0$. Then we have: $Y_{t_n}^a = Y_{ct_n}^a = (\alpha/\psi)^{\alpha/(1-\alpha)} A_{ct_n}$, which yields $S_t^a \geq S_t$ for all $t \geq t_n$ since the alternative policy either reduces or maintains dirty input production relative to the original policy. Moreover, we have: $C_{t_n}^a = (1-\alpha)Y_{t_n}^a \geq (1-\alpha)\left(\alpha/\psi\right)^{\alpha/(1-\alpha)} A_{ct_n} > 2^{\varepsilon/(\varepsilon-1)}M' > C_{t_n}$, whereas consumption in periods $t \neq t_n$ remains unchanged. Thus the alternative policy leads to (weakly) higher consumption and environmental quality in all periods, and to strictly higher consumption in periods $t = t_n$, thus overall to strictly higher welfare, than the original policy. Hence the original policy is not optimal, using a contradiction. This in turn establishes that on the optimal path $\liminf Y_{ct} = \infty$ and therefore $\lim Y_{ct} = \infty$.

Part 4: From Part 3 we know that on the optimal path $Y_{ct}/Y_{dt} \to \infty$, that is $(1+\tau_t)^{1-\varepsilon}(A_{ct}/A_{dt})^{\varphi} \to 0$. Now from (B.4) and (B.5), one can reexpress consumption as a function of the carbon tax and technologies:

$$C_t = \left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}} \frac{A_{ct}A_{dt}}{\left((1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi} + A_{dt}^{\varphi}\right)^{\frac{1}{\varphi}}} \left(1-\alpha + \frac{\tau_t A_{ct}^{\varphi}}{A_{ct}^{\varphi} + (1+\tau_t)^{\varepsilon} A_{dt}^{\varphi}}\right); \tag{B.6}$$

Since $(1+\tau_t)(A_{ct}/A_{dt})^{1-\alpha} \to \infty$, we get

$$\lim \frac{C_t}{A_{ct}} = \left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha)$$

Now by contradiction let us suppose that $\liminf s_{ct} = s < 1$.Then for any $\tilde{T}$ there exists $\upsilon > \tilde{T}$, such that $s_{c\upsilon} < (1+s)/2$. Now, as $\lim(C_t/A_{ct}) = (\alpha/\psi)^{\alpha/(1-\alpha)}(1-\alpha)$, there exists some T such that for any $t > T$, we have $C_t < (\alpha/\psi)^{\alpha/(1-\alpha)}(1-\alpha)A_{ct}\left(1+\gamma\eta_c\right)/\left(1+\gamma\eta_c(1+s)/2\right)$. Then take υ sufficiently large that $\upsilon > T$ and $s_{c\upsilon} < (1+s)/2$, and consider the following alternative policy: the alternative policy is identical to the original policy up to time $\upsilon - 1$, then at υ, the alternative policy allocates all research to the clean sector, and for $t > \upsilon$, the allocation of research is identical to the original policy, and for $t \geq \upsilon$, the carbon tax is infinite. Then under the alternative policy, there is no pollution for $t \geq \upsilon$ so the quality of the environment is weakly better than under the original policy. Moreover: $A_{ct}^{a} = \left(1+\gamma\eta_c\right)A_{ct}/\left(1+\gamma\eta_c s_{c\upsilon}\right)$, for all $t \geq \upsilon$ (where the superscript a indicates the alternative policy schedule). Thus for $t \geq \upsilon$:

$$\begin{aligned} C_t^{a} &= \left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)A_{ct}^{a} > \left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)\frac{1+\gamma\eta_c}{1+\gamma\eta_c s_{c\upsilon}}A_{ct} \\ &> \left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)\frac{1+\gamma\eta_c}{1+\gamma\eta_c\left(\frac{1+s}{2}\right)}A_{ct} > C_t, \end{aligned}$$

so that the alternative policy brings higher welfare. This in turn contradicts the optimality of the original policy. Hence $\liminf s_{ct} = 1$, so $\lim s_{ct} = 1$, and consequently, $\lim(A_{ct}^{\varphi}/A_{dt}^{\varphi}) = 0$.

Part 5: First note that (B.5) and (B.6) can be rewritten as:

(B.7)

$$\begin{aligned} \ln(C_t) - \ln\left(\left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}}\right) &= \ln(A_{ct}) + \ln(A_{dt}) \\ &\quad -\frac{1}{\varphi}\ln\left(\left((1+\tau_t)^{1-\varepsilon}A_{ct}^{\varphi} + A_{dt}^{\varphi}\right)\right) + \ln\left(1-\alpha+\frac{\tau_t A_{ct}^{\varphi}}{A_{ct}^{\varphi}+(1+\tau_t)^{\varepsilon}A_{dt}^{\varphi}}\right), \end{aligned}$$

(B.8)
$$\ln(Y_{dt}) - \ln\left(\left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}}\right) = (\alpha+\varphi)\ln(A_{ct}) + \ln(A_{dt}) - \frac{\alpha}{\varphi}\ln\left(\left(A_{dt}^{\varphi} + (1+\tau_t)^{1-\varepsilon}A_{ct}^{\varphi}\right)\right) - \ln\left(\left(A_{ct}^{\varphi} + (1+\tau_t)^{\varepsilon}A_{dt}^{\varphi}\right)\right).$$

Now, suppose that s_{ct} does not reach 1 in finite time. Then for any T, there exists $\upsilon > T$, such that $s_{c\upsilon} < 1$. For T arbitrarily large $s_{c\upsilon}$ becomes arbitrarily close to 1, so that $1 - s_{c\upsilon}$ becomes infinitesimal and is accordingly denoted ds. We then consider the following thought experiment: let us increase the allocation of researchers to clean innovation at υ from $s_{c\upsilon} < 1$ to 1, but leave this allocation unchanged in all subsequent periods. Meanwhile, let us adjust the tax τ_t in all periods after υ in order to leave Y_{dt} unchanged. Then using superscript a to denote the value of technologies under the alternative policy, we have for $t \geq \upsilon$:

$$A_{ct}^{a} = \frac{1+\gamma\eta_c}{1+\gamma\eta_c s_{c\upsilon}}A_{ct}.$$

A first-order Taylor expansion of the logarithm of the productivity around $s_{c\upsilon} = 1$ yields:

$$d(\ln(A_{ct})) = \frac{\gamma\eta_c ds}{1+\gamma\eta_c} + o(ds), \tag{B.9}$$

and similarly,

$$d(\ln(A_{dt})) = -\gamma\eta_d ds + o(ds).$$

Using the fact that that $d(\ln(A_{ct}))$ and $d(\ln(A_{dt}))$ are of the same order as ds, first-order Taylor expansions of (B.7) and (B.8) give:

(B.10)
$$\begin{aligned} d(\ln(C_t)) = {} & d(\ln(A_{ct})) + d(\ln(A_{dt})) \\ & - \frac{(1+\tau)^{1-\varepsilon}A_{ct}^{\varphi}\left(\varphi d(\ln(A_{ct})) + (1-\varepsilon)d(\ln(1+\tau_t))\right) + \varphi A_{dt}^{\varphi}d(\ln(A_{dt}))}{\varphi\left((1+\tau_t)^{1-\varepsilon}A_{ct}^{\varphi} + A_{dt}^{\varphi}\right)} \\ & + \frac{1}{1-\alpha+\frac{\tau_t A_{ct}^{\varphi}}{A_{ct}^{\varphi}+(1+\tau_t)^{\varepsilon}A_{dt}^{\varphi}}} \frac{(1+\tau_t)A_{ct}^{\varphi}d(\ln(1+\tau_t)) + \varphi\tau_t A_{ct}^{\varphi}d(\ln(A_{ct}))}{A_{ct}^{\varphi}+(1+\tau_t)^{\varepsilon}A_{dt}^{\varphi}} \\ & - \frac{\tau_t A_{ct}^{\varphi}}{1-\alpha+\frac{\tau_t A_{ct}^{\varphi}}{A_{ct}^{\varphi}+(1+\tau_t)^{\varepsilon}A_{dt}^{\varphi}}} \frac{\varphi A_{ct}^{\varphi}d(\ln(A_{ct})) + (1+\tau_t)^{\varepsilon}A_{dt}^{\varphi}\left(\varphi d(\ln(A_{dt})) + \varepsilon d(\ln(1+\tau_t))\right)}{\left(A_{ct}^{\varphi}+(1+\tau_t)^{\varepsilon}A_{dt}^{\varphi}\right)^2} \\ & + o(ds) + o(d(\ln(1+\tau_t))), \end{aligned}$$

and

$$
\begin{aligned}
d(\ln(Y_{dt})) &= (\alpha+\varphi)\, d(\ln(A_{ct})) + d(\ln(A_{dt})) \\
&- \frac{(1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi}\left(\varphi d(\ln(A_{ct})) + (1-\varepsilon)\, d(\ln(1+\tau_t))\right) + \varphi A_{dt}^{\varphi} d(\ln(A_{dt}))}{\varphi\alpha^{-1}\left((1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi} + A_{dt}^{\varphi}\right)} \\
&- \frac{\varphi A_{ct}^{\varphi} d(\ln(A_{ct})) + (1+\tau_t)^{\varepsilon} A_{dt}^{\varphi}\left(\varphi d(\ln(A_{dt})) + \varepsilon d(\ln(1+\tau_t))\right)}{A_{ct}^{\varphi} + (1+\tau_t)^{\varepsilon} A_{dt}^{\varphi}} + o(ds) + o(d(\ln(1+\tau_t))).
\end{aligned}
$$

Then, using the fact that in the variation in question, taxes are adjusted to keep production of the dirty input constant, the previous equation gives:

$$
\begin{aligned}
&\left(\frac{\varepsilon(1+\tau_t)^{\varepsilon} A_{d}^{\varphi}}{A_{ct}^{\varphi} + (1+\tau_t)^{\varepsilon} A_{dt}^{\varphi}} + \frac{\alpha}{\varphi}\frac{(1-\varepsilon)(1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi}}{(1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi} + A_{dt}^{\varphi}}\right) d(\ln(1+\tau_t)) \\
= \;& (\alpha+\varphi)\, d(\ln(A_{ct})) + d(\ln(A_{dt})) - \frac{\alpha}{\varphi}\frac{\varphi(1+\tau)_t^{1-\varepsilon} A_{ct}^{\varphi} d(\ln(A_{ct})) + \varphi A_{dt}^{\varphi} d(\ln(A_{dt}))}{(1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi} + A_{dt}^{\varphi}} \\
&- \frac{\varphi A_{ct}^{\varphi} d(\ln(A_{ct})) + \varphi(1+\tau)^{\varepsilon} A_{dt}^{\varphi} d(\ln(A_{dt}))}{A_{ct}^{\varphi} + (1+\tau_t)^{\varepsilon} A_{dt}^{\varphi}} + o(ds) + o(d(\ln(1+\tau_t))).
\end{aligned}
$$

Now recall the following: (i) $\lim_{t\to\infty} A_{ct}^{\varphi}/A_{dt}^{\varphi} = 0$; (ii) the term

$$
\frac{\varepsilon(1+\tau_t)^{\varepsilon} A_{d}^{\varphi}}{A_{ct}^{\varphi} + (1+\tau_t)^{\varepsilon} A_{dt}^{\varphi}} + \frac{\alpha}{\varphi}\frac{(1-\varepsilon)(1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi}}{(1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi} + A_{dt}^{\varphi}}
$$

is bounded and bounded away from 0; (iii) the terms in front of $d(\ln(A_{dt}))$ and $d(\ln(A_{ct}))$ are bounded. Therefore, we can rewrite (B.10) as:

$$
\begin{aligned}
d(\ln(C_t)) &= d(\ln(A_{ct})) + \frac{(1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi} A_{dt}^{-\varphi}}{(1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi} A_{dt}^{-\varphi} + 1}\left(d(\ln(A_{dt})) - d(\ln(A_{ct})) - (1-\alpha)^{-1} d(\ln(1+\tau_t))\right) \\
&+ \frac{1}{1-\alpha + \frac{\tau_t(1+\tau_t)^{-\varepsilon} A_{ct}^{\varphi} A_{dt}^{-\varphi}}{A_{ct}^{\varphi} A_{dt}^{-\varphi}(1+\tau_t)^{-\varepsilon}+1}}\frac{(1+\tau_t)^{1-\varepsilon} A_{ct}^{\varphi} A_{dt}^{-\varphi}}{(1+\tau_t)^{-\varepsilon} A_{ct}^{\varphi} A_{dt}^{-\varphi} + 1}\left(d(\ln(1+\tau_t)) + \varphi\frac{\tau_t}{1+\tau_t} d(\ln(A_{ct}))\right) \\
&- \frac{\tau_t(1+\tau_t)^{-\varepsilon} A_{ct}^{\varphi} A_{dt}^{-\varphi}}{1-\alpha + \frac{\tau_t(1+\tau_t)^{-\varepsilon} A_{ct}^{\varphi} A_{dt}^{-\varphi}}{A_{ct}^{\varphi} A_{dt}^{-\varphi}(1+\tau_t)^{-\varepsilon}+1}}\frac{\varphi A_{ct}^{\varphi} A_{dt}^{-\varphi}(1+\tau_t)^{-\varepsilon} d(\ln(A_{ct})) + \varphi d(\ln(A_{dt})) + \varepsilon d(\ln(1+\tau_t))}{\left((1+\tau_t)^{-\varepsilon} A_{ct}^{\varphi} A_{dt}^{-\varphi} + 1\right)^2} + o(ds)
\end{aligned}
$$

Using again the fact that $\lim_{t\to\infty} A_{ct}^{\varphi}/A_{dt}^{\varphi} = 0$ and (B.9), the previous expression becomes

$$
d(\ln(C_t)) = \left(\frac{\gamma\eta_c}{1+\gamma\eta_c} + O\left(\frac{A_{ct}^{\varphi}}{A_{dt}^{\varphi}}\right)\right) ds + o(ds),
$$

which implies that for T sufficiently large, $O\left(A_{ct}^{\varphi}/A_{dt}^{\varphi}\right)$ will be smaller than $\gamma\eta_c/\left(1+\gamma\eta_c\right)$, and thus consumption increases. This implies that the alternative policy raises consumption for all periods after υ, and does so without affecting the quality of the environment, hence the original policy cannot be optimal. This contradiction establishes that s_{ct} reaches 1 in finite time.

Part 6: Thus the optimal allocation must involve $s_{ct} = 1$ for all $t \geq \tilde{T}$ (for some $\tilde{T} < \infty$) and $A_{ct}/A_{dt} \to \infty$. Then, note that (A.17) implies that even if $\tau_t = q_t = 0$, the equilibrium allocation of scientists involves $s_{ct} = 1$ for all $t \geq T$ for some T sufficiently large. This is sufficient to establish that $q_t = 0$ for all $t \geq T$ is consistent with an optimal allocation. Finally, equation (B.5) implies that when $\varepsilon > 1/(1-\alpha)$, $Y_{dt} \to 0$, which together with (12), implies that S_t reaches $\overline{S}$ in finite time. But then the assumption that $\partial u\left(C, \overline{S}\right)/\partial S = 0$ combined with (23) implies that the optimal input tax reaches 0 in finite time. On the contrary, when $\varepsilon \leq 1/(1-\alpha)$, even when all research ends up being directed towards clean technologies, (B.5) shows that without imposing a positive input tax we have $Y_{dt} \to \infty$ and thus $S_t = 0$ in finite time, which cannot be optimal. So in this case, taxation must be permanent at the optimum.

B5. Equilibrium profit ratio with exhaustible resources

We first analyze how the static equilibrium changes when we introduce the limited resource constraint. The description of clean sectors remains exactly as before. Profit maximization by producers of machines in the dirty sector now leads to the equilibrium price $p_{dit} = \psi/\alpha_1$ (as α_1 is the share of machines in the production of dirty input). The equilibrium output level for machines is then given by:

$$x_{dit} = \left((\alpha_1)^2 \psi^{-1} p_{dt} R_t^{\alpha_2} L_{dt}^{1-\alpha}\right)^{\frac{1}{1-\alpha_1}} A_{dit}. \tag{B.11}$$

Profit maximization by the dirty input producer leads to the following demand equation for the resource: $p_{dt}\alpha_2 R_t^{\alpha_2 - 1} L_{dt}^{1-\alpha} \int_0^1 A_{dit}^{1-\alpha_1} x_{dit}^{\alpha_1} di = c\left(Q_t\right)$, plugging in the equilibrium output level of machines (B.11) yields:

$$R_t = \left(\frac{(\alpha_1)^2}{\psi}\right)^{\frac{\alpha_1}{1-\alpha}} \left(\frac{\alpha_2 A_{dt}}{c(Q_t)}\right)^{\frac{1-\alpha_1}{1-\alpha}} p_{dt}^{\frac{1}{1-\alpha}} L_{dt} \tag{B.12}$$

which in turn, together with (5), leads to the following expression for the equilibrium production of dirty input:

$$Y_{dt} = \left(\frac{(\alpha_1)^2}{\psi}\right)^{\frac{\alpha_1}{1-\alpha}} \left(\frac{\alpha_2 A_{dt}}{c(Q_t)}\right)^{\frac{\alpha_2}{1-\alpha}} p_{dt}^{\frac{\alpha}{1-\alpha}} L_{dt} A_{dt}, \tag{B.13}$$

while equilibrium profits from producing machine i in the dirty sector becomes:

$$\pi_{dit} = (1-\alpha_1)\, \alpha_1^{\frac{1+\alpha_1}{1-\alpha_1}} \left(\frac{1}{\psi^{\alpha_1}}\right)^{\frac{1}{1-\alpha_1}} p_{dt}^{\frac{1}{1-\alpha_1}} R_t^{\frac{\alpha_2}{1-\alpha_1}} L_{dt}^{\frac{1-\alpha}{1-\alpha_1}} A_{dit}. \tag{B.14}$$

The production of the clean input and the profits of the producer of machine i in the clean sector are still given by (A.4) and (15). Now, labor market clearing requires that the marginal product of labor be equalized across sectors; this, together with (B.13) and (A.4) for $j = c$, leads to the equilibrium price ratio:

$$\frac{p_{ct}}{p_{dt}} = \frac{\psi^{\alpha_2} (\alpha_1)^{2\alpha_1} (\alpha_2)^{\alpha_2} A_{dt}^{1-\alpha_1}}{c(Q_t)^{\alpha_2} \alpha^{2\alpha} A_{ct}^{1-\alpha}}, \tag{B.15}$$

thus a higher extraction cost will bid up the price of the dirty input. Profit maximization by final good producers still yields (13) which, together with (B.15), (B.13) and (A.4) for $j = c$, yield the relative employment in the two sectors:

$$\frac{L_{ct}}{L_{dt}} = \left(\frac{c(Q_t)^{\alpha_2} \alpha^{2\alpha}}{\psi^{\alpha_2} \alpha_1^{2\alpha_1} (\alpha_2)^{\alpha_2}} \right)^{(\varepsilon-1)} \frac{A_{ct}^{-\varphi}}{A_{dt}^{-\varphi_1}}, \tag{B.16}$$

with $\varphi_1 \equiv (1 - \alpha_1)(1 - \varepsilon)$. Hence, the higher the extraction cost, the higher the amount of labor allocated to the clean industry when $\varepsilon > 1$.

Using (15) for $j = c$, (B.14), (B.12), (B.15), (B.16), the ratio of expected profits from undertaking innovation at time t in the clean versus the dirty sector, is then equal to:

$$\begin{aligned} \frac{\Pi_{ct}}{\Pi_{dt}} &= \frac{\eta_c}{\eta_d} \frac{(1-\alpha_1)\, \alpha_1^{\frac{1+\alpha_1}{1-\alpha_1}} \left(\frac{1}{\psi^{\alpha_1}}\right)^{\frac{1}{1-\alpha_1}}}{(1-\alpha)\, \alpha^{\frac{1+\alpha}{1-\alpha}} \left(\frac{1}{\psi}\right)^{\frac{\alpha}{1-\alpha}}} \frac{p_{ct}^{\frac{1}{1-\alpha}} L_{ct}}{p_{dt}^{\frac{1}{1-\alpha_1}} R_t^{\frac{\alpha_2}{1-\alpha_1}} L_{dt}^{\frac{1-\alpha}{1-\alpha_1}}} \frac{A_{ct-1}}{A_{dt-1}} \\ &= \kappa \frac{\eta_c}{\eta_d} \frac{c(Q_t)^{\alpha_2(\varepsilon-1)} \left(1 + \gamma\, \eta_c s_{ct}\right)^{-\varphi-1} A_{ct-1}^{-\varphi}}{\left(1 + \gamma\, \eta_d s_{dt}\right)^{-\varphi_1-1} A_{dt-1}^{-\varphi_1}} \end{aligned}$$

where we let $\kappa \equiv \frac{(1-\alpha)\alpha}{(1-\alpha_1)\alpha_1^{(1+\alpha_2-\alpha_1)/(1-\alpha_1)}} \left(\frac{\alpha^{2\alpha}}{\psi^{\alpha_2} \alpha_1^{2\alpha_1} \alpha_2^{\alpha_2}} \right)^{(\varepsilon-1)}$. This establishes (25).

B6. Proof of Proposition 7

First, we derive the equilibrium production of R_t and Y_{dt}.

Using the expression for the equilibrium price ratio (B.15), together with the choice of the final good as the numeraire (9), we get:

$$p_{ct} = \frac{\psi^{\alpha_2} (\alpha_1)^{2\alpha_1} (\alpha_2)^{\alpha_2} A_{dt}^{1-\alpha_1}}{\left(\left(\alpha^{2\alpha} c\, (Q_t)^{\alpha_2}\right)^{1-\varepsilon} A_{ct}^{\varphi} + \left(\psi^{\alpha_2} (\alpha_1)^{2\alpha_1} (\alpha_2)^{\alpha_2}\right)^{1-\varepsilon} A_{dt}^{\varphi_1} \right)^{\frac{1}{1-\varepsilon}}}$$

$$p_{dt} = \frac{\alpha^{2\alpha}\,(c(Q_t))^{\alpha_2}\,A_{ct}^{1-\alpha}}{\left(\left(\alpha^{2\alpha}c\,(Q_t)^{\alpha_2}\right)^{1-\varepsilon}A_{ct}^{\varphi} + \left(\psi^{\alpha_2}\,(\alpha_1)^{2\alpha_1}\,(\alpha_2)^{\alpha_2}\right)^{1-\varepsilon}A_{dt}^{\varphi_1}\right)^{\frac{1}{1-\varepsilon}}}$$

Similarly, using the expression for the equilibrium labor ratio (B.16), and labor market clearing (7), we obtain:

$$L_{dt} = \frac{\left(c(Q_t)^{\alpha_2}\alpha^{2\alpha}\right)^{(1-\varepsilon)}A_{ct}^{\varphi}}{\left(c(Q)^{\alpha_2}\alpha^{2\alpha}\right)^{(1-\varepsilon)}A_{ct}^{\varphi} + \left(\psi^{\alpha_2}\alpha_1^{2\alpha_1}\,(\alpha_2)^{\alpha_2}\right)^{(1-\varepsilon)}A_{dt}^{\varphi_1}}$$

$$L_{ct} = \frac{\left(\psi^{\alpha_2}\alpha_1^{2\alpha_1}\,(\alpha_2)^{\alpha_2}\right)^{(1-\varepsilon)}A_{d}^{\varphi_1}}{\left(c(Q_t)^{\alpha_2}\alpha^{2\alpha}\right)^{(1-\varepsilon)}A_{ct}^{\varphi} + \left(\psi^{\alpha_2}\alpha_1^{2\alpha_1}\,(\alpha_2)^{\alpha_2}\right)^{(1-\varepsilon)}A_{dt}^{\varphi_1}}$$

Next, using the above expressions for equilibrium prices and labor allocation, and plugging them in (B.12) and (B.13), we obtain:

$$\text{(B.17)}\qquad Y_{dt} = \frac{\left(\frac{\alpha_1^2}{\psi}\right)^{\frac{\alpha_1}{1-\alpha}}\alpha_2^{\frac{\alpha_2}{1-\alpha}}\alpha^{2\alpha\left(\frac{1}{1-\alpha}-\varepsilon\right)}c(Q_t)^{-\varepsilon\alpha_2}A_{ct}^{\alpha+\varphi}A_{dt}^{\frac{1-\alpha_1}{1-\alpha}}}{\left(\left(c(Q_t)^{\alpha_2}\alpha^{2\alpha}\right)^{(1-\varepsilon)}A_{ct}^{\varphi} + \left(\psi^{\alpha_2}\alpha_1^{2\alpha_1}\,(\alpha_2)^{\alpha_2}\right)^{(1-\varepsilon)}A_{dt}^{\varphi_1}\right)^{\frac{\alpha+\varphi}{\varphi}}}$$

and

$$R_t = \frac{\alpha^{2\alpha\left(\frac{1}{1-\alpha}+1-\varepsilon\right)}\alpha_1^{2\frac{\alpha_1}{1-\alpha}}\alpha_2^{\frac{1-\alpha_1}{1-\alpha}}\psi^{-\frac{\alpha_1}{1-\alpha}}\,(c(Q_t))^{\alpha_2-1-\alpha_2\varepsilon}A_{ct}^{1+\varphi}A_{dt}^{\frac{1-\alpha_1}{1-\alpha}}}{\left(\left(c(Q_t)^{\alpha_2}\alpha^{2\alpha}\right)^{(1-\varepsilon)}A_{ct}^{\varphi} + \left(\psi^{\alpha_2}\alpha_1^{2\alpha_1}\,(\alpha_2)^{\alpha_2}\right)^{(1-\varepsilon)}A_{dt}^{\varphi_1}\right)^{\frac{1+\varphi}{\varphi}}},$$

so that the ratio of resource consumed per unit of dirty input is:

$$\frac{R_t}{Y_{dt}} = \frac{\alpha_2\alpha^{2\alpha}\,(c(Q_t))^{\alpha_2-1}}{\left(\left(\alpha^{2\alpha}c\,(Q_t)^{\alpha_2}\right)^{1-\varepsilon} + \left(\psi^{\alpha_2}\,(\alpha_1)^{2\alpha_1}\,(\alpha_2)^{\alpha_2}\right)^{1-\varepsilon}\frac{A_{dt}^{\varphi_1}}{A_{ct}^{\varphi}}\right)^{\frac{1}{1-\varepsilon}}}.$$

When $\varepsilon > 1$, production of the dirty input is not essential to final good production. Thus, even if the stock of exhaustible resource gets fully depleted, it is still possible to achieve positive long-run growth. For a disaster to occur for any initial value of the environmental quality, it is necessary that Y_{dt} grow at a positive rate, while R_t must converge to 0. This implies that R_t/Y_{dt} must converge to 0. This in turn means that the expression

$$\left(\alpha^{2\alpha}c\,(Q_t)^{\alpha_2}\right)^{1-\varepsilon} + \left(\psi^{\alpha_2}\,(\alpha_1)^{2\alpha_1}\,(\alpha_2)^{\alpha_2}\right)^{1-\varepsilon}\frac{A_{dt}^{\varphi_1}}{A_{ct}^{\varphi}}$$

must converge to zero, which is impossible since $c\,(Q_t)$ is bounded above. Therefore, for sufficiently high initial quality of the environment, a disaster will be avoided.

Next, one can show that innovation will always end up occurring in the clean sector only. This is obvious if the resource gets depleted in finite time, so let us consider the case where it never gets depleted. Recall that the ratio of expected profits in clean versus dirty innovation is given by (25), so that to prevent innovation from occurring asymptotically in the clean sector only, it must be the case that $A_{ct}^{-\varphi}$ does not grow faster then $A_{dt}^{-\varphi_1}$. In this case $R = O\left(A_{dt}^{\frac{1-\alpha_1}{1-\alpha}}\right)$. But $A_{dt}^{\frac{1-\alpha_1}{1-\alpha}}$ grows at a positive rate over time, so that the resource gets depleted in finite time after all. This completes the proof of Proposition 7.

The case where $\varepsilon < 1$: It is also straightforward to derive the corresponding results for the case where $\varepsilon < 1$. In particular, when $\varepsilon < 1$, Y_{dt} is now essential for production and thus so is the resource flow R_t. Consequently, it is necessary that Q_t does not get depleted in finite time in order to get positive long-run growth. Recall that innovation takes place in both sectors if and only if $\kappa \frac{\eta_c}{\eta_d} \frac{c(Q_t)^{\alpha_2(\varepsilon-1)}\left(1+\gamma\, \eta_c s_{ct}\right)^{-\varphi-1} A_{ct-1}^{-\varphi}}{\left(1+\gamma\, \eta_d s_{dt}\right)^{-\varphi_1-1} A_{dt-1}^{-\varphi_1}} = 1$, and positive long-run growth requires (positive) growth of both dirty and clean inputs. This requires that innovation occurs in both sectors, so $A_{dt}^{1-\alpha_1}$ and $A_{ct}^{1-\alpha}$ should be of same order.

But then:

$$R = O\left(A_{dt}^{\frac{1-\alpha_1}{1-\alpha}}\right),$$

so that R_t grows over time. But this in turn leads to the resource stock being fully exhausted in finite time, thereby also shutting down the production of dirty input, which here prevents positive long-run growth.

B7. *Proof of Proposition 8*

We denote the Lagrange multiplier for equation (6) by $\widetilde{m}_t$. We can use (6) to rewrite the condition $Q_t \geq 0$ for all t, as:

$$\sum_{\upsilon=0}^{\infty} R_{\upsilon} \leq Q(0).$$

Denoting the Lagrange multiplier for this constraint by $\nu \geq 0$, the first-order condition with respect to R_t gives:

$$\alpha_2 \widehat{p}_{dt} R_t^{\alpha_2-1} L_{dt}^{1-\alpha} \int_0^1 A_{dit}^{1-\alpha_1} x_{dit}^{\alpha_1} di = \frac{\widetilde{m}_t + \nu}{\lambda_t} + c\left(Q_t\right),$$

where recall that $\widehat{p}_{jt} = \lambda_{jt}/\lambda_t$. The wedge $\left(\widetilde{m}_t + \nu\right)/\lambda_t$ is the value, in time t units of final good, of one unit of resource at time t.

The law of motion for the shadow value of one unit of natural resource at time t is then determined by the first-order condition with respect to Q_t, namely

$$\widetilde{m}_t = \widetilde{m}_{t-1} + \lambda_t c'\left(Q_t\right) R_t,$$

where $\widetilde{m}_t \geq 0$. Letting $m_t = \widetilde{m}_t + \nu$ we obtain:

$$m_t = m_\infty + \sum_{\upsilon=t+1}^{\infty} \lambda_\upsilon \left(-c'\left(Q_\upsilon\right)\right) R_\upsilon,$$

where $m_\infty > 0$ is the limit of m_t as $t \to \infty$.

Thus the social optimum requires a resource tax equal to

$$\text{(B.18)} \qquad \theta_t = \frac{m_t}{\lambda_t c\left(Q_t\right)} = \frac{(1+\rho)^t m_\infty - \sum_{\upsilon=t+1}^{\infty} \frac{1}{(1+\rho)^{\upsilon-t}} c'\left(Q_\upsilon\right) R_\upsilon \partial u\left(C_\upsilon, S_\upsilon\right)/\partial C}{c\left(Q_t\right) \partial u\left(C_t, S_t\right)/\partial C}.$$

In particular, the optimal resource tax is always positive.

B8. Proof of Proposition 9

The proof consists of three parts: in Part 1, we prove that when $\ln(1+\rho) > (1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$, then in the long run innovation must occur in the clean sector only. In Part 2, we show that if $\ln(1+\rho) < (1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$ and innovation occurs in the dirty sector only or in both sectors in the long run, then a disaster necessarily occurs. Finally, in Part 3, we derive the asymptotic growth rate of dirty input production when innovation occurs in the clean sector only.

First, note that the expressions for Y_{jt}, derived above for the case where there are no well-defined property rights to the resource, still hold provided one replaces the unit extraction cost $c\left(Q_t\right)$ by the resource price P_t. So that (B.17) now becomes:

$$\text{(B.19)} \qquad Y_{dt} = \frac{\left(\frac{\alpha_1^2}{\psi}\right)^{\frac{\alpha_1}{1-\alpha}} \alpha_2^{\frac{\alpha_2}{1-\alpha}} \alpha^{2\alpha\left(\frac{1}{1-\alpha}-\varepsilon\right)} P_t^{-\varepsilon\alpha_2} A_{ct}^{\alpha+\varphi} A_{dt}^{\frac{1-\alpha_1}{1-\alpha}}}{\left(\left(P_t^{\alpha_2}\alpha^{2\alpha}\right)^{(1-\varepsilon)} A_{ct}^{\varphi} + \left(\psi^{\alpha_2}\alpha_1^{2\alpha_1}\left(\alpha_2\right)^{\alpha_2}\right)^{(1-\varepsilon)} A_{dt}^{\varphi_1}\right)^{\frac{\alpha+\varphi}{\varphi}}}.$$

Similarly

$$\text{(B.20)} \qquad Y_{ct} = \frac{\left(\psi^{\alpha_2}\left(\alpha_1\right)^{2\alpha_1}\left(\alpha_2\right)^{\alpha_2}\right)^{\frac{\alpha+\varphi}{1-\alpha}} A_{ct} A_{dt}^{\frac{1-\alpha_1}{1-\alpha}(\alpha+\varphi)}}{\left(\left(\alpha^{2\alpha} P_t^{\alpha_2}\right)^{1-\varepsilon} A_c^{\varphi} + \left(\psi^{\alpha_2}\left(\alpha_1\right)^{2\alpha_1}\left(\alpha_2\right)^{\alpha_2}\right)^{1-\varepsilon} A_{dt}^{\varphi_1}\right)^{\frac{\alpha+\varphi}{\varphi}}},$$

and we can rewrite (25) as:

$$\text{(B.21)} \qquad \frac{\Pi_{ct}}{\Pi_{dt}} = \kappa \frac{\eta_c}{\eta_d} \frac{P_t^{\alpha_2(\varepsilon-1)}\left(1+\gamma\,\eta_c s_{ct}\right)^{-\varphi-1} A_{ct-1}^{-\varphi}}{\left(1+\gamma\,\eta_d s_{dt}\right)^{-\varphi_1-1} A_{dt-1}^{-\varphi_1}}.$$

Part 1: Let us assume that $\ln(1+\rho) > (1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$. We want to show that innovation then ends up occurring in the clean sector only in the long run. Here, we shall reason by contradiction, and assume, first that innovation ends up occurring in the dirty sector only in the long run, and second that innovation keeps occurring in both sectors forever, and each time we shall generate a contradiction.

Part 1.a: Assume that innovation ends up occurring in the dirty sector only. Then, from (B.21), the ratio of expected profits from innovating clean to expected profits from innovating dirty, is asymptotically proportional to $\left(P_t^{\alpha_2}/A_{dt}^{1-\alpha_1}\right)^{\varepsilon-1}$, i.e.,

$$\Pi_{ct}/\Pi_{dt} = O\left(P_t^{\alpha_2}/A_{dt}^{1-\alpha_1}\right)^{\varepsilon-1}. \tag{B.22}$$

Thus, for innovation to take place only in the dirty sector in the long run, it is necessary for $A_{dt}^{1-\alpha_1}$ to grow faster than $P_t^{\alpha_2}$. Assume that this is the case, then using (B.19) we obtain

$$Y_{dt} = O\left(A_{dt}^{1-\alpha_1}/P_t^{\alpha_2}\right)^{\frac{1}{1-\alpha}} \tag{B.23}$$

so that the asymptotic growth rate of the economy g satisfies:

$$\ln(1+g) = \frac{(1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right) - \alpha_2\ln(1+r)}{(1-\alpha)}.$$

Combining this with (28) gives:

$$\ln(1+g) = \frac{(1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right) - \alpha_2\ln(1+\rho)}{1-\alpha+\alpha_2\sigma}. \tag{B.24}$$

Since $\ln(1+\rho) > [(1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$, this equation implies $g < 0$, and therefore the ratio of expected profits Π_{ct}/Π_{dt} goes to infinity over time. Thus innovation only in the dirty sector in the long run cannot be an equilibrium, yielding a contradiction.

Part 1.b: Assume now that innovation occurs in both sectors forever. Using (B.21) we obtain:

$$\Pi_{ct}/\Pi_{dt} = O\left(P_t^{\alpha_2}A_{ct}^{1-\alpha}/A_{dt}^{1-\alpha_1}\right)^{\varepsilon-1},$$

so that $P_t^{\alpha_2}A_{ct}^{1-\alpha}$ and $A_{dt}^{1-\alpha_1}$ must grow asymptotically at the same rate. Then from (B.19) and (B.20), we have

$$Y_{dt} = O(A_{ct}) \text{ and } Y_{ct} = O\left(A_{ct}\right), \tag{B.25}$$

so that $g = \gamma\,\eta_c s_c$, where s_c is the asymptotic fraction of scientists working on clean research.

For $P_t^{\alpha_2} A_{ct}^{1-\alpha}$ and $A_{dt}^{1-\alpha_1}$ to grow at the same rate, it is then necessary (using (28)) that:

$$\frac{\alpha_2}{1-\alpha_1}\left(\ln(1+\rho)+\sigma\ln\left(1+\gamma\,\eta_c s_c\right)\right)+\frac{1-\alpha}{1-\alpha_1}\ln\left(1+\gamma\,\eta_c s_c\right)=\ln\left(1+\gamma\,\eta_d\,(1-s_c)\right)$$

which in turn is impossible if $\ln(1+\rho) > (1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$ (the above equation would then imply that $s_c < 0$, which cannot be).

This concludes Part 1, namely we have shown that if $\ln(1+\rho) > (1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$ then innovation occurs in the clean sector only in the long run.

Part 2: We now show that if innovation does not switch to the clean sector in finite time then a disaster is bound to occur when $\ln(1+\rho) < (1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$. Indeed, suppose that innovation does not switch to the clean sector in finite time. Then, either innovation ends up occurring in the dirty sector only, or innovation keeps occurring in both sectors forever. In the former case, dirty input production must grow at rate g given by (B.24), which is strictly positive if $\ln(1+\rho) < (1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$. In the latter case, (B.25) implies that Y_{dt} will grow over time, again leading to a disaster.

Part 3: We now assume that innovation occurs in the clean sector only. Using (B.20) we get $g=\gamma\,\eta_c$ and using (B.19) we get:

$$Y_{dt}=O(P_t^{-\varepsilon\alpha_2}A_{ct}^{\alpha+\varphi}).$$

Thus overall Y_{dt} grows at rate g_{Y_d} satisfying:

$$\ln\left(1+g_{Y_d}\right)=(1-\varepsilon\,(1-\alpha))\ln\left(1+\gamma\,\eta_c\right)-\varepsilon\alpha_2\left(\ln(1+\rho)+\sigma\ln\left(1+\gamma\,\eta_c\right)\right).$$

Now, if $g_{Y_d} > 0$, then a disaster cannot be avoided. However, when $g_{Y_d} < 0$, and provided that the initial environmental quality is sufficiently large, a disaster is avoided.

In conclusion, Part 1 shows that when $\ln(1+\rho) > (1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$, innovation must eventually occur in the clean sector only. Part 3 then shows that in that case and provided that $(1-\varepsilon\,(1-\alpha))\ln\left(1+\gamma\,\eta_c\right)-\varepsilon\alpha_2\left(\ln(1+\rho)+\sigma\ln\left(1+\gamma\,\eta_c\right)\right) < 0$, a disaster is indeed avoided for sufficiently large initial environmental quality. This last condition in turn is met whenever $\varepsilon > 1/\left(2-\alpha-\alpha_1\right)$ if $\ln(1+\rho) > (1-\alpha_1)\ln\left(1+\gamma\ \max\left\{\eta_d,\eta_c\right\}\right)/\alpha_2$. This proves the first claim of Proposition 9. Then Part 2 establishes that when innovation does not occur in the clean sector only in the long run, then a disaster is bound to occur if $\ln(1+\rho)\neq(1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$ (when $\ln(1+\rho) > (1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$, we know that innovation has to occur in the clean sector asymptotically). Finally, Part 3 shows that even when innovation ends up occurring in the clean sector only, yet a disaster occurs if
$(1-\varepsilon\,(1-\alpha))\ln\left(1+\gamma\,\eta_c\right)-\varepsilon\alpha_2\left(\ln(1+\rho)+\sigma\ln\left(1+\gamma\,\eta_c\right)\right) > 0$ or equivalently if $\ln(1+\rho) < (1/\varepsilon-(1-\alpha)-\alpha_2\sigma)\ln\left(1+\gamma\,\eta_c\right)/\alpha_2$. Thus no matter where innovation occurs asymptotically, if $\ln(1+\rho) < (1/\varepsilon-(1-\alpha)-\alpha_2\sigma)\ln\left(1+\gamma\,\eta_c\right)/\alpha_2$ and $\ln(1+\rho)\neq(1-\alpha_1)\ln\left(1+\gamma\,\eta_d\right)/\alpha_2$, a disaster will necessarily happen. This proves the second claim of Proposition 9.

B9. Perfect competition in the absence of innovation

Here we show how our results are slightly modified if, instead of having monopoly rights randomly attributed to "entrepreneurs" when innovation does not occur, machines are produced competitively. There are two types of machines. Those where innovation occurred at the beginning of the period are produced monopolistically with demand function

$$x_{jit} = x^m_{jit} = \left(\frac{\alpha^2 p_{jt}}{\psi}\right)^{\frac{1}{1-\alpha}} L_{jt} A_{jit}.$$

Those for which innovation failed are produced competitively. In this case, machines are priced at marginal cost ψ, which leads to a demand for competitively produced machines equal to $x_{jit} = x^c_{jit} = \left(\frac{\alpha p_{jt}}{\psi}\right)^{\frac{1}{1-\alpha}} L_{jt} A_{jit}$. The number of machines produced under monopoly is simply given by $\eta_j s_{jt}$ (the number of successful innovation).

Hence the equilibrium production of input j is given by

$$\begin{aligned} Y_{jt} &= L^{1-\alpha}_{jt} \int_0^1 A^{1-\alpha}_{jit} \left(\eta_j s_{jt} \left(x^m_{jit}\right)^\alpha + \left(1 - \eta s_{jt}\right) \left(x^c_{jit}\right)^\alpha\right) di \\ &= \left(\frac{\alpha p_{jt}}{\psi}\right)^{\frac{\alpha}{1-\alpha}} \left(\eta_j s_{jt} \left(\alpha^{\frac{\alpha}{1-\alpha}} - 1\right) + 1\right) A_{jt} L_{jt} \\ &= \left(\frac{\alpha p_{jt}}{\psi}\right)^{\frac{\alpha}{1-\alpha}} \widetilde{A_{jt}} L_{jt} \end{aligned}$$

where s_j is the number of scientists employed in clean industries and

$$\widetilde{A_{jt}} = \left(\eta_j s_{jt} \left(\alpha^{\frac{\alpha}{1-\alpha}} - 1\right) + 1\right) A_{jt}$$

is the average corrected productivity level in sector j (taking into account that some machines are produced by monopolists and others are not).

The equilibrium price ratio is now equal to:

$$\frac{p_{ct}}{p_{dt}} = \left(\frac{\widetilde{A_{ct}}}{\widetilde{A_{dt}}}\right)^{-(1-\alpha)},$$

and the equilibrium labor ratio becomes:

$$\frac{L_{ct}}{L_{dt}} = \left(\frac{\widetilde{A_{ct}}}{\widetilde{A_{dt}}}\right)^{-\varphi}.$$

The ratio of expected profits from innovation in clean versus dirty sector now becomes

$$\begin{aligned}\frac{\Pi_{ct}}{\Pi_{dt}} &= \frac{\eta_c}{\eta_d}\left(\frac{p_{ct}}{p_{dt}}\right)^{\frac{1}{1-\alpha}}\frac{L_{ct}}{L_{dt}}\frac{A_{ct-1}}{A_{dt-1}}\\ &= \frac{\eta_c}{\eta_d}\left(\frac{\left(\eta_c s_{ct}\left(\alpha^{\frac{\alpha}{1-\alpha}}-1\right)+1\right)\left(1+\gamma\,\eta_c s_{ct}\right)}{\left(\eta_d s_{dt}\left(\alpha^{\frac{\alpha}{1-\alpha}}-1\right)+1\right)\left(1+\gamma\,\eta_d s_{dt}\right)}\right)^{-\varphi-1}\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}\end{aligned}$$

This yields the modified lemma:

LEMMA 2: *In the decentralized equilibrium, innovation at time t can occur in the clean sector only when*

$$\eta_c A_{ct-1}^{-\varphi} > \eta_d\left(\left(1+\gamma\,\eta_c\right)\left(\left(\eta_c\left(\alpha^{\frac{\alpha}{1-\alpha}}-1\right)+1\right)\right)\right)^{\varphi+1}A_{dt-1}^{-\varphi};$$

in the dirty sector only when

$$\eta_c\left(\left(1+\gamma\,\eta_d\right)\left(\left(\eta_d\left(\alpha^{\frac{\alpha}{1-\alpha}}-1\right)+1\right)\right)\right)^{\varphi+1}A_{ct-1}^{-\varphi} < \eta_d A_{dt-1}^{-\varphi};$$

and can occur in both when

$$\begin{aligned}&\eta_c\left(\left(\eta_d s_{dt}\left(\alpha^{\frac{\alpha}{1-\alpha}}-1\right)+1\right)\left(1+\gamma\,\eta_d s_{dt}\right)\right)^{\varphi+1}A_{ct-1}^{-\varphi}\\ =\;&\eta_d\left(\left(\eta_c s_{ct}\left(\alpha^{\frac{\alpha}{1-\alpha}}-1\right)+1\right)\left(1+\gamma\,\eta_c s_{ct}\right)\right)^{\varphi+1}A_{dt-1}^{-\varphi}.\end{aligned}$$

This modified lemma can then be used to prove the analogs of Propositions 1, 2 and 3 in the text. The results with exhaustible resource can similarly be generalized to this case.

B10. Quantitative example for the exhaustible resource case

We now perform a quantitative evaluation for the exhaustible resource case similar to that presented in Section V. As in the text, a time period corresponds to 5 years, $\gamma = 1$ and $\alpha = 1/3$, and Y_{c0} and Y_{d0} are still identified with the world production of energy from non-fossil and from fossil fuel origins respectively between 2002 and 2006. The definitions of S, ξ, and δ, and the utility function $u(C, S)$ are also the same as in the baseline calibration. To map our model, which has one exhaustible resource, to data, we focus on oil use and we compute the share of world energy produced from crude oil in the total amount of energy produced from fossil fuels from 2002 to 2006 (still according to the EIA). We then convert units of crude oil production and stock into units of total fossil production and stock by dividing the former by the share of world energy produced with oil relative to the world energy produced by any fossil fuel. We approximate the price for the exhaustible resource in our model by the refiner acquisition cost of imported crude oil in the United States (measured in 2000 chained dollars and again taken from the EIA). We extract the trend from the price series between 1970 and 2007 using the HP filter with the smoothing parameter of 100. We then restrict attention to the period 1995-2007 (during which the filtered real price of oil increases) and parameterize this price trend as a quadratic function of the estimated reserves of fossil resource. The estimated price of the fossil resource in 2002, combined with the consumption of fossil resource between 2002 and 2006 together with the value of world GDP from 2002 to 2006 from the World Bank, and the initial values of Y_{c0} and Y_{d0}, then allow us to compute α_2, A_{c0} and A_{d0} and the cost function $c(Q)$ as the price of the exhaustible resource in units of the final good. This procedure gives $\alpha_2 = 0.0491$. Finally η_c is still taken to be 2% per year, but η_d needs to be rescaled. Indeed, if innovation occurs in the dirty sector only, output in the long-run—abstracting from the exhaustion of the natural resource—will be proportional to $A_d^{\frac{1-\alpha_1}{1-\alpha}}$ instead of A_d, so we compute η_d such that innovation in the dirty sector still corresponds to the same long-run annual growth rate of 2% after making this correction.

We now show how the optimal policy with exhaustible resource compares with that in the baseline case for the four configurations of (ε, ρ) (ε taking the high value of 10 and the low value of 3, ρ taking the high value of 0.015 and the low value of 0.001).

As illustrated by Figure 2B, the switch towards clean innovation again occurs immediately for $[\varepsilon = 10, \rho = 0.001]$, $[\varepsilon = 10, \rho = 0.015]$ and $[\varepsilon = 10, \rho = 0.001]$. The switch to clean innovation occurs slightly later in the exhaustible resource case when $[\varepsilon = 3, \rho = 0.015]$. The reason for this slight delay is that even though the growth prospects in the dirty sector are hampered by the depletion of the resource (this pushes towards an earlier switch to clean innovation), we also have that less dirty input is being produced in the exhaustible resource case, which in turn can accommodate a later switch to clean innovation. Which effect dominates in practice depends on parameters.

Moreover, with the exhaustible resource, the clean research subsidy does not need to be as high as in the baseline case to induce the switch because of the costs of the resource (see Figure 2A). For the same reason, the carbon tax does not need to be as high either (Figure 2C) and the switch to clean production occurs earlier than in the baseline, except when $[\varepsilon = 3, \rho = 0.015]$, whereby the later switch in innovation mitigates the effect of the increase in the extraction cost so that the switch to clean production occurs around the same time (Figure 2D). The figure also shows that when ε is smaller, the resource tax needs to be higher, as more of the resource ends up being extracted at any point in time, and that temperature increases less over time with the exhaustible resource.

B11. Equilibrium and optimal policy with productivity-enhancing and pollution-reducing innovations

We now characterize the laissez-faire equilibrium and optimal policy under the alternative technology, sketched in the text in subsection II.E, where innovations are either productivity-enhancing or pollution-reducing. Recall that in this case there are no clean and dirty technologies, and instead the final good is produced as

$$Y_t = L^{1-\alpha} \int_0^1 A_{it}^{1-\alpha} x_{it}^{\alpha} di,$$

where x_i is the amount of machines i produced and A_i is their productivity. The dynamics of the environment stock are given by

$$S_{t+1} = -\xi \int_0^1 e_{it}^{1-\alpha} x_{it}^{\alpha} di + (1+\delta) S_t,$$

where e_{it} measures how dirty machine i is at time t. Innovation can be directed at either increasing productivity, A_{it}, or reducing the pollution content, e_{it}, as specified below. To simplify notation, in this part of the Appendix, we normalize the total supply of labor to $L = 1$. As in the baseline model, all machines are again produced monopolistically with marginal cost $\psi = \alpha^2$ in terms of the final good. To facilitate comparison with the social optimum, and without any substantive implications, we assume that the optimal subsidy of $1-\alpha$ to the use of machines is always present. We also suppose that there is a "carbon tax" imposed on pollution generated at the rate $\tau \geq 0$. Then, the equilibrium demand for machine of type i at time t satisfies

$$x_{it} = \alpha^{-\frac{1}{1-\alpha}} \left(A_{it}^{1-\alpha} - \tau_t e_{it}^{1-\alpha} \right)^{\frac{1}{1-\alpha}},$$

and generates monopoly profits for producer i of

$$\pi_{it} = \alpha^{-\frac{\alpha}{1-\alpha}} (1-\alpha) \left(A_{it}^{1-\alpha} - \tau_t e_{it}^{1-\alpha} \right)^{\frac{1}{1-\alpha}}.$$

Innovation is directed at either increasing A_{it} or decreasing e_{it}. The technology of innovation is the same as in our baseline model: if a fraction s of the available research resources is directed at pollution reduction and a fraction $1-s$ at increasing productivity, then A_{it} will increase by a factor $(1 + \gamma (1-s))$ (with $\gamma > 1$) and e_{it} will be reduced by a factor $(1 - \zeta s)$ (with $\zeta < 1$).

We consider two alternative specifications. In the first specification, there is no "creative destruction" and thus an incumbent monopolist is the only one who will innovate over its current technology until its patent expires. We assume that the probability that the patent expires υ periods after innovation is $\iota_\upsilon \in [0, 1]$. The special case of one-period patents corresponds to $\iota_1 = 0$. Until the patent expires, the monopolist retains permanent monopoly rights over the production of that machine. After it expires, other scientists can innovate over its technology. In the second specification, we model knowledge spillovers resulting from creative destruction building on the shoulders of giants in a simple way. We assume that a new scientist can always improve over an existing machine. If, when this happens, the incumbent monopolist's patent has expired, the new scientist becomes the monopolist. If the incumbent still has a valid patent, we assume that the new inventor makes a patent payment equal to the profits the incumbent would have obtained with its existing technology. One could have alternatively assumed a knowledge spillover from pollution-reduction activities go from one machine variety to others. Our specification here is simpler notationally and closer to our baseline model.

As in the baseline model, the allocation of scientists to machines is random, so that if scientists devote a fraction s of their time to work towards reducing the pollution content of existing machines, each of them will innovate over a randomly selected machine and this machine will have $(1+\gamma\ (1-s))$ times its initial productivity and $(1-\zeta s)$ times its pollution content.[19] Throughout, we focus on a symmetric equilibrium where $A_{it} \equiv A_t$ and $e_{it} \equiv e_t$ for all i.

Equilibrium. Suppose that there is an input tax τ_t and a subsidy to clean research q_t, and denote the interest rate at time t by r_t. Under the first specification of research technology (without knowledge spillovers/creative destruction), the monopolist will allocate research in order to maximize the payoffs of future profits, that is the equilibrium allocation research effort by incumbents, $\{s_{t+k}\}_{k=0}^{\infty}$, must solve

(B.26)
$$\max_{\{s_{t+k}\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} \prod_{\upsilon=1}^{k} \left(\frac{1-\iota_\upsilon}{1+r_{t+\upsilon}} \right) \left(\alpha^{-\frac{\alpha}{1-\alpha}} (1-\alpha) \left(A_{t+k}^{1-\alpha} - \tau_{t+k} e_{t+k}^{1-\alpha} \right)^{\frac{1}{1-\alpha}} + q_{t+k} s_{t+k} \right),$$

where

$$\begin{aligned} A_{t+k} &= (1+\gamma\ (1-s_{t+k}))\, A_{t+k-1}, \text{ and} \\ e_{t+k} &= (1-\zeta s_{t+\tau})\, A_{t+k-1}. \end{aligned}$$

Under the second specification (with knowledge spillovers/creative destruction), instead, we have

$$\max_{s_t} \sum_{k=0}^{\infty} \prod_{\upsilon=1}^{k} \left(\frac{1-\iota_\upsilon}{1+r_{t+\upsilon}} \right) \alpha^{-\frac{\alpha}{1-\alpha}} (1-\alpha) \left(A_t^{1-\alpha} - \tau_{t+k} e_t^{1-\alpha} \right)^{\frac{1}{1-\alpha}} + q_t s_t, \tag{B.27}$$

since in this case the incumbent will only innovate once at the beginning and will then obtain rents from that innovation until it expires. From the consumer maximization problem, the interest rate in both cases satisfies

$$1+r_t = (1+\rho) \frac{\frac{\partial U}{\partial C} (C_{t-1}, S_{t-1})}{\frac{\partial U}{\partial C} (C_t, S_t)}.$$

Social optimum. Using the symmetry across all varieties of machines, the social planner solves (under both specifications),

$$\max_{\{s_t, C_t, S_t, Y_t, A_t, X_t\}_{t=0}^{\infty}} \sum_{k=0}^{\infty} \frac{1}{(1+\rho)^t} U(C_t, S_t)$$

[19] This specification is equivalent to one where all scientists (which have, recall, size normalized to 1), including the incumbent inventor, attempt to innovate on all machines but one, and only one, succeeds.

subject to

$$C_t = Y_t - \alpha^2 X_t,$$
$$Y_t = A_t^{1-\alpha} X_t^{\alpha},$$
$$S_{t+1} = (1+\delta) S_t - e_t^{1-\alpha} X_t^{\alpha};$$

$$\begin{aligned} A_{t+1} &= (1+\gamma\ (1-s_{t+1})) A_t \\ e_{t+1} &= (1-\zeta s_{t+1}) e_t, \text{ and} \\ s_t &\geq 0 \text{ and } s_t \leq 1. \end{aligned}$$

We denote the respective Lagrangian multipliers of these constraints by χ_t, λ_t, ω_{t+1}, μ_{dt+1}, μ_{ct+1}, ν_{0t} and ν_{1t}. Then, the first-order condition with respect to C_t gives

$$\frac{1}{(1+\rho)^t} \frac{\partial U}{\partial C} (C_t, S_t) = \lambda_t = \chi_t$$

where the second equality uses the first-order condition with respect to Y_t. The first-order condition with respect to X_t gives

$$\alpha^{-\frac{1}{1-\alpha}} \left(A_t^{1-\alpha} - \frac{\omega_{t+1}}{\lambda_t} e_t^{1-\alpha} \right)^{\frac{1}{1-\alpha}} = X_t,$$

which is the level of production in the decentralized equilibrium in the presence of a tax $\tau_t = \frac{\omega_{t+1}}{\lambda_t}$ (under both specifications) and the subsidy to the use of all machines of $1-\alpha$.

Now turning to the optimal allocation of research, the first-order condition with respect to A_t gives

$$\mu_{dt} = \lambda_t \alpha^{-\frac{\alpha}{1-\alpha}} (1-\alpha) A_t^{-\alpha} \left(A_t^{1-\alpha} - \frac{\omega_{t+1}}{\lambda_t} e_t^{1-\alpha} \right)^{\frac{\alpha}{1-\alpha}} + (1+\gamma\ (1-s_{t+1})) \mu_{dt+1}$$

and the first-order condition with respect to e_t gives

$$\mu_{ct} = -\omega_t \alpha^{-\frac{\alpha}{1-\alpha}} (1-\alpha) \frac{1}{\alpha^{\frac{\alpha}{1-\alpha}}} e_t^{-\alpha} \left(A_t^{1-\alpha} - \frac{\omega_{t+1}}{\lambda_t} e_t^{1-\alpha} \right)^{\frac{\alpha}{1-\alpha}} + (1-\zeta s_{t+1}) \mu_{ct+1}$$

Thus, using the expression for interest rates in the laissez-faire equilibrium, maximizing social welfare with respect to the allocation of scientists s_t is equivalent to the following problem:

(B.28)

$$\begin{aligned} & \max \mu_{dt} (1+\gamma\ (1-s_t)) A_{t-1} + \mu_{ct} (1-\zeta s_t) e_{t-1} \\ = \ & \max \lambda_t \alpha^{-\frac{\alpha}{1-\alpha}} (1-\alpha) \left(A_t^{1-\alpha} - \frac{\omega_{t+1}}{\lambda_t} e_t^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \\ & + (1+\gamma\ (1-s_{t+1})) \mu_{dt+1} A_t + \mu_{ct+1} (1-\zeta s_{t+1}) e_t \\ = \ & \lambda_t \max \sum_{k=0}^{\infty} \prod_{\upsilon=1}^{k} \left(\frac{1}{1+r_{t+\upsilon}} \right) \alpha^{-\frac{\alpha}{1-\alpha}} (1-\alpha) \left(A_{t+k}^{1-\alpha} - \tau_{t+k} e_{t+k}^{1-\alpha} \right)^{\frac{1}{1-\alpha}} . \end{aligned}$$

Now the comparison of (B.28) to (B.26) and (B.27) establishes the claims in the text. First, note that if $\iota_t = 0$ for all t, meaning that there is full perpetual patent enforcement and we are under the first specification (without knowledge spillovers/creative destruction), then a carbon tax is sufficient (together with the subsidy to machines) to decentralize the social optimum as can be seen by comparing (B.28) and (B.26) with $q_t = 0$ for all t. This is no longer true, however, either when $\iota_t > 0$ for some t or if there is creative destruction with knowledge spillovers, as can be seen by comparing (B.28) and (B.27). In this case, the laissez-faire equilibrium will typically involve too little pollution-reducing activity (too low s_t) and hence additional clean research subsidies, $q_t > 0$, are necessary as part of optimal environmental regulation.

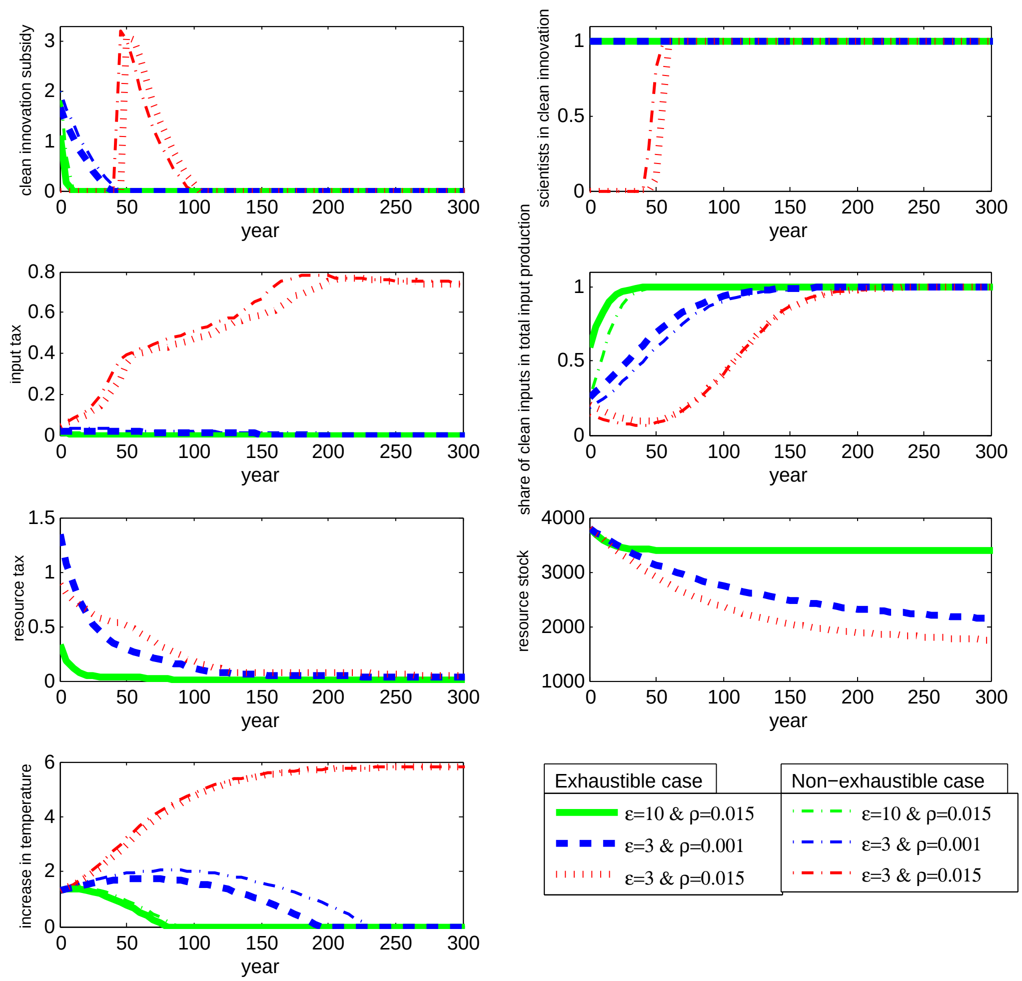

FIGURE B1. OPTIMAL POLICY FOR $\varepsilon = 10$ OR 3 AND $\rho = 0.015$ OR 0.001, IN EXHAUSTIBLE AND NON EXHAUSTIBLE CASES.