

A A Non-Constructible Equilibrium¹

The example depicts a separable contest with three players and one prize of common value 1 (so $v_i(\cdot) = 1 - c_i(\cdot)$). I construct an equilibrium (C, G, G) of the contest, in which player 1's best-response set is the Cantor set. Let $c_1(x) = x$. Let $F(x) = x$, and modify $F(x)$ by mimicking the construction of the Cantor set on $[0, 1]$ in the following way. For every removed interval (a, b) and every $x \in (a, b)$, let $F(x) = a + (x - a)^2 / (b - a)$. Denote the resulting function by \tilde{F} . Then $\tilde{F}(0) = 0$, $\tilde{F}(1) = 1$, and \tilde{F} is continuous, strictly increasing, equals x precisely on the Cantor set, and is strictly lower than x on its complement. In particular, if player 1's probability of winning a prize when choosing x is $\tilde{F}(x)$, then his best-response set is the Cantor set. To achieve this, let $G(x)$ satisfy $\tilde{F}(x) = 1 - (1 - G(x))^2$ for all $x \in [0, 1]$. That is,

$$G(x) = 1 - \sqrt{1 - \tilde{F}(x)}.$$

Then $G(x)$ is continuous and strictly increasing, with $G(0) = 0$ and $G(1) = 1$. To define the cost functions of players 2 and 3, let C be the Cantor function, and recall that it is continuous and weakly increasing, with $C(0) = 0$ and $C(1) = 1$. Let

$$c_2(x) = c_3(x) = 1 - (1 - G(x))(1 - C(x)) = 1 - \left(\sqrt{1 - \tilde{F}(x)} \right) (1 - C(x)).$$

Then c_2 and c_3 are continuous and strictly increasing, with $c_2(0) = c_3(0) = 0$ and $c_2(1) = c_3(1) = 1$. It is straightforward to verify that (C, G, G) is an equilibrium of the contest and player 1's best-response set is the Cantor set.

B The Example of Figure 2

Players' cost functions are $c_2(x) = \frac{3x}{4}$,

$$c_1(x) = \begin{cases} \frac{x}{100} & \text{if } 0 \leq x \leq 0.31948 \\ \frac{x}{100} + 1.0581(x - (0.31948))^2 & \text{if } 0.31948 < x \leq 1 \\ 0.5 + 1.45(x - 1) & \text{if } 1 < x \end{cases},$$

¹I thank George Mailath for encouraging me to provide this example.

and

$$c_3(x) = \begin{cases} \frac{x}{12} & \text{if } 0 \leq x \leq 0.31948 \\ \frac{x}{12} + 1.9794(x - 0.31948)^2 & \text{if } 0.31948 < x \leq 0.7259 \\ 0.38744 + 1.6923(x - 0.7259) + 25(x - 0.7259)^2 & \text{if } 0.7259 < x \leq 0.85 \\ 0.98247 + \frac{(1-0.98247)}{0.15}(x - 0.85) & \text{if } 0.85 < x \end{cases}.$$

These cost functions give powers of 0 , $\frac{1}{4}$ and $\frac{1}{2}$. Perturbing the cost functions slightly does not change the qualitative aspects of the equilibrium.

C Proofs of Results from Section 3

C.1 Proof of Theorem 9

Players $m+2, \dots, n$ do not participate because the conditions of Theorem 2 hold. Denote by s_i^l the infimum of the scores at which player i is active. Because $v_i(\cdot) = V_i - \gamma_i c_i(\cdot)$ and $\gamma_i > 0$, the conditions of Theorem 5 hold, so $s_i^l < T$ for every $i \leq m+1$ and there are no atoms at the threshold. Because $m, m+1 \in \mathcal{CP}(0)$, we have $s_m^l = s_{m+1}^l = 0$. Now, suppose that for $i, j \leq m$, $s_i^l \leq s_j^l$. Denote by $P_k(\cdot)$ and $u_k(\cdot)$ player k 's probability of winning and payoff in the equilibrium as a function of his score. Since players i and j have positive powers and are not active below s_i^l , $G_i(s_i^l) = G_j(s_i^l) = 0$. Thus, $P_i(s_i^l) = P_j(s_i^l)$.² Therefore,

$$\begin{aligned} \frac{u_i(s_i^l)}{V_i} - \frac{u_j(s_i^l)}{V_j} &= P_i(s_i^l)(1 - \alpha)c(s_i^l)(a_j - a_i) + \alpha c(s_i^l)(a_j - a_i) \\ &= (a_j - a_i)c(s_i^l)(\alpha + P_i(s_i^l)(1 - \alpha)). \end{aligned}$$

Also, $w_i/V_i = 1 - a_i c(T)$ and $w_j/V_j = 1 - a_j c(T)$, so $w_i/V_i - w_j/V_j = (a_j - a_i)c(T)$. Since $w_i/V_i = u_i(s_i^l)/V_i$ and $w_j/V_j \geq u_j(s_i^l)/V_j$, we have

$$(1) \quad 0 \geq \left(\frac{w_i}{V_i} - \frac{w_j}{V_j} \right) - \left(\frac{u_i(s_i^l)}{V_i} - \frac{u_j(s_i^l)}{V_j} \right) = (a_j - a_i)(c(T) - c(s_i^l)(\alpha + P_i(s_i^l)(1 - \alpha))).$$

²If $s_i^l = 0$, consider the limit of the probabilities of winning as the score approaches 0 from above, and similarly for $u_i(s_i^l)$ and $u_j(s_i^l)$.

Because $\alpha > 0$ and $P_i(s_i^l) \leq 1$, we have $(\alpha + P_i(s_i^l)(1 - \alpha)) > 0$, so (1) holds if and only if

$$(a_j - a_i)(c(T) - c(s_i^l)) \leq 0.$$

Since $c(s_i^l) < c(T)$, this implies that $a_j \leq a_i$.

It remains to show that if a player becomes active at some score, then he remains active until the threshold. To do this, let us derive some properties of players' semi-elasticities. First normalize players' payoffs so that the prize value is 1 for all players. To this end, note that the contest is strategically equivalent to a contest in which all valuations equal 1 and in which player i 's cost is $a_i c$ instead of $\gamma_i c$. We then have

$$q_i(x) = \frac{v_i(x) - v_i(T)}{v_i(x) + c_i(x)} = \frac{a_i \left(\frac{1}{a_{m+1}} - c(x) \right)}{1 - (1 - \alpha) a_i c(x)}$$

and

$$\varepsilon_i(x) = -\frac{q_i'(x)}{q_i(x)} = \frac{c'(x)(a_i(\alpha - 1) + a_{m+1})}{(1 - c(x)a_{m+1})(a_i c(x)(\alpha - 1) + 1)}.$$
³

Viewed as a function of a_i , we obtain

$$\frac{\partial \varepsilon_i(x)}{\partial a_i} = -\frac{c'(x)(1 - \alpha)}{(a_i c(x)(\alpha - 1) + 1)^2} \leq 0,$$

so at every score players with a higher reach have higher semi-elasticities. This means that when a new player becomes active all existing active players remain active, and that whether an active player remains active depends only on his semi-elasticity and those of players with lower reaches. In particular, players m and $m + 1$ are always active, since their semi-elasticities are always the lowest. To show that players $1, \dots, m - 1$ are active on an interval, observe that the ratio of semi-elasticities of players $j > i$ is non-decreasing in score,

$$\begin{aligned} \left(\frac{\varepsilon_j(x)}{\varepsilon_i(x)} \right)' &= \left(\frac{(a_j(\alpha - 1) + a_{m+1})(a_i c(x)(\alpha - 1) + 1)}{(a_i(\alpha - 1) + a_{m+1})(a_j c(x)(\alpha - 1) + 1)} \right)' \\ &= \frac{a_{m+1} - (1 - \alpha)a_j}{a_{m+1} - (1 - \alpha)a_i} \frac{(1 - \alpha)(a_j - a_i)c'(x)}{(a_j c(x)(\alpha - 1) + 1)^2} \geq 0, \end{aligned}$$

³For expositional simplicity, throughout the proof I ignore the possibility that some players' semi-elasticities may be 0 at some scores. Because players' costs are strictly increasing, piecewise analyticity implies that there are at most a finite number of such scores, so only slight modifications are needed to deal with such scores.

and also that this ratio is at most 1 (since players with a higher reach have higher semi-elasticities). Suppose in contradiction that there is a player who is not active on an interval, and let $i \leq m - 1$ be the player with the highest index among such players. Suppose that player i is active at $x > 0$. Denote by $H^{i+1, \dots, m+1}(\cdot)$ the positive fixed point of the “supply function” defined using the semi-elasticities of players $i + 1, \dots, m + 1$, and consider some $x' \in [x, T]$. By definition of player i and because players with higher reaches become active at higher scores, players $m + 1, \dots, i + 1$ are active at x' . So, because the semi-elasticities of all players $1, \dots, i - 1$ are weakly higher than that of player i , $\varepsilon_i(x') \leq H(x')$ if and only if $\varepsilon_i(x') \leq H^{i+1, \dots, m+1}(x')$. Let $b = \varepsilon_i(x') / \varepsilon_i(x)$. For all $j > i$, since $\varepsilon_j(x') / \varepsilon_i(x') \geq \varepsilon_j(x) / \varepsilon_i(x)$, we have $\varepsilon_j(x') / \varepsilon_j(x) \geq b$. Therefore,

$$H^{i+1, \dots, m+1}(x') = \frac{1}{m-i} \sum_{j=i+1}^{m+1} \varepsilon_j(x') \geq \frac{1}{m-i} \sum_{j=i+1}^{m+1} b\varepsilon_j(x) = bH^{i+1, \dots, m+1}(x).$$

Because player i is active at s_i , $\varepsilon_i(x) \leq H(x)$ so $\varepsilon_i(x) \leq H^{i+1, \dots, m+1}(x)$. Therefore, $\varepsilon_i(x') = b\varepsilon_i(x) \leq bH^{i+1, \dots, m+1}(x) \leq H^{i+1, \dots, m+1}(x')$ so $\varepsilon_i(x') \leq H(x')$. This shows that once a player becomes active he remains active until the threshold.

C.2 Proof of Corollary 5

Because $s_j^l \leq s_i^l$, we have $G_i(x) \leq G_j(x)$ for any $x \leq s_i^l$. By Theorem 9, both players are active on $[s_i^l, T]$. Also, $\varepsilon_i \geq \varepsilon_j$, so because both players are active on $[s_i^l, T]$ the algorithm shows us that $h_i \leq h_j$ on $[s_i^l, T]$. So player i starts being active later and has a lower hazard rate than player j , which means that G_i FOSD G_j . To see this, recall that $h_i(x) = -(1 - G_i(x))' / (1 - G_i(x))$, so $h_i \leq h_j$ implies that $(1 - G_i)' / (1 - G_i) \geq (1 - G_j)' / (1 - G_j)$. This implies that for $y \in [s_i^l, T]$,

$$\begin{aligned} 0 &\leq \int_{s_i^l}^y \left(\frac{(1 - G_i(x))'}{1 - G_i(x)} - \frac{(1 - G_j(x))'}{1 - G_j(x)} \right) dx = \ln \left(\frac{1 - G_i(x)}{1 - G_j(x)} \right) \Big|_{s_i^l}^y \\ &= \ln \left(\frac{1 - G_i(y)}{1 - G_j(y)} \right) - \ln \left(\frac{1 - G_i(s_i^l)}{1 - G_j(s_i^l)} \right). \end{aligned}$$

Because $G_i(s_i^l) \leq G_j(s_i^l)$, we have $(1 - G_i(s_i^l)) / (1 - G_j(s_i^l)) \geq 1$, so by taking exponents the previous inequality implies $\frac{1 - G_i(y)}{1 - G_j(y)} \geq 1$, or $G_j(y) \geq G_i(y)$, as required.

This FOSD implies that player i 's probability of winning a prize is higher than that of player j for any given score, and hence also in expectation. To see this, note that by choosing $x > 0$ player i chooses a higher score than player j with probability $G_j(x)$, whereas by choosing x player j chooses a higher score than player i with probability $G_i(x)$. Therefore, because $G_j(x) \geq G_i(x)$, for any given score player i wins a prize with at least as high a probability as player j does, i.e., $P_i(x) \geq P_j(x)$. Therefore, $P_i = \int P_i(x) dG_i \geq \int P_j(x) dG_i$, and because $P_j(\cdot)$ is non-decreasing, by FOSD $\int P_j(x) dG_i \geq \int P_j(x) dG_j = P_j$.

C.3 Proof of Theorem 10

Let

$$f(\alpha, x) = \frac{1}{(1 - a_{m+1}c(x))} \frac{\prod_{k=i+1}^{m+1} (1 - a_k c(x) (1 - \alpha))}{(1 - a_i c(x) (1 - \alpha))^{m-i}},$$

and note that f is differentiable in $\alpha < 1$ and $x < T$. Denote by $s_i^l(\alpha)$ the lowest x such that $f(\alpha, x) = (\prod_{k=i+1}^m a_k) / a_i^{m-i}$. Note that $(\prod_{k=i+1}^m a_k) / a_i^{m-i} > 1$ (because a_k increases in k) and $f(\alpha, 0) = 1$ (since $c(0) = 0$). Suppose that when α increases to α' the value of f at $s_i^l(\alpha)$ increases. Then, because $f(\alpha', s_i^l(\alpha)) > (\prod_{k=i+1}^m a_k) / a_i^{m-i}$ and $f(\alpha', 0) < (\prod_{k=i+1}^m a_k) / a_i^{m-i}$, the intermediate value theorem shows that $s_i^l(\alpha') < s_i^l(\alpha)$. Therefore, to show that s_i^l decreases in α it suffices to show that $\partial f(a, x) / \partial \alpha > 0$ for $x \in (0, T)$. By considering the expression for $\partial f(a, x) / \partial \alpha$, one can verify that it suffices to show that

$$\begin{aligned} & \left(\sum_{j=i+1}^{m+1} a_j c(x) \prod_{k \in \{i+1, \dots, m+1\} \setminus j} (1 - a_k c(x) (1 - \alpha)) \right) (1 - a_i c(x) (1 - \alpha)) \\ & > \prod_{k=i+1}^{m+1} (1 - a_k c(x) (1 - \alpha)) ((m - i) a_i c(x)). \end{aligned}$$

For this inequality to hold, it suffices that for every $j = i + 1, \dots, m + 1$,

$$a_j c(x) > a_i c(x) \quad \text{and} \quad 1 - a_i c(x) (1 - \alpha) > 1 - a_j c(x) (1 - \alpha),$$

and this holds because a_k increases in k . Therefore, s_i^l decreases in $\alpha < 1$ for every player $i < m$. Now consider what happens to s_i^l as α approaches 0. For $x < T$,

$$f(0, x) = \frac{1}{(1 - a_{m+1}c(x))} \frac{\prod_{k=i+1}^{m+1} (1 - a_k c(x))}{(1 - a_i c(x))^{m-i}} = \frac{\prod_{k=i+1}^m (1 - a_k c(x))}{(1 - a_i c(x))^{m-i}} \leq 1.$$

Therefore, by uniform continuity of $f(\alpha, x)$ on $[0, \tilde{\alpha}] \times [0, x]$ for any $\tilde{\alpha} \in (0, 1)$, s_i^l approaches T as α approaches 0.

C.4 Proof of Corollary 6

Choose $\beta < 1$. By Theorem 10 there exist $\tilde{x} < T$ and $\tilde{\alpha} > 0$ such that for all $\alpha < \tilde{\alpha}$ and $i < m$, $s_i^l > \tilde{x}$. Choose such \tilde{x} and $\tilde{\alpha}$ that also satisfy

$$(2) \quad \frac{V_m - \gamma_m \frac{V_{m+1}}{\gamma_{m+1}} + \alpha \gamma_m c(\tilde{x})}{V_m - (1 - \alpha) \gamma_m c(\tilde{x})} > \beta$$

and

$$(3) \quad \frac{\alpha \gamma_{m+1} c(\tilde{x})}{V_{m+1} - (1 - \alpha) \gamma_{m+1} c(\tilde{x})} < 1 - \beta$$

for every $\alpha < \tilde{\alpha}$. Consider the unique equilibrium \mathbf{G} of a such a simple contest with $\alpha < \tilde{\alpha}$. Since $G_i(\tilde{x}) = 0$ for $i < m$ and $G_i(0) = 1$ for $i > m + 1$, Corollary 3 shows that the CDFs of players $m + 1$ and m on $[0, \tilde{x}]$ are given by (2) and (3). Since $s_i^l > \tilde{x}$ for $i < m$, each of these $m - 1$ players chooses a higher scores than player $m + 1$, and therefore wins a prize, with probability higher than β . Player m chooses scores higher than \tilde{x} with probability higher than β , and therefore wins a prize with probability higher than β^2 .