

An Exploration of Technology Diffusion: Online Appendix

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This Appendix contains further mathematical details behind the model and estimation procedures presented in the paper.

Derivation of equation (8):

The demand for capital of a particular vintage is given by the factor demand equation

$$(41) \quad R_v K_v = \alpha P_v Y_v.$$

Since revenue generated from the output produced with the vintage is determined by the demand function (7), we can write

$$(42) \quad R_v K_v = \alpha Y P^{\frac{\mu}{\mu-1}} P_v^{-\frac{1}{\mu-1}}.$$

Moreover, the price of the output produced with this vintage is given by the equilibrium unit production cost, such that we can write

$$(43) \quad R_v K_v = \alpha Y Z_v^{\frac{1}{\mu-1}} \left(\frac{(1-\alpha)}{W} \right)^{\frac{1-\alpha}{\mu-1}} \left(\frac{\alpha}{R_v} \right)^{\frac{\alpha}{\mu-1}},$$

such that

$$(44) \quad K_v = Y_\tau Z_v^{\frac{1}{\mu-1}} \left(\frac{(1-\alpha)}{W} \right)^{\frac{1-\alpha}{\mu-1}} \left(\frac{\alpha}{R_v} \right)^\epsilon,$$

where

$$(45) \quad \epsilon = \frac{\mu}{\mu-1} - \frac{1-\alpha}{\mu-1} = 1 + \frac{\alpha}{\mu-1},$$

which is equation (8).

Derivation of equation (9):

The Lagrangian associated with the dynamic profit maximization problem of the supplier of capital good v at time t equals

$$(46) \quad \mathcal{L}_{vt} = \int_t^\infty e^{-\int_t^s \tilde{r}_{s'} ds'} H_{vs} ds,$$

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where H_{vs} is the current value Hamiltonian. We will drop the time subscript s in what follows. Here

$$(47) \quad H_v = (R_v K_v - Q I_v) + \lambda_v \left(R_v K_v - \alpha Y Z_v^{\frac{1}{\mu-1}} \left(\frac{(1-\alpha)}{W} \right)^{\frac{1-\alpha}{\mu-1}} \left(\frac{\alpha}{R_v} \right)^{\epsilon-1} \right) + \nu_v (I_v - \delta K_v).$$

Here λ_v is the co-state variable associated with the demand function that the capital goods supplier faces and ν_v is the co-state variable associated with the capital accumulation equation.

The resulting optimality conditions read

$$(48) \quad \begin{aligned} \text{w.r.t. } R_v: & \quad (1 + \lambda_v) K_v + (\epsilon - 1) \lambda_v K_v = 0. \\ \text{w.r.t. } I_v: & \quad \nu_v = 1. \\ \text{w.r.t. } K_v: & \quad (1 + \lambda_v) R_v = \tilde{r} \nu_v - \dot{\nu}_v. \end{aligned}$$

The first optimality condition yields that

$$(49) \quad \lambda_v = -\frac{1}{\epsilon},$$

while the second and third yield that

$$(50) \quad \begin{aligned} R_v &= \frac{1}{(1 + \lambda_v)} \tilde{r} \\ &= \frac{\epsilon}{\epsilon - 1} \tilde{r}, \end{aligned}$$

which is (9). Note that the resulting flow profits satisfy

$$(51) \quad \pi_v = \frac{1}{\epsilon - 1} \tilde{r} K_v = \frac{1}{\epsilon} R_v K_v.$$

Derivation of equation (12):

Under the one-sector model assumptions, the price of intermediates produced with capital goods of vintage v and the aggregate price level equal

$$(52) \quad P_v = \frac{1}{Z_v} \left(\frac{W}{1-\alpha} \right)^{1-\alpha} \left(\frac{R}{\alpha} \right)^\alpha \quad \text{and} \quad P = \frac{1}{A} \left(\frac{W}{1-\alpha} \right)^{1-\alpha} \left(\frac{R}{\alpha} \right)^\alpha.$$

As a consequence, the relative price of output produced with vintage v is given by the relative TFP level, i.e.

$$(53) \quad \frac{P_v}{P} = P_v = \frac{A}{Z_v}.$$

From the demand function we obtain that the revenue from output produced with capital

goods of vintage v is given by

$$(54) \quad P_v Y_v = \left(\frac{P}{P_v} \right)^{\frac{1}{\mu-1}} Y = \left(\frac{Z_v}{A} \right)^{\frac{1}{\mu-1}} Y.$$

The flow profits that the capital goods producer of vintage v makes are equal to

$$(55) \quad \pi_v = \frac{\alpha}{\epsilon} P_v Y_v = \frac{\alpha}{\epsilon} \left(\frac{Z_v}{A} \right)^{\frac{1}{\mu-1}} Y.$$

This means that the market value of each of the capital goods suppliers of vintage v , at time t equals the present discounted value of the above flow profits. That is,

$$(56) \quad \begin{aligned} M_{v,t} &= \int_t^\infty e^{-\int_t^s \tilde{r}_{s'} ds'} \pi_{v,s} ds \\ &= \left(\frac{Z_v}{A_t} \right)^{\frac{1}{\mu-1}} \frac{\alpha}{\epsilon} \int_t^\infty e^{-\int_t^s \tilde{r}_{s'} ds'} \left(\frac{A_t}{A_s} \right)^{\frac{1}{\mu-1}} Y_s ds \\ &= \left(\frac{Z_v}{A_t} \right)^{\frac{1}{\mu-1}} \left[\frac{\alpha}{\epsilon} \int_t^\infty e^{-\int_t^s \tilde{r}_{s'} ds'} \left(\frac{A_t}{A_s} \right)^{\frac{1}{\mu-1}} \frac{Y_s}{Y_t} ds \right] Y_t \\ &= \left(\frac{Z_v}{A_t} \right)^{\frac{1}{\mu-1}} \Psi_t Y_t. \end{aligned}$$

Derivation of equilibrium adoption lag, (15):

The optimal adoption of technology vintages implies that the best vintage adopted at each instant satisfies

$$(57) \quad \Gamma_v = M_v.$$

The adoption costs satisfy

$$(58) \quad \begin{aligned} \Gamma_{vt} &= \bar{\Psi} (1+b) \left(\frac{Z_v}{Z_t} \right)^{\frac{\vartheta}{\mu-1}} P_v Y_v \\ &= \bar{\Psi} (1+b) \left(\frac{Z_v}{Z_t} \right)^{\frac{1+\vartheta}{\mu-1}} \left(\frac{Z_t}{A_t} \right)^{\frac{1}{\mu-1}} Y_t. \end{aligned}$$

Combining this with the market value of the capital goods supplier of capital good v , we obtain that the vintage that satisfies (57), solves

$$(59) \quad \left(\frac{Z_v}{Z_t} \right)^{\frac{\vartheta}{\mu-1}} = \min \left\{ 1, \frac{1}{1+b} \left(\frac{\Psi_t}{\bar{\Psi}} \right) \right\},$$

such that

$$(60) \quad \ln Z_v - \ln Z_t = \min \left\{ 0, -\frac{\mu-1}{\vartheta} \{ \ln(1+b) - \ln \Psi_t - \ln \bar{\Psi} \} \right\},$$

which means that the adoption lag equals

$$(61) \quad D_t = \max \left\{ \frac{\mu - 1}{\gamma \vartheta} \{ \ln(1 + b) - \ln \Psi_t - \ln \bar{\Psi} \}, 0 \right\} = D_t.$$

and constant across vintages, v .

Frictions and other adoption costs in (15):

Next, we show that the effect of several potential frictions in the economy on the adoption lags can be subsumed in the adoption cost b . We start by considering a tax on the rental price of the capital that embodies new technologies. Let $\phi_R R_v$ be the after tax price of the rental price of capital associated to vintage v net of taxes (i.e. $\phi_R \geq 1$). We assume for simplicity of exposition that ϕ_R is constant across vintages and over time.

Equations (52) and (8) now are:

$$(62) \quad P_v = \frac{1}{Z_v} \left(\frac{W}{1 - \alpha} \right)^{1 - \alpha} \left(\frac{\phi_R R_v}{\alpha} \right)^\alpha$$

and

$$(63) \quad K_v = Y Z_v^{\frac{1}{\mu - 1}} \left(\frac{1 - \alpha}{W} \right)^{\frac{1 - \alpha}{\mu - 1}} \left(\frac{\alpha}{\phi_R R_v} \right)^\epsilon.$$

The markup charged by capital goods producers is not affected by ϕ_R .

The profits flow of the capital producer of vintage v is:

$$(64) \quad \pi_v = \frac{\alpha}{\epsilon \phi_R} P_v Y_v = \frac{\alpha}{\epsilon \phi_R} \left(\frac{Z_v}{A} \right)^{\frac{1}{\mu - 1}} Y.$$

The market value of each capital goods supplier equals the present discounted value of the flow profits. That is,

$$(65) \quad M_{v,t} = \int_t^\infty e^{-\int_t^s \tilde{r}_{s'} ds'} \pi_{v,s} ds = \left(\frac{Z_v}{Z_t} \right)^{\frac{1}{\mu - 1}} \left(\frac{Z_t}{A_t} \right)^{\frac{1}{\mu - 1}} \frac{\Psi_t Y_t}{\phi_R},$$

where

$$(66) \quad \Psi_t = \frac{\alpha}{\epsilon} \int_t^\infty e^{-\int_t^s \tilde{r}_{s'} ds'} \left(\frac{A_t}{A_s} \right)^{\frac{1}{\theta - 1}} \left(\frac{Y_s}{Y_t} \right) ds$$

is the stockmarket capitalization to GDP ratio in the absence of taxes to the rental of capital goods (i.e. if $\phi_{R=1}$).

Optimal adoption implies that the adoption lag equals

$$(67) \quad D_v = \max \left\{ \frac{\mu - 1}{\gamma \vartheta} \{ \ln(1 + b) + \ln(\phi_R) - \ln V + \ln \bar{V} \}, 0 \right\} = D,$$

where it is clear that b and ϕ_R enter symmetrically in the adoption lag.

A similar friction of some interest is the expropriation risk. Suppose that the capital good producer faced a probability ϕ_E of being expropriated from the right to future profits right after having incurred in the cost of adopting a new vintage. Again, for simplicity let's assume that this probability is the same across vintages and over time. Then, the optimal adoption condition is

$$(68) \quad \Gamma_v \leq (1 - \phi_E)M_v.$$

This yields an adoption lag

$$(69) \quad D_v = \max \left\{ \frac{\theta - 1}{\gamma\vartheta} \{ \ln(1 + b) - \ln(1 - \phi_E) - \ln V + \ln \bar{V} \}, 0 \right\} = D,$$

where it is clear that b and ϕ_E enter symmetrically.

Note that the adoption lag we obtain would be the same if rather than a one time expropriation, the capital good producer faced a probability ϕ_E of being expropriated from the instantaneous profits every period.

Best vintage adopted:

In the main text, we present the equilibrium dynamics of the model for the particular case in which, at every instant, there are some vintages adopted. This does not have to be the case along all equilibrium paths of this economy. Here, in the appendix, we derive the general equilibrium dynamics of the model and subsequently explain how the main text is a special case.

For these general dynamics, we define \bar{v}_t as the best vintage adopted until time t . This means that if $\bar{v}_t > t - D_t$, then, at instant t , there will be no additional vintages adopted. In the main text, we limited ourselves to the case in which, at any point in time, $\bar{v}_t = t - D_t$.

Derivation of aggregate TFP, (16):

This allows us to write aggregate total factor productivity as

$$(70) \quad \begin{aligned} A_t &= \left(\int_{-\infty}^{\bar{v}_t} Z_v^{\frac{1}{\mu-1}} dv \right)^{\mu-1} \\ &= Z_0 \left(\int_{-\infty}^{\bar{v}_t} e^{\frac{\gamma}{\mu-1}v} dv \right)^{\mu-1} \\ &= Z_0 \left(\frac{\mu-1}{\gamma} \right)^{\mu-1} e^{\gamma\bar{v}_t} \\ &= A_0 e^{\gamma\bar{v}_t} \end{aligned}$$

which, under the assumption that $\bar{v}_t = t - D_t$, equals

$$(71) \quad A_t = A_0 e^{\gamma(t-D_t)}.$$

Derivation of aggregate adoption costs, (17):

We derive the aggregate adoption costs at each instant of time by taking the limit of the adoption cost at a period of time of length dt starting at time t for dt going to zero. The

total adoption costs between time t and $t + dt$ in the economy are given by:

$$\begin{aligned}
 (72) \quad \Gamma_t dt &= \int_{\bar{v}_t}^{\bar{v}_t+dt} \bar{\Psi} (1+b) \left(\frac{Z_v}{Z_t} \right)^{\frac{1+\vartheta}{\mu-1}} \left(\frac{Z_t}{A_t} \right)^{\frac{1}{\mu-1}} Y_t dv \\
 &= \bar{\Psi} (1+b) \left[\int_{\bar{v}_t}^{\bar{v}_t+dt} Z_v^{\frac{1+\vartheta}{\mu-1}} dv \right] \left(\frac{1}{Z_t} \right)^{\frac{1+\vartheta}{\mu-1}} \left(\frac{Z_t}{A_t} \right)^{\frac{1}{\mu-1}} Y_t \\
 &= \bar{\Psi} (1+b) \left(\frac{1}{Z_0} \right)^{\frac{\vartheta}{\mu-1}} e^{-\frac{\vartheta}{\mu-1} \gamma t} \left(\frac{1}{A_t} \right)^{\frac{1}{\mu-1}} Y_t \int_{\bar{v}_t}^{\bar{v}_t+dt} Z_v^{\frac{1+\vartheta}{\mu-1}} dv.
 \end{aligned}$$

Note that

$$(73) \quad \lim_{dt \downarrow 0} \left[\int_{\bar{v}_t}^{\bar{v}_t+dt} Z_v^{\frac{1+\vartheta}{\mu-1}} dv \right] / dt = Z_{\bar{v}_t}^{\frac{1+\vartheta}{\mu-1}} \dot{\bar{v}}_t.$$

This means that

$$\begin{aligned}
 (74) \quad &\lim_{dt \downarrow 0} \left[\bar{\Psi} (1+b) \left(\frac{1}{Z_0} \right)^{\frac{\vartheta}{\mu-1}} e^{-\frac{\vartheta}{\mu-1} \gamma t} \left(\frac{1}{A_t} \right)^{\frac{1}{\mu-1}} Y_t \int_{\bar{v}_t}^{\bar{v}_t+dt} Z_v^{\frac{1+\vartheta}{\mu-1}} dv \right] / dt \\
 &= \bar{\Psi} (1+b) \left(\frac{1}{Z_0} \right)^{\frac{\vartheta}{\mu-1}} e^{-\frac{\vartheta}{\mu-1} \gamma t} \left(\frac{1}{A_t} \right)^{\frac{1}{\mu-1}} Y_t \dot{\bar{v}}_t Z_{\bar{v}_t}^{\frac{1+\vartheta}{\mu-1}} \\
 &= \bar{\Psi} (1+b) e^{-\frac{\vartheta}{\mu-1} \gamma (t-\bar{v}_t)} \left(\frac{Z_0 e^{\gamma \bar{v}_t}}{A_t} \right)^{\frac{1}{\mu-1}} Y_t \dot{\bar{v}}_t.
 \end{aligned}$$

Hence, the aggregate adoption cost at each instant in time are given by

$$\begin{aligned}
 (75) \quad \Gamma_t &= \bar{\Psi} (1+b) e^{-\frac{\vartheta}{\mu-1} \gamma (t-\bar{v}_t)} \left(\frac{Z_0 e^{\gamma \bar{v}_t}}{A_t} \right)^{\frac{1}{\mu-1}} Y_t \dot{\bar{v}}_t \\
 &= \bar{\Psi} (1+b) \left(\frac{\gamma}{\mu-1} \right) e^{-\frac{\vartheta}{\mu-1} \gamma (t-\bar{v}_t)} Y_t \dot{\bar{v}}_t.
 \end{aligned}$$

Equilibrium:

Equilibrium in this case consists of the consumption Euler equation

$$(76) \quad \frac{\dot{C}_t}{C_t} = \left(\alpha \frac{\epsilon - 1}{\epsilon} \frac{Y_t}{K_t} - \rho \right)$$

where we have used that the real interest rate is related to the marginal product of capital as follows;

$$(77) \quad \tilde{r}_t = \alpha \frac{\epsilon - 1}{\epsilon} \frac{Y_t}{K_t};$$

The resource constraint

$$(78) \quad Y_t = C_t + I_t + \Gamma_t;$$

The capital accumulation equation

$$(79) \quad \dot{K}_t = I_t;$$

The production function

$$(80) \quad Y_t = A_t K_t^\alpha;$$

The aggregate TFP equation

$$(81) \quad A_t = A_0 e^{\gamma \bar{v}_t};$$

The adoption cost equation

$$(82) \quad \Gamma_t = \bar{\Psi} (1 + b) \left(\frac{\gamma}{\mu - 1} \right) e^{-\frac{\theta}{\mu-1} \gamma (t - \bar{v}_t)} Y_t \dot{\bar{v}}_t;$$

The adoption lag equation

$$(83) \quad D_t = \max \left\{ \frac{\mu - 1}{\gamma \theta} \{ \ln(1 + b) - \ln V_t - \ln \bar{V} \}, 0 \right\};$$

And the market value equation

$$(84) \quad \Psi_t = \frac{\alpha}{\epsilon} \int_t^\infty e^{-\int_t^s \tilde{r}_{s'} ds'} \left(\frac{A_t}{A_s} \right)^{\frac{1}{\theta-1}} \frac{Y_s}{Y_t} ds,$$

which is best written in changes over time

$$(85) \quad \frac{\dot{\Psi}_t}{\Psi_t} = \left\{ \alpha \frac{\epsilon - 1}{\epsilon} \frac{Y_t}{K_t} + \frac{1}{\mu - 1} \frac{\dot{A}_t}{A_t} - \frac{\dot{Y}_t}{Y_t} \right\} - \frac{\alpha}{\epsilon} \frac{1}{\Psi_t};$$

and the technology adoption equation

$$(86) \quad \dot{\bar{v}}_t = \begin{cases} \max \left\{ 1 - D_t, 0 \right\} & \text{if } \bar{v}_t = t - D_t \\ 0 & \text{if } \bar{v}_t > t - D_t \end{cases}.$$

Because, in the main text we assumed that $\bar{v}_t = t - D_t$ for all t , the dynamic equilibrium equations in the main text are based on the assumption that along the equilibrium paths considered $\dot{\bar{v}}_t = 1 - D_t$, and thus that $\dot{D}_t < 1$.

Balanced growth path:

We will consider the balanced growth path in this economy in deviation from the trend

$$(87) \quad \bar{A}_t = A_0 e^{\gamma t}.$$

The nine transformed/detrended variables on the balanced growth path are

$$(88) \quad C_t^* = \frac{C_t}{\bar{A}_t^{1-\alpha}}, Y_t^* = \frac{Y_t}{\bar{A}_t^{1-\alpha}}, I_t^* = \frac{I_t}{\bar{A}_t^{1-\alpha}}, K_t^* = \frac{K_t}{\bar{A}_t^{1-\alpha}}, \Gamma_t^* = \frac{\Gamma_t}{\bar{A}_t^{1-\alpha}}, \text{ and } A_t^* = \frac{A_t}{\bar{A}_t},$$

as well as

$$(89) \quad D_t, \Psi_t, \text{ and } \bar{v}_t^* = \bar{v}_t - t.$$

Derivation of transformed dynamic system:

The resulting dynamic equations that define the transitional dynamics of the economy around the balanced growth path are the following Euler equation

$$(90) \quad \frac{\dot{C}_t^*}{C_t^*} = \left(\alpha \frac{\epsilon - 1}{\epsilon} \frac{Y_t^*}{K_t^*} - \rho \right) - \frac{\gamma}{1 - \alpha};$$

The resource constraint

$$(91) \quad Y_t^* = C_t^* + I_t^* + \Gamma_t^*;$$

The capital accumulation equation

$$(92) \quad \frac{\dot{K}_t^*}{K_t^*} = -\frac{\gamma}{1 - \alpha} + \frac{I_t^*}{K_t^*};$$

The production function

$$(93) \quad Y_t^* = A_t^* (K_t^*)^\alpha;$$

The trend adjusted productivity level

$$(94) \quad A_t^* = e^{\gamma \bar{v}_t^*};$$

The aggregate adoption cost

$$(95) \quad \Gamma_t^* = \bar{\Psi} (1 + b) \left(\frac{\gamma}{\mu - 1} \right) e^{-\frac{\vartheta}{\mu - 1} \gamma (t - \bar{v}_t^*)} Y_t^* \left(\bar{v}_t^* + 1 \right);$$

The adoption lag

$$(96) \quad D_t = \max \left\{ \frac{\mu - 1}{\vartheta \gamma} \{ \ln(1 + b) - \ln \Psi_t + \ln \bar{\Psi} \}, 0 \right\};$$

and the market value transitional equation

$$(97) \quad \frac{\dot{\Psi}_t}{\Psi_t} = \left\{ \alpha \frac{\epsilon - 1}{\epsilon} \frac{Y_t^*}{K_t^*} + \frac{1}{\mu - 1} \left\{ \frac{\dot{A}_t^*}{A_t^*} + \gamma \right\} - \left\{ \frac{\dot{Y}_t^*}{Y_t^*} + \frac{\gamma}{1 - \alpha} \right\} \right\} - \frac{\alpha}{\epsilon} \frac{1}{\Psi_t};$$

as well as the adoption law of motion

$$(98) \quad \dot{\bar{v}}_t^* = \begin{cases} \max \left\{ -\dot{D}_t, -1 \right\} & \text{if } \bar{v}_t^* = -D_t \\ -1 & \text{if } \bar{v}_t^* > -D_t \end{cases}.$$

Steady state equations:

The steady state is defined by the following equations

$$(99) \quad 0 = \left(\alpha \frac{\epsilon - 1}{\epsilon} \frac{\bar{Y}^*}{\bar{K}^*} - \rho \right) - \frac{\gamma}{1 - \alpha};$$

The resource constraint

$$(100) \quad \bar{Y}^* = \bar{C}^* + \bar{I}^* + \bar{\Gamma}^*;$$

The capital accumulation equation

$$(101) \quad 0 = -\frac{\gamma}{1 - \alpha} + \frac{\bar{I}^*}{\bar{K}^*};$$

The production function

$$(102) \quad \bar{Y}^* = \bar{A}^* (\bar{K}^*)^\alpha;$$

The trend adjusted productivity level

$$(103) \quad \bar{A}^* = e^{-\gamma \bar{D}};$$

The aggregate adoption cost

$$(104) \quad \bar{\Gamma}^* = \bar{\Psi} (1 + b) \left(\frac{\gamma}{\mu - 1} \right) e^{\frac{\vartheta}{\mu - 1} \gamma \bar{v}^*} \bar{Y}^*;$$

The steady state adoption lag, assuming that $b \geq 0$, equals

$$(105) \quad \bar{D} = \frac{\mu - 1}{\vartheta \gamma} \ln(1 + b);$$

and the market value transitional equation

$$(106) \quad 0 = \left\{ \alpha \frac{\epsilon - 1}{\epsilon} \frac{Y_t^*}{K_t^*} + \frac{\gamma}{\mu - 1} - \frac{\gamma}{1 - \alpha} \right\} - \frac{\alpha}{\epsilon} \frac{1}{\bar{\Psi}};$$

as well as

$$(107) \quad \bar{v}^* = -\bar{D}.$$

Steady state solution:

Combining the Euler equation with the market cap to GDP equation, we obtain that

$$(108) \quad 0 = \left\{ \rho + \frac{\gamma}{\mu - 1} \right\} - \frac{\alpha}{\epsilon} \frac{1}{\bar{\Psi}},$$

such that the steady state market cap to GDP ratio equals

$$(109) \quad \bar{\Psi} = \frac{\alpha}{\epsilon} \frac{1}{\left\{ \rho + \frac{\gamma}{\mu - 1} \right\}}.$$

The steady state trend adjusted level of productivity equals

$$(110) \quad \bar{A}^* = \left[\left(\frac{1}{1+b} \right)^{\frac{1}{\vartheta}} \right]^{(\mu-1)}.$$

When we insert this into the Euler equation, we find that

$$(111) \quad 0 = \left(\alpha \frac{\epsilon - 1}{\epsilon} \left[\left(\frac{1}{1+b} \right)^{\frac{1}{\vartheta}} \right]^{(\mu-1)} \left(\frac{1}{\bar{K}^*} \right)^{1-\alpha} - \delta - \rho \right) - \frac{\gamma}{1-\alpha},$$

which allows us to solve for the steady state capital stock

$$(112) \quad \bar{K}^* = \left[\frac{\alpha \frac{\epsilon - 1}{\epsilon} \bar{A}^*}{\rho + \frac{\gamma}{1-\alpha}} \right]^{\frac{1}{1-\alpha}} = \left[\frac{\alpha \frac{\epsilon - 1}{\epsilon} \left[\left(\frac{1}{1+b} \right)^{\frac{1}{\vartheta}} \right]^{(\mu-1)}}{\rho + \frac{\gamma}{1-\alpha}} \right]^{\frac{1}{1-\alpha}},$$

such that

$$(113) \quad \begin{aligned} \bar{I}^* &= \frac{\gamma}{1-\alpha} \bar{K}^* \\ &= \frac{\gamma}{1-\alpha} \left[\frac{\alpha \frac{\epsilon - 1}{\epsilon} \left[\left(\frac{1}{1+b} \right)^{\frac{1}{\vartheta}} \right]^{(\mu-1)}}{\rho + \frac{\gamma}{1-\alpha}} \right]^{\frac{1}{1-\alpha}}, \end{aligned}$$

and output equals

$$(114) \quad \bar{Y}^* = \bar{A}^* (\bar{K}^*)^\alpha = \left\{ \left[\left(\frac{1}{1+b} \right)^{\frac{1}{\vartheta}} \right]^{(\mu-1)} \right\}^{\frac{1}{1-\alpha}} \left[\frac{\alpha \frac{\epsilon-1}{\epsilon}}{\rho + \frac{\gamma}{1-\alpha}} \right]^{\frac{\alpha}{1-\alpha}},$$

while the aggregate adoption cost is

$$(115) \quad \bar{\Gamma}^* = \bar{V} (1+b) \left(\frac{\gamma}{\mu-1} \right)^{\frac{1}{\theta-1}} e^{\mu-1 \gamma \bar{v}^*} \bar{Y}^*,$$

and steady state consumption equals

$$(116) \quad \bar{C}^* = \bar{Y}^* - \bar{I}^* - \bar{\Gamma}^*.$$

Note that, for steady state consumption to be positive, we need a restriction on the parameters, such that the total adoption costs do not fully exhaust productive capacity.

Transitional dynamics:

The next thing is to linearize the transitional dynamics around the steady state. Note that this model has only one state variable, namely the capital stock K_t . The stock market capitalization to GDP ratio, V_t , is a jump variable and so are the adoption lag, D_t , the best vintage adopted, \bar{v}_t , and the trend adjusted productivity level, A_t^* .

The log-linearized equations are the Euler equation

$$(117) \quad \dot{\hat{C}}_t^* = \alpha \frac{\epsilon-1}{\epsilon} \frac{\bar{Y}^*}{\bar{K}^*} \hat{Y}_t^* - \alpha \frac{\epsilon-1}{\epsilon} \frac{\bar{Y}^*}{\bar{K}^*} \hat{K}_t^*;$$

The resource constraint

$$(118) \quad 0 = \hat{Y}_t^* - \frac{\bar{C}^*}{\bar{Y}^*} \hat{C}_t^* - \frac{\bar{I}^*}{\bar{Y}^*} \hat{I}_t^* - \frac{\bar{\Gamma}^*}{\bar{Y}^*} \hat{Y}_t^*;$$

The capital accumulation equation

$$(119) \quad \dot{\hat{K}}_t^* = \frac{\bar{I}^*}{\bar{K}^*} \hat{I}_t^* - \frac{\bar{I}^*}{\bar{K}^*} \hat{K}_t^*;$$

The production function

$$(120) \quad 0 = \hat{Y}_t^* - \hat{A}_t^* - \alpha \hat{K}_t^*;$$

The trend adjusted productivity level

$$(121) \quad 0 = \hat{A}_t^* - \gamma (\bar{v}_t^* - \bar{v}^*);$$

The adoption lag equation

$$(122) \quad 0 = (D_t - \bar{D}) + \frac{\mu-1}{\vartheta\gamma} \hat{V}_t;$$

The aggregate adoption cost

$$(123) \quad \dot{\bar{v}}_t^* = \hat{\Gamma}_t^* - \frac{\vartheta}{\mu - 1} \gamma (\bar{v}_t^* - \bar{\bar{v}}^*) - \hat{Y}_t^*,$$

and the market capitalization equation

$$(124) \quad \dot{\hat{\Psi}}_t = \alpha \frac{\epsilon - 1}{\epsilon} \frac{\bar{Y}^*}{\bar{K}^*} \hat{Y}_t^* - \alpha \frac{\epsilon - 1}{\epsilon} \frac{\bar{Y}^*}{\bar{K}^*} \hat{K}_t^* + \frac{\gamma}{\mu - 1} \dot{\bar{v}}_t^* - \gamma \dot{\bar{v}}_t^* - \alpha \dot{\hat{K}}_t^* + \frac{\alpha}{\epsilon} \frac{1}{\bar{\Psi}} \hat{\Psi}_t,$$

which simplifies to

$$(125) \quad \dot{\hat{\Psi}}_t + \left(1 - \frac{1}{\theta - 1}\right) \gamma \dot{\bar{v}}_t^* + \alpha \dot{\hat{K}}_t^* = \alpha \frac{\epsilon - 1}{\epsilon} \frac{\bar{Y}^*}{\bar{K}^*} \hat{Y}_t^* - \alpha \frac{\epsilon - 1}{\epsilon} \frac{\bar{Y}^*}{\bar{K}^*} \hat{K}_t^* + \frac{\alpha}{\epsilon} \frac{1}{\bar{\Psi}} \hat{\Psi}_t,$$

where we have assumed that, all along the equilibrium path $\bar{v}_t^* = -D_t$, such that

$$(126) \quad \dot{\bar{v}}_t^* = \begin{cases} \max \left\{ -\frac{\mu - 1}{\vartheta \gamma} \dot{\hat{\Psi}}_t, -1 \right\} & \text{if } \bar{v}_t^* = -D_t \\ -1 & \text{if } \bar{v}_t^* > -D_t \end{cases}.$$

For our examples, we limit ourselves to the part of the transitional path for which $\bar{v}_t^* = -D_t$ for all t . On that path, the transitional dynamics simplify, because then

$$(127) \quad \bar{v}_t^* - \bar{\bar{v}}^* = -(D_t - \bar{D})$$

and

$$(128) \quad \dot{\bar{v}}_t^* = -\dot{D}_t = \frac{\mu - 1}{\vartheta \gamma} \dot{\hat{\Psi}}_t,$$

which allows us to write

$$(129) \quad 0 = \hat{A}_t^* + \gamma (D_t - \bar{D}),$$

and

$$(130) \quad \frac{\mu - 1}{\vartheta \gamma} \dot{\hat{\Psi}}_t = \hat{\Gamma}_t^* + \frac{\vartheta}{\mu - 1} \gamma (D_t - \bar{D}) - \hat{Y}_t^*,$$

as well as

$$(131) \quad \left[1 + (\mu - 2) \frac{\gamma}{\vartheta \gamma}\right] \dot{\hat{\Psi}}_t + \alpha \dot{\hat{K}}_t^* = \alpha \frac{\epsilon - 1}{\epsilon} \frac{\bar{Y}^*}{\bar{K}^*} \hat{Y}_t^* - \alpha \frac{\epsilon - 1}{\epsilon} \frac{\bar{Y}^*}{\bar{K}^*} \hat{K}_t^* + \frac{\alpha}{\epsilon} \frac{1}{\bar{\Psi}} \hat{\Psi}_t$$

Derivation of intermediate technology aggregation results:

The factor demands for each of the vintage specific output types satisfy

$$(132) \quad WL_\tau = W \int_{v \in V_\tau} L_v dv = (1 - \alpha) \int_{v \in V_\tau} P_v Y_v dv = (1 - \alpha) P_\tau Y_\tau,$$

and

$$(133) \quad RK_\tau = R \int_{v \in V_\tau} K_{v\tau} dv = \alpha \int_{v \in V_\tau} P_v Y_v dv = \alpha P_\tau Y_\tau.$$

Hence relative factor demands are the same as relative revenue levels

$$(134) \quad \frac{P_v Y_v}{P_\tau Y_\tau} = \left(\frac{Y_v}{Y_\tau} \right)^{\frac{1}{\mu}} = \frac{L_v}{L_\tau} = \frac{K_v}{K_\tau},$$

which allows us to write

$$(135) \quad \begin{aligned} Y_v &= Z_v K_v^\alpha L_v^{1-\alpha} = Z_v \left(\frac{Y_v}{Y_\tau} \right)^{\frac{1}{\mu}} K_\tau^\alpha L_\tau^{1-\alpha} \\ &= (Z_v)^{\frac{\mu}{\mu-1}} \left(\frac{1}{Y_\tau} \right)^{\frac{1}{\mu}} (K_\tau^\alpha L_\tau^{1-\alpha})^{\frac{\mu}{\mu-1}}. \end{aligned}$$

We obtain that

$$(136) \quad \begin{aligned} Y_\tau &= \left(\int_{v \in V_\tau} Y_v^{\frac{1}{\mu}} dv \right)^\mu = \left(\int_{v \in V_\tau} Z_v^{\frac{1}{\mu-1}} dv \right)^\mu \left(\frac{1}{Y_\tau} \right)^{\frac{1}{\mu}} (K_\tau^\alpha L_\tau^{1-\alpha})^{\frac{\mu}{\mu-1}} \\ &= \left(\int_{v \in V_\tau} Z_v^{\frac{1}{\mu-1}} dv \right)^{\mu-1} (K_\tau^\alpha L_\tau^{1-\alpha}) = A_\tau K_\tau^\alpha L_\tau^{1-\alpha}. \end{aligned}$$

The value of the unit production cost follows from the unit production cost of a Cobb-Douglas production function. The aggregation results at the highest level of aggregation can be derived in a similar way.