

THE EVOLUTION OF TIME PREFERENCE WITH AGGREGATE UNCERTAINTY

Technical Appendix: Details of Proofs

Not for publication

Equation numbers such as (17) refer to equations in the paper, while (A1) denotes an equation in the technical appendix.

Proof of Proposition 6

We begin the argument with equations (25)–(26) of the paper. These indicate that we have increasing impatience if, for $\tau = 2, \dots, T - 2$,

$$-\frac{dx_{\tau+1}}{dx_{\tau}} < -\frac{dx_{\tau+2}}{dx_{\tau+1}}$$

or

$$\begin{aligned} \frac{\frac{d\Lambda}{dx_{\tau}}}{\frac{d\Lambda}{dx_{\tau+1}}} &= \frac{\alpha\Phi^{T-\tau} + \epsilon^2\beta(\tau-1)\Phi^{T-\tau} - \epsilon^2\gamma K_{\tau} + O(\epsilon^3)}{\alpha\Phi^{T-\tau-1} + \epsilon^2\beta\tau\Phi^{T-\tau-1} - \epsilon^2\gamma K_{\tau+1} + O(\epsilon^3)} \\ &< \frac{\alpha\Phi^{T-\tau-1} + \epsilon^2\beta\tau\Phi^{T-\tau-1} - \epsilon^2\gamma K_{\tau+1} + O(\epsilon^3)}{\alpha\Phi^{T-\tau-2} + \epsilon^2\beta(\tau+1)\Phi^{T-\tau-2} - \epsilon^2\gamma K_{\tau+2} + O(\epsilon^3)} = \frac{\frac{d\Lambda}{dx_{\tau+1}}}{\frac{d\Lambda}{dx_{\tau+2}}}. \end{aligned}$$

For ϵ sufficiently small, this inequality is implied by⁴⁵

$$\begin{aligned} &\epsilon^2\alpha\Phi^{T-\tau}\beta(\tau+1)\Phi^{T-\tau-2} - \epsilon^2\alpha\Phi^{T-\tau}\gamma K_{\tau+2} \\ &\epsilon^2\alpha\Phi^{T-\tau-2}\beta(\tau-1)\Phi^{T-\tau} - \epsilon^2\alpha\Phi^{T-\tau-2}\gamma K_{\tau} \\ &< 2\epsilon^2\alpha\Phi^{T-\tau-1}\beta\tau\Phi^{T-\tau-1} - 2\epsilon^2\alpha\Phi^{T-\tau-1}\gamma K_{\tau+1}. \end{aligned}$$

The terms involving β cancel one another. Then dividing by $-\epsilon^2\alpha\gamma$, it suffices that

$$(A3) \quad \Phi^{T-\tau} K_{\tau+2} + \Phi^{T-\tau-2} K_{\tau} > 2\Phi^{T-\tau-1} K_{\tau+1}.$$

⁴⁵Cross multiplication gives identical terms of order ϵ^0 on both sides. The next largest terms, of order ϵ^2 , are collected below.

Dividing by $\Phi^{T-\tau}$ and substituting for K , this is

$$\begin{aligned} & 2 \sum_{j=1}^T \Phi^{2T-(\tau+2)-j} \mathbf{C}_{\tau+2,j} + \Phi^{-2} \left(2 \sum_{j=1}^T \Phi^{2T-\tau-j} \mathbf{C}_{\tau,j} \right) \\ & > \Phi^{-1} \left(4 \sum_{j=1}^T \Phi^{2T-(\tau+1)-j} \mathbf{C}_{\tau+1,j} \right). \end{aligned}$$

Recalling our assumption that each covariance is equal to C and each variance equal to V , we see that the terms in the summation corresponding to values of j other than τ , $\tau + 1$ and $\tau + 2$ cancel. Dividing by $\Phi^{2T-\tau}$, it suffices that

$$\begin{aligned} & 2\Phi^{-\tau-2}C + 2\Phi^{-\tau-3}C + 2\Phi^{-\tau-4}V \\ & + 2\Phi^{-\tau-2}V + 2\Phi^{-\tau-3}C + 2\Phi^{-\tau-4}C \\ & > 4\Phi^{-\tau-2}C + 4\Phi^{-\tau-3}V + 4\Phi^{-\tau-4}C. \end{aligned}$$

Dividing by $2\Phi^{-\tau}$, we can rearrange to obtain the sufficient condition

$$2\Phi^{-3}C + \Phi^{-4}V + \Phi^{-2}V > \Phi^{-2}C + \Phi^{-4}C + 2\Phi^{-3}V.$$

Multiplying by Φ^4 , this is equivalent to

$$2\Phi C + V + \Phi^2 V > \Phi^2 C + C + 2\Phi V$$

or

$$V(\Phi - 1)^2 > C(\Phi - 1)^2,$$

and so the result follows. ■

Proof of Proposition 5

We begin with equation (28) of the paper, giving

$$\Lambda = \ln \phi + E \ln \tilde{S} - \ln S - \frac{\epsilon^2 u_T^2 v_1^2}{2\phi^2} E \left(\frac{S}{\tilde{S}} \right)^2 K.$$

Then, analogously to (22) of the paper, we are interested in derivatives of the form (where $d\Lambda/d\phi$ is derived analogously to (21))

$$\frac{d\Lambda}{dx_\tau} = \frac{D\Lambda}{D\phi} \Phi^{T-\tau} \frac{d\phi}{dx_T} + \frac{d\Lambda}{dv_1} \frac{dv_1}{dx_\tau} - \frac{\epsilon^2 u_T^2 v_1^2}{2\phi^2} E \left(\frac{S}{\tilde{S}} \right)^2 \frac{dK}{dx_\tau}.$$

Then following the reasoning that took us from steps (22) of the paper to (A3), we have decreasing marginal rates of substitution if⁴⁶

$$\Phi^{T-\tau} \frac{dK}{dx_{\tau+2}} + \Phi^{T-\tau-2} \frac{dK}{dx_{\tau}} < 2\Phi^{T-\tau-1} \frac{dK}{dx_{\tau+1}},$$

or

$$(A4) \quad \Phi^2 \frac{dK}{dx_{\tau+2}} + \frac{dK}{dx_{\tau}} < 2\Phi \frac{dK}{dx_{\tau+1}}.$$

To verify (A4), we must first calculate dK/dx_{τ} . This is (see (27) for k_i)

$$\begin{aligned} \frac{dK}{dx_{\tau}} &= \sum_{j=1}^T \Phi^{2T-\tau-j} x_j \mathbf{C}_{\tau j} + \sum_{i=1}^T \Phi^{2T-i-\tau} x_i \mathbf{C}_{i\tau} + \sum_{i=1}^{T-1} \sum_{j=1}^{T-1} \Phi^{2T-i-j} \left(\frac{dk_i}{dx_{\tau}} k_j + \frac{dk_j}{dx_{\tau}} k_i \right) C_{ij} \\ &\quad + \sum_{j=1}^{T-1} \Phi^{2T-\tau-j} k_j \mathbf{C}_{\tau j} + \sum_{i=1}^T \sum_{j=1}^{T-1} \Phi^{2T-i-j} x_i \frac{dk_j}{dx_{\tau}} \mathbf{C}_{ij} \\ &= \Phi^{2T} \left(2 \sum_{j=1}^T \Phi^{-\tau-j} x_j \mathbf{C}_{\tau j} + 2 \sum_{i=1}^{T-1} \sum_{j=1}^{T-1} \Phi^{-i-j} \frac{dk_i}{dx_{\tau}} k_j C_{ij} \right. \\ &\quad \left. + \sum_{j=1}^{T-1} \Phi^{-\tau-j} k_j \mathbf{C}_{\tau j} + \sum_{i=1}^T \sum_{j=1}^{T-1} \Phi^{-i-j} x_i \frac{dk_j}{dx_{\tau}} \mathbf{C}_{ij} \right) \\ &= \Phi^{2T} \left(2 \sum_{j=1}^T \Phi^{-\tau-j} x_j \mathbf{C}_{\tau j} + 2 \sum_{i=1}^{\tau-1} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-(\tau-i)} k_j C_{ij} \right. \\ &\quad \left. + \sum_{j=1}^{T-1} \Phi^{-\tau-j} k_j \mathbf{C}_{\tau j} + \sum_{i=1}^T \sum_{j=1}^{\tau-1} \Phi^{-i-j} x_i \Phi^{-(\tau-j)} \mathbf{C}_{ij} \right), \end{aligned}$$

where the first equality collects like terms and the second uses (27) to take derivatives of k_i . Inserting in (A4) we then have increasing impatience if

⁴⁶At this point, we simply write dK/dx_{τ} rather than taking the derivative explicitly; the corresponding derivative in moving from (20) to (A3) is $2 \sum_{j=1}^T \Phi^{2T-\tau-j} x_j \mathbf{C}_{\tau j}$, the notation for which we subsequently simplify by letting $K_{\tau} \equiv 2 \sum_{j=1}^T \Phi^{2T-\tau-j} \mathbf{C}_{\tau j}$ when deriving (A3).

$$\begin{aligned}
& 2\Phi^2 \sum_{j=1}^T \Phi^{-(\tau+2)-j} x_j \mathbf{C}_{\tau+2,j} + 2\Phi^2 \sum_{i=1}^{\tau+1} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-((\tau+2)-i)} k_j C_{ij} \\
& + \Phi^2 \sum_{j=1}^{T-1} \Phi^{-(\tau+2)-j} k_j \mathbf{C}_{\tau+2,j} + \Phi^2 \sum_{i=1}^T \sum_{j=1}^{\tau+1} \Phi^{-i-j} x_i \Phi^{-((\tau+2)-j)} \mathbf{C}_{ij} \\
& + 2 \sum_{j=1}^T \Phi^{-\tau-j} x_j \mathbf{C}_{\tau j} + 2 \sum_{i=1}^{\tau-1} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-(\tau-i)} k_j C_{ij} \\
& + \sum_{j=1}^{T-1} \Phi^{-\tau-j} k_j \mathbf{C}_{\tau j} + \sum_{i=1}^T \sum_{j=1}^{\tau-1} \Phi^{-i-j} x_i \Phi^{-(\tau-j)} \mathbf{C}_{ij} \\
< & 4\Phi \sum_{j=1}^T \Phi^{-(\tau+1)-j} x_j \mathbf{C}_{\tau+1,j} + 4\Phi \sum_{i=1}^{\tau} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-((\tau+1)-i)} k_j C_{ij} \\
& + 2\Phi \sum_{j=1}^{T-1} \Phi^{-(\tau+1)-j} k_j \mathbf{C}_{\tau+1,j} + 2\Phi \sum_{i=1}^T \sum_{j=1}^{\tau} \Phi^{-i-j} x_i \Phi^{-((\tau+1)-j)} \mathbf{C}_{ij}.
\end{aligned}$$

It is then helpful to tackle this inequality in parts. We begin with the first and fifth terms on the left, and the first on the right. These are precisely the terms that entered the calculations in proving Proposition 6, leading to increasing impatience. In this case, given our assumption that the random variables \hat{Z}_τ , for $\tau = 1, \dots, T$, are perfectly correlated, these terms cancel.

Now we work on the second and sixth terms on the left and the second on the right.

We have

$$\begin{aligned}
& 2\Phi^2 \sum_{i=1}^{\tau+1} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-((\tau+2)-i)} k_j C_{ij} + 2 \sum_{i=1}^{\tau-1} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-(\tau-i)} k_j C_{ij} \\
< & 4\Phi \sum_{i=1}^{\tau} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-((\tau+1)-i)} k_j C_{ij}
\end{aligned}$$

if

$$\begin{aligned}
& 2 \sum_{i=1}^{\tau+1} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-(\tau-i)} k_j C_{ij} + 2 \sum_{i=1}^{\tau-1} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-(\tau-i)} k_j C_{ij} \\
< & 4 \sum_{i=1}^{\tau} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-(\tau-i)} k_j C_{ij}
\end{aligned}$$

if

$$2 \sum_{i=\tau}^{\tau+1} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-(\tau-i)} k_j C_{ij} < 4 \sum_{i=\tau}^{\tau} \sum_{j=1}^{T-1} \Phi^{-i-j} \Phi^{-(\tau-i)} k_j C_{ij}$$

if

$$2 \sum_{j=1}^{T-1} \Phi^{-\tau+j} k_j C_{\tau+1,j} < 2 \sum_{j=1}^{T-1} \Phi^{-\tau-j} k_j C_{\tau j}$$

if

$$\Phi^{-\tau} k_{\tau} C + \Phi^{-(\tau+1)} k_{\tau+1} V < \Phi^{-\tau} k_{\tau} V + \Phi^{-(\tau+1)} k_{\tau+1}$$

if

$$2\Phi^{-2\tau} C \left(k_{\tau} - \frac{k_{\tau+1}}{\Phi} \right) < 2\Phi^{-2\tau} V \left(k_{\tau} - \frac{k_{\tau+1}}{\Phi} \right)$$

if

$$(A5) \quad 2\Phi^{-2\tau} x_{\tau+1} C < 2\Phi^{-2\tau} x_{\tau+1} V,$$

which follows from $C < V$ given $x_{\tau+1} > 0$.

Now we turn to the third and seventh term on the left and the third on the right.

Here we have

$$\Phi^2 \sum_{j=1}^{T-1} \Phi^{-(\tau+2)-j} k_j \mathfrak{C}_{\tau+2,j} + \sum_{j=1}^{T-1} \Phi^{-\tau-j} k_j \mathfrak{C}_{\tau j} = 2\Phi \sum_{j=1}^{T-1} \Phi^{-(\tau+1)-j} k_j \mathfrak{C}_{\tau+1,j}.$$

Taking out a factor $\Phi^{-\tau}$, this holds if

$$\sum_{j=1}^{T-1} \Phi^{-j} k_j \mathfrak{C}_{\tau+2,j} + \sum_{j=1}^{T-1} \Phi^{-j} k_j \mathfrak{C}_{\tau j} = 2 \sum_{j=1}^{T-1} \Phi^{-j} k_j \mathfrak{C}_{\tau+1,j}.$$

However, each of the terms $\mathfrak{C}_{\tau\tau'}$ represents the common covariance C between one of the random variables $\hat{Z}_1, \dots, \hat{Z}_{\tau-1}$ and the \hat{Z}_0 . Hence, this equality holds.

Finally, we work on the fourth and eighth terms on the left, and the fourth term on the right. Here, we have

$$\Phi^2 \sum_{i=1}^T \sum_{j=1}^{\tau+1} \Phi^{-i-j} x_i \Phi^{-((\tau+2)-j)} \mathfrak{C}_{ij} + \sum_{i=1}^T \sum_{j=1}^{\tau-1} \Phi^{-i-j} x_i \Phi^{-(\tau-j)} \mathfrak{C}_{ij} = 2\Phi \sum_{i=1}^T \sum_{j=1}^{\tau} \Phi^{-i-j} x_i \Phi^{-(\tau+1)-j} \mathfrak{C}_{ij}.$$

Taking out $\Phi^{-\tau}$, this holds if

$$\sum_{i=1}^T \sum_{j=1}^{\tau+1} x_i \Phi^{-i} \mathfrak{C}_{ij} + \sum_{i=1}^T \sum_{j=1}^{\tau-1} x_i \Phi^{-i} \mathfrak{C}_{ij} = \sum_{i=1}^T \sum_{j=1}^{\tau} x_i \Phi^{-i} \mathfrak{C}_{ij}.$$

Once again, each of the terms $\mathfrak{C}_{\tau\tau'}$ represents the common covariance C between one of the random variables $\hat{Z}_1, \dots, \hat{Z}_{\tau-1}$ and the \hat{Z}_0 . These terms are thus constant in j , and hence the equality can be verified by simply counting the number of terms on each side. The desired result (A4) then follows from (A5). ■