# Online Appendix

This appendix provides supplementary material for "Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match." The numbering of sections parallels portions of the Appendix (i.e. A2' corresponds to Appendix A2).

## A2': Description of New York City High School Admissions

The following table summarizes the distribution of assignments from students in Round 1 and Round 2.

Table A2: Distribution of Assignments
from Round 1 and Round 2 in New York City

| Choice | 2003-04 | 2004-05 | 2005-06 | 2006-07 |
|---|---|---|---|---|
| 1 | 31,021 (40.2%) | 33,083 (41.4%) | 38,727 (47.0%) | 37,270 (46.5%) |
| 2 | 12,504 (16.2%) | 14,818 (18.6%) | 16,524 (20.1%) | 15,898 (19.8%) |
| 3 | 8,713 (11.3%) | 9,929 (12.4%) | 9,882 (12.0%) | 9,845 (12.3%) |
| 4 | 6,587 (8.5%) | 6,927 (8.7%) | 6,308 (7.7%) | 6,369 (7.9%) |
| 5 | 4,893 (6.3%) | 4,739 (5.9%) | 3,984 (4.8%) | 4,051 (5.1%) |
| 6 | 3,652 (4.7%) | 3,415 (4.3%) | 2,699 (3.3%) | 2,532 (3.2%) |
| 7 | 2,682 (3.5%) | 2,246 (2.8%) | 1,603 (1.9%) | 1,629 (2.0%) |
| 8 | 2,160 (2.8%) | 1,651 (2.1%) | 1,054 (1.3%) | 978 (1.2%) |
| 9 | 1,635 (2.1%) | 1,149 (1.4%) | 688 (0.8%) | 681 (0.8%) |
| 10 | 1,376 (1.8%) | 786 (1.0%) | 440 (0.5%) | 436 (0.5%) |
| 11 | 1,063 (1.4%) | 600 (0.8%) | 291 (0.4%) | 275 (0.3%) |
| 12 | 877 (1.1%) | 476 (0.6%) | 205 (0.2%) | 184 (0.2%) |

## A3': Relationship between the Model and Actual NYC System

This section describes the relationship between the model in the main text and the actual New York City high school assignment process.

## A3.1': Students may rank no more than 12 choices

The following table shows the distribution of the length of the rank order list in Round 1 across years.

Table A3.1: Length of Applicant ROLs in Round 1

| | 2003-04 | 2004-05 | 2005-06 | 2006-07 |
|---|---|---|---|---|
| 1 | 7,907 (8.47) | 6,123 (6.59) | 6,648 (7.18) | 6,786 (7.48) |
| 2 | 4,967 (5.32) | 4,369 (4.70) | 4,808 (5.20) | 4,683 (5.16) |
| 3 | 6,332 (6.79) | 6,048 (6.51) | 6,694 (7.23) | 6,615 (7.29) |
| 4 | 6,722 (7.2) | 6,697 (7.21) | 7,670 (8.29) | 7,490 (8.26) |
| 5 | 6,817 (7.31) | 7,159 (7.71) | 8,109 (8.76) | 8,098 (8.93) |
| 6 | 6,504 (6.97) | 7,480 (8.05) | 8,194 (8.86) | 8,115 (8.95) |
| 7 | 5,607 (6.01) | 6,320 (6.81) | 6,990 (7.55) | 7,026 (7.75) |
| 8 | 5,386 (5.77) | 5,798 (6.24) | 6,123 (6.62) | 6,336 (6.99) |
| 9 | 4,808 (5.15) | 4,841 (5.21) | 4,971 (5.37) | 5,286 (5.83) |
| 10 | 5,741 (6.15) | 4,952 (5.33) | 4,804 (5.19) | 5,025 (5.54) |
| 11 | 8,647 (9.27) | 5,561 (5.99) | 5,261 (5.69) | 5,269 (5.81) |
| 12 | 23,875 (25.59) | 27,524 (29.64) | 22,260 (24.06) | 19,952 (22.00) |

## A3.2': Top 2% Priority at Educational Option Programs

**Proposition:** In the student-proposing deferred acceptance mechanism where a student can rank at most $k$ schools, if a student is guaranteed a placement at a school only if she ranks it first, then she can do no better than

- either ranking that program as her first choice, and submit the rest of her preferences according to her true preference ordering, or
- submitting her preferences by selecting at most $k$ schools among the set of schools she prefers to being unassigned and ranking them according to her true preference ordering.

**Proof:** Consider a student with a guaranteed placement at a school. Given her preferences, partition her set of strategies into two sets: The first set consists of preference list of at most $k$ schools that rank her guaranteed school as first choice. The second set consists of all other preference lists of at most $k$ schools. We will show that her optimal strategy lies either in the first or the second set.

She is indifferent among all the preference lists in the first set, as she is guaranteed her guaranteed school by submitting any of those preference lists. So, there is no loss of generality in considering a particular strategy from this set, namely the one that ranks the guaranteed school as her first choice, and ranks the rest of her preferences according to her true preference ordering.

By the proposition above, her optimal strategy among the ones in the second set ranks schools in her true preference ordering, yielding the desired conclusion. ⋄

# A4': Ex Ante Comparison of DA-STB and DA-MTB

Let $p_i^k$ be the probability that student $i$ receives her $k$th choice. An allocation is a vector of probabilities $p_i = (p_i^1, ..., p_i^n)$ for each item on the rank order list $P_i$ such that $\sum_{k=1}^n p_i^k = 1$. We will say that an allocation $p_i$ **ordinally dominates** an allocation $q_i$ for student $i$, if for all $m = 1, .., n$,

$$\sum_{k=1}^{m} p_i^k \geq \sum_{k=1}^{m} q_i^k,$$

with strict inequality for some $m$. An allocation vector $p = (p_i)$ stochastically dominates $q = (q_i)$ if $p_i$ stochastically dominates $q_i$ for some $i$, and does no worse for all $i$.

**Proposition.** There is no ordinal dominance relationship between DA-STB and DA-MTB.

**Proof.** We present an example where there is no ordinal dominance relationship. Consider an economy with three students $i_1, i_2, i_3$ and three schools, $s_1, s_2, s_3$, each with one seat. Suppose student preferences are:

$$i_1 : s_1 \succ s_2 \succ s_3$$
$$i_2 : s_3 \succ s_1 \succ s_2$$
$$i_3 : s_1 \succ s_3 \succ s_2$$

Suppose three schools are indifferent between all applicants. Then DA-STB induces the following distribution over matchings:

$$\frac{1}{3} \cdot \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_3 & s_2 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_3 & s_1 \end{pmatrix} + \frac{1}{6} \cdot \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

DA-MTB induces the following distribution over matchings:

$$\frac{1}{4}\cdot\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_3 & s_2 \end{pmatrix}+\frac{1}{2}\cdot\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_3 & s_1 \end{pmatrix}+\frac{1}{6}\cdot\begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}+\frac{1}{12}\cdot\begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{pmatrix}.$$

Student $i_3$ is more likely to receive her first or second choice under DA-MTB than DA-STB, while student $i_1$ is more likely to receive her first or second choice under DA-STB than DA-MTB. Therefore, there is no ordinal dominance relationship between the two mechanisms. ◇

# A5': Implementation of the Stable Improvement Cycles Algorithm

This section describes the Stable Improvement Cycles algorithm of Erdil and Ergin (2008) and explain its implementation. The data we use for New York is for all 8th grade applicants in Round 1 of the New York City High School match. If an applicant is marked as a student who receives top 2% priority at an Educational Option school and ranks the school as their top choice, we do not include the applicant in these tables. The data we use for Boston is all elementary (Grade K2), middle (Grade 6), and high school (Grade 9) applicants in Round 1 for 2005-06 and 2006-07, when Boston employed a student-proposing deferred acceptance algorithm to place students. These students will be receiving their top choice and thus will not be affected by a stable improvement cycle. If an applicant ranked 12 schools, we work with the stated rank order list. Given a stable matching $\mu$, define the following: Let $A_s$ be the set of students assigned to school $s$ under $\mu$; $B_s$ be the set of students who are ranked highest by $s$ among all who prefer $s$ to their assignment. Formally,

$$A_s = \{i \in I : \mu(i) = s\},$$
$$B_s = \{i \in I : sP_i\mu(i) \text{ and } iR_sj \text{ for all } j \text{ such that } sP_j\mu(j)\}.$$

A stable improvement cycle is a list of distinct students $i_1, ..., i_n \equiv i_0$, $n \geq 2$, such that $\mu(i_l) \in S$ and $i_l \in B_{\mu(i_{l+1})}$ for $l = 0, ..., n-1$. We implement a stable improvement cycle by forming a new matching $\mu'$ as

$$\mu'(i) = \begin{cases} \mu(i) \text{ if } i \notin \{i_0, ..., i_{n-1}\} \\ \mu(i_{l+1}) \text{ if } i = i_l \text{ for some } l = 0, ..., n-1 \end{cases}$$

We start with a single tie breaking rule and matching produced by the associated DA-STB.

Given a stable matching, we construct a directed graph as follows: The nodes of the graph are schools. We draw an edge from school $s$ to school $s'$ if there is a student $i$ such that $\mu(i) = s$ and $i \in B_{s'}$. We also associate that edge with the set of all such students, denoted by $E_{ss'}$. Formally,

$$E_{ss'} = \{i \in I : \mu(i) = s \text{ and } i \in B_{s'}\}$$

Students in $E_{ss'}$ are sorted according to the given tie breaking rule. Let $E_s$ be the set of edges originating from $s$. During a search for a cycle, schools are tried in the alphabetical order. In particular, we start the search for a stable improvement cycle with the first school in the alphabetical order. If we cannot find a cycle after starting the search with a school, we restart the search with the next school in the alphabetical order. When we reach a school $s$ in our search, we continue our search with the schools in $E_s$ in the alphabetical order. When a student is to be moved from $s$ to $s'$ in cycle, the last student $i$ in $E_{ss'}$ is moved from $s$ to $s'$. Then $i$ is removed from all $E_{ss''}$ for every $s'' \in S\backslash\{s\}$. We find and implement all the cycles in the graph. Then we repeat these steps with the new matching until no cycle is found.

# A6': Tradeoff between Stability and Efficiency

As we mention in the text, we take only students' preferences into account for welfare considerations. In order to measure the cost of stability associated with a student-optimal stable matching $\mu$, we find a Pareto efficient matching that Pareto dominates $\mu$.

If a matching is not Pareto efficient, we find a Pareto efficient matching that Pareto dominates it from the perspective of students via Gale's top trading cycle algorithm as follows:

If a matching of students to schools, $\mu$, is not Pareto efficient, then there exits a cycle of students $i_1, i_2, ..., i_{n+1} \equiv i_1$, $n \geq 2$, such that $i_l$ prefers $i_{l+1}$'s matched school over her match, that is $\mu(i_{l+1}) P_{i_l} \mu(i_l)$, $l = 1, ..., n$. A new matching $TTC(\mu)$ can be obtained by picking an arbitrary cycle $i_1, i_2, ..., i_{n+1} \equiv i_1$, and transferring every $i_l$ to $i_{l+1}$'s matched school:

$$TTC(\mu)(i) = \begin{cases} \mu(i_{l+1}) \text{ if } i = i_l \text{ for some } l = 1, ..., n \\ \mu(i) \text{ otherwise} \end{cases}$$

$TTC(\mu)$ Pareto dominates $\mu$. Therefore, a Pareto efficent matching that Pareto dominates $\mu$ can be found as the limit of $\mu^{t+1} = TTC(\mu^t)$ where $\mu^0 = \mu$. The limit is obtained in finite steps by finiteness of the model.