Appendix to: "Conversations Among Competitors"

Jeremy C. Stein
Harvard University and NBER
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The proof of Proposition 4 builds on that of Proposition 5, so these two results are proved in reverse order.

A. PROOF OF PROPOSITION 5:

Consider a given path of play in the equilibrium described in Proposition 3. Until there are more than k^* signals available for decoding, a new conversation is initiated each time a conversant fails to generate an additional signal. Since player A is assumed to start with an initial signal, c is the random number of failures in a series of independent Bernoulli (p) trials before there are $I(k^*)$ successes. A random variable such as c with

(A.1) $$\Pr(c=i)=\binom{i+I(k^*)-1}{I(k^*)-1}p^{I(k^*)}(1-p)^i \text{ for } i=0,1,2,\ldots.$$

is said to follow the negative binomial distribution. It is then straightforward to show that $E(c) = I(k^*)(1-p)/p$ (see George Casella and Roger L. Berger (2002), p. 95-96).

Next, note that the random variable k follows a geometric distribution with

(A.2) $$\Pr(k=i)=(1-p)\,p^{i-1} \text{ for } i=1,2,\ldots.$$

A simple calculation confirms that E(k) = 1/(1 – p) (see Casella and Berger (2002), p. 97).

B. PROOF OF PROPOSITION 4:

First, recall that cov(c,k) = E(ck) – E(c)E(k). By Proposition 5, we have E(c)E(k)= $I(k^*)/p$. To calculate E(ck), we can use the fact that E(ck) = E[E($c|k$)k] where E($c|k$) denotes the conditional expectation of c given k. Next note that:

(A.3) $$E(c\,|\,k)=\begin{cases}1+\left(I(k^*)+1-k\right)\dfrac{(1-p)}{p} & k\le I(k^*)\\ 0 & k\ge I(k^*)+1.\end{cases}$$

Thus, for $k\le I(k^*)$, we carry out Bernoulli (p) trials until there are $I(k^*)+1-k$ successes, so the expression for E($c|k$) follows from Proposition 5 (we must add one since we are effectively conditioning on $c\ge 1$); while for $k\ge I(k^*)+1$, B will not initiate a conversation with C so that E($c|k$) = 0. Note that E($c|k$) is decreasing in k. Calculating, we then obtain:

(A.4) $$E[E(c\,|\,k)k]=\sum_{i=1}^{I(k^*)}\left[\left(1+\left(I(k^*)+1-i\right)\frac{(1-p)}{p}\right)i\right](1-p)\,p^{i-1}=\frac{I(k^*)}{p}-\frac{1-p^{I(k^*)}}{1-p},$$

so that $\operatorname{cov}(c,k) = -(1-p^{I(k^*)})/(1-p) < 0$. After computing var(*c*) and var(*k*) (see Casella and Berger (2002), p. 95-97), and rearranging the resulting expression for $\rho_{ck} = \operatorname{cov}(c,k)/\sqrt{\operatorname{var}(c)\operatorname{var}(k)}$, we have:

(A.5) $$\rho_{ck} = -\left(1-p^{I(k^*)}\right)\sqrt{\frac{p}{I(k^*)(1-p)}} < 0 .$$

C. COMPARATIVE STATICS PROPERTIES OF E(*c*) and Φ{E(*c*)}:

Let us begin with the comparative statics of k^* with respect to p and θ. Let

(A.6) $$F(k,p,\beta,\theta) = \left[\frac{\theta(1-p)+p(1-\beta)}{1-p\beta}\right]\beta^k - \theta$$

and note that the function $k^*(p,\beta,\theta)$ is defined implicitly by $F(k^*(p,\beta,\theta),p,\beta,\theta) = 0$.

Since $\partial F(k,p,\beta,\theta)/\partial k < 0$, applying the Implicit Function Theorem and noting that

(A.7) $$\frac{\partial F(k^*,p,\beta,\theta)}{\partial p} = \frac{(1-\theta)(1-\beta)}{(1-p\beta)^2}\beta^{k*} > 0 \text{, and}$$

(A.8) $$\frac{\partial F(k^*,p,\beta,\theta)}{\partial \theta} = \frac{(1-p)}{(1-p\beta)}\beta^{k*} - 1 < 0 ,$$

we see that $\partial k^* / \partial p > 0$ and $\partial k^* / \partial \theta < 0$. Since $I(k^*)$ is weakly increasing in k^*, it follows immediately from the above results that $E(c) = I(k^*)(1 - p)/p$ is weakly decreasing in θ.

Finally, recall that the upper bound on $E[c]$ is given by $\Phi\{E[c]\} = k^*(1-p)/p$. Differentiating with respect to p yields

$$\text{(A.9)} \qquad \frac{\partial \Phi\{E[c]\}}{\partial p} = \frac{\partial k^*}{\partial p}\frac{(1-p)}{p} - \frac{k^*}{p^2} \ .$$

Computing $\partial k^* / \partial p$ and noting that $(1-p)/(1-p\beta) < 1$, we have

$$\text{(A.10)} \qquad \begin{aligned} \frac{\partial k^*}{\partial p}\frac{(1-p)}{p} &= \left[-\frac{1}{\ln[\beta]}\frac{(1-\theta)(1-\beta)}{(1-p\beta)\left[\theta(1-p)+p(1-\beta)\right]}\right]\frac{(1-p)}{p} \\ &< -\frac{1}{p\ln[\beta]}\left[\frac{(1-\theta)(1-\beta)}{\theta(1-p)+p(1-\beta)}\right]. \end{aligned}$$

Finally since $-\ln[x] > 1-x$ for $0 < x < 1$, it follows that

$$\text{(A.11)} \qquad \begin{aligned} \frac{k^*}{p^2} &= \frac{1}{p^2}\frac{\log\left[\frac{(1-p\beta)\theta}{\theta(1-p)+p(1-\beta)}\right]}{\ln[\beta]} \\ &> -\frac{1}{p^2\ln[\beta]}\left[1-\frac{(1-p\beta)\theta}{\theta(1-p)+p(1-\beta)}\right] \\ &= -\frac{1}{p\ln[\beta]}\left[\frac{(1-\theta)(1-\beta)}{\theta(1-p)+p(1-\beta)}\right]. \end{aligned}$$

Combining these inequalities, it follows that $\partial \Phi\{E[c]\} / \partial p < 0$.

Reference

Casella, George, and Roger L. Berger. 2002. *Statistical Inference*. Duxbury Press.