

Supplementary Appendices

(to be posted on the web only)

Appendix S1: The Consumption-Savings Model

I. We first give a result for general per-period utility functions in the simple savings model. Consider the problem of maximizing

$$U(\vec{a}) = \sum_{t=1}^{\infty} \delta^{t-1} [(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t)]$$

over all feasible plans \vec{a} , i.e. plans that satisfy $a_t \in [0, 1]$ and the wealth equation

$y_t = Ra_{t-1}y_{t-1}$. We suppose that u is non-decreasing and continuous on $(0, \infty)$; we do not require continuity on $[0, \infty)$ because we want to allow for the logarithmic case where $u(0) = \lim_{c \rightarrow 0} u(c) = -\infty$. Let \bar{U} be the supremum in this problem.

Proposition S1.1: *Suppose $R > 1$ and*

$$(A.1) \quad \sum_{t=1}^{\infty} \delta^{t-1} u(R^{t-1}y_1) < \infty.$$

Then: (i) For any feasible plan the sum defining U has a well defined value in the sense that either the sum converges absolutely or converges to $-\infty$.

(ii) The supremum \bar{U} of the feasible values satisfies $-\infty < \bar{U} < \infty$

(iii) If feasible $\vec{a}^n \rightarrow \vec{a}^*$ in the product topology then \vec{a}^* is feasible. If in addition $U(\vec{a}^n) \rightarrow \bar{U}$ then $U(\vec{a}^*) = \bar{U}$.

(iv) An optimal plan exists. That is, there is a feasible plan that attains \bar{U} .

Proof: (i) Define $\chi_+(x) = \max\{x, 0\}$ and $\chi_-(x) = -\chi_+(-x)$.

We can write any finite sum as the sum of negative and positive parts, so for any sequence (\vec{a}, \vec{y}) we have

$$\begin{aligned} & \sum_{t=1}^T \delta^{t-1} [(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t)] = \\ & \sum_{t=1}^T \delta^{t-1} \chi_+ [(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t)] + \\ & \sum_{t=1}^T \delta^{t-1} \chi_- [(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t)] \end{aligned}$$

The positive part of the sum is summable from (A.1), since

$$(1 + \gamma)u((1 - a_t)y_t) - \gamma u(y_t) \leq u((1 - a_t)y_t) \leq u(R^{t-1}y_1).$$

The negative part is monotone decreasing in T , so it either converges absolutely or converges to $-\infty$. In the former case the entire sum converges absolutely; in the latter case the sum converges to $-\infty$.

(ii) Part (i) already shows that $\bar{U} < \infty$. To see that $\bar{U} > -\infty$, note that it is feasible to set $a_t = 1/R$ for all t , and that for $R > 1$ this plan yields a finite value.

(iii) Consider a sequence of feasible plans $\vec{a}^n \rightarrow \vec{a}^*$. Because the constraints are period by period and closed, it is clear that \vec{a}^* satisfies the constraints, so it is feasible.

Now suppose in addition that $U(\vec{a}^n) \rightarrow \bar{U}$. Choose any $\varepsilon > 0$ and pick N large enough that $\bar{U} - U(\vec{a}^n) < \varepsilon/2$ for all $n > N$. If we now pick T that

$$\sum_{t=T+1}^{\infty} \delta^{t-1} |u(R^{t-1}y_1)| < \varepsilon / 2,$$

we know that

$$\bar{U} - \sum_{t=1}^T \delta^{t-1} [(1 + \gamma)u((1 - a_t^n)y_t^n) - \gamma u(y_t^n)] \leq \varepsilon$$

for all $n > N$ and $\tau > T$. Since per-period payoffs are continuous at any (a, y) with $a > 0$, $\vec{a}^n \rightarrow \vec{a}^*$, and $y_t^n \rightarrow y_t^*$, it follows that

$$\bar{U} - \sum_{t=1}^{\tau} \delta^{t-1} [(1 + \gamma)u((1 - a_t^*)y_t^*) - \gamma u(y_t^*)] \leq \varepsilon \text{ for all } \tau > T.$$

Since this is true for any $\varepsilon > 0$ and we know that $U(\vec{a}^*) \leq \bar{U}$, we conclude that

$$U(\vec{a}^*) = \bar{U}.$$

(iv) Now consider a feasible sequence (\vec{a}^n, \vec{y}^n) with $U(\vec{a}^n) \rightarrow \bar{U}$. Each savings rate a_t must lie in the compact interval $[0, 1]$ and each y_t must lie in the compact interval $[0, R^{t-1}y_1]$, so the sequence (\vec{a}^n, \vec{y}^n) has an accumulation point (\vec{a}^*, \vec{y}^*) in the product topology. This accumulation point is a maximum by part (iii).

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II. Now we specialize to the CRRA utility functions

$$u(c) = \frac{c^{1-\rho} - 1}{1 - \rho}$$

and $u(c) = \ln(c)$, which corresponds to the case $\rho = 1$. Assuming $\delta < R^{\rho-1}$ implies

$$\sum_{t=1}^{\infty} \delta^{t-1} u(R^{t-1} y_1) < \infty.$$

It follows from Proposition S1.1 that an optimum \vec{a}^* exists.

Proposition S1.2: *With CRRA utility a stationary optimum with $a_t = a$ exists.*

Proof: Suppose that \vec{a}^* is an optimal plan. By homogeneity of the objective function, and the fact that plans are defined in terms of savings rates, \vec{a}^* is also an optimal plan starting in period 2 (for any initial condition). Note that the plan $\vec{a}^2 = (a_1^*, a_1^*, a_2^*, a_3^*, \dots)$ yields wealth in period 2 of $a_1^* R y_1$, and let $\bar{U}(y_1)$ denote the maximized utility when starting in the second period with wealth y_1 . Then

$$U(\vec{a}^2) = (1 + \gamma)u((1 - a_1^*)y_1) - \gamma u(y_1) + \delta \bar{U}(a_1^* R y_1) = \bar{U}$$

where the first equality follows because \vec{a}^* is optimal from period 2 on, and the second equality because \vec{a}^* is optimal from the first period. Proceeding in this way we can construct sequence of feasible plans $\vec{a}^n = (a_1^*, a_1^*, \dots, a_1^*, a_2^*, a_3^*, \dots)$ that play a_1^* for the first n periods such that $U(\vec{a}^n) = U(a^*) = \bar{U}$. Clearly \vec{a}^n converges in the product topology to the plan of choosing the fixed savings rate a_1^* . From Proposition A.1 (iii), this limiting plan is feasible and gives utility \bar{U} ; that is, it is optimal.

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III. We have shown that it is sufficient to compute the present value utility from a fixed savings rate a , and maximize over savings rates. We have present value utility

$$\begin{aligned}
U &= \frac{y_1^{1-\rho}}{1-\rho} \sum_{t=1}^{\infty} (\delta(Ra)^{1-\rho})^{t-1} [(1+\gamma)(1-a)^{1-\rho} - \gamma] - \frac{1}{(1-\delta)(1-\rho)} \\
&= \frac{y_1^{1-\rho}}{1-\rho} \frac{[(1+\gamma)(1-a)^{1-\rho} - \gamma]}{1-\delta(Ra)^{1-\rho}} - \frac{1}{(1-\delta)(1-\rho)}
\end{aligned}$$

Since the optimal savings rate cannot be 0 or 1, we may differentiate with respect to the saving rate to find

$$dU / da \propto [(1+\gamma)(1-a) - \gamma(1-a)^\rho] \delta R(Ra)^{-\rho} - (1+\gamma)(1-\delta(Ra)^{1-\rho})$$

which gives the first-order condition for an optimumⁱ

$$(1+\gamma)a_\gamma^{*\rho} = R^{1-\rho} \delta ((1+\gamma) - \gamma(1-a_\gamma^*)^\rho).$$

When $\gamma = 0$ we get the usual solution $a^* = R^{(1-\rho)/\rho} \delta^{1/\rho}$. Thus we can rewrite the first order condition as

$$(a_\gamma^* / a^*)^\rho = ((1+\gamma) - \gamma(1-a_\gamma^*)^\rho) / (1+\gamma).$$

IV: Turning to the simple banking model, utility starting in the second period is the

$\gamma = 0$ solution

$$\begin{aligned}
U_2(y_2) &= \frac{y_2^{1-\rho}}{1-\rho} \frac{(1-a^*)^{1-\rho}}{1-\delta(Ra^*)^{1-\rho}} - \frac{1}{(1-\delta)(1-\rho)} \\
&= \frac{y_2^{1-\rho}}{1-\rho} \frac{1}{(1-a^*)^\rho} - \frac{1}{(1-\delta)(1-\rho)} \\
&= \frac{y_2^{1-\rho}}{1-\rho} \frac{1}{(1-\delta^{1/\rho} R^{(1-\rho)/\rho})^\rho} - \frac{1}{(1-\delta)(1-\rho)}
\end{aligned}$$

The utility of both selves in the first period is

$$(1 + \gamma) \frac{c_1^{1-\rho} - 1}{1 - \rho} - \gamma \frac{(x_1 + z_1)^{1-\rho} - 1}{1 - \rho},$$

and so the overall objective of the long-run self is to maximize

$$(1 + \gamma) \frac{c_1^{1-\rho} - 1}{1 - \rho} - \gamma \frac{(x_1 + z_1)^{1-\rho} - 1}{1 - \rho} + \frac{(R(y_1 + z_1 - c_1))^{1-\rho}}{1 - \rho} \frac{\delta}{(1 - \delta^{1/\rho} R^{(1-\rho)/\rho})^\rho} - \frac{\delta}{(1 - \delta)(1 - \rho)}$$

The first order condition for optimal consumption is

$$\frac{c_1^*}{R(y_1 + z_1 - c_1^*)} = \frac{(1 + \gamma)^{1/\rho} (1 - \delta^{1/\rho} R^{(1-\rho)/\rho})}{(\delta R)^{1/\rho}}.$$

If there are one or more solutions that satisfy the constraint $c_1^* \leq x_1 + z_1$ then one of them represents the optimum; otherwise the optimum is to consume all pocket cash.

Note that x_1 is the solution for $\gamma = 0$, so it satisfies

$$\frac{x_1}{R(y_1 - x_1)} = \frac{(1 - \delta^{1/\rho} R^{(1-\rho)/\rho})}{R^{1/\rho}}.$$

Thus we can write the first order condition as

$$\frac{c_1^*}{y_1 + z_1 - c_1^*} = (1 + \gamma)^{1/\rho} \frac{x_1}{y_1 - x_1}$$

or

$$\begin{aligned}c_1^* &= \frac{(1 + \gamma)^{1/\rho} x_1}{y_1 - x_1 + (1 + \gamma)^{1/\rho} x_1} (y_1 + z_1) \\ &= \frac{1 + [(1 + \gamma)^{1/\rho} - 1]}{1 + [(1 + \gamma)^{1/\rho} - 1](1 - a^*)} (1 - a^*) (y_1 + z_1) \\ &= B(1 - a^*) (y_1 + z_1).\end{aligned}$$

Appendix S2: Hyperbolic Procrastination and Delay

Theorem 4:

a) If $v > \mu$ and $\bar{x} > \delta\beta v$, the “sophisticated quasi-hyperbolic model” has a stationary equilibrium with $\underline{x} < x^{**} < \bar{x}$.

b) There is an open set of parameters that satisfies the restrictions of part a) and for which there are other equilibria.

proof: Recall equations (10) into (11) from the paper:

$$W^{**} = \frac{P(x^{**})(\delta V) + (1 - P(x^{**}))E(x | x > x^{**})}{1 - \delta + \delta P(x^{**})}. \quad (10)$$

$$x^{**} = \delta\beta(V - W^{**}) \quad (11).$$

Substituting (10) into (11) we see that

$$\begin{aligned} x^{**} &= \delta\beta \left(\frac{V - \delta V + \delta P(x^{**})V - P(x^{**})(\delta V) - (1 - P(x^{**}))E(x | x > x^{**})}{1 - \delta + \delta P(x^{**})} \right) \\ &= \delta\beta \left(\frac{v - (1 - P(x^{**}))E(x | x > x^{**})}{1 - \delta + \delta P(x^{**})} \right). \end{aligned}$$

$$\text{Let } F(x^{**}) = x^{**} - \delta\beta \left(\frac{v - (1 - P(x^{**}))E(x | x > x^{**})}{1 - \delta + \delta P(x^{**})} \right).$$

To prove the theorem it suffices to show that there is an x^{**} where $F(x^{**}) = 0$. The assumption that $v > \mu$ implies that $F(0) = 0 - \delta\beta\left(\frac{v - \mu}{1 - \delta}\right) < 0$, and the assumption that $\bar{x} > \delta\beta v$ implies that $F(\bar{x}) = \bar{x} - \delta\beta v > 0$.

b) Suppose that $x^1 = \underline{x}$ and $x^2 = \bar{x}$ so that the odd-numbered agents never act and even-numbered agents always act. Then the equilibrium payoff of an even-numbered agent is $\delta\beta V$, and the payoff of an even-numbered agent with cost x_t who chooses to wait is

$x_t + \delta\beta\mu + \delta^3\beta V$, so the even agents' strategy is a best response for all x_t if $\delta\beta V(1 - \delta^2) > \bar{x} + \delta\beta\mu$, or equivalently

$$\delta\beta(v(1 + \delta) - \mu) > \bar{x}. \quad (\text{S2.1})$$

The payoff of the odd-numbered agents who wait is $x_t + \delta^2\beta V$, and the payoff to acting is $\delta\beta V$, so waiting is better for all x_t if

$$\underline{x} > \delta\beta(1 - \delta)V = \delta\beta v. \quad (\text{S2.2})$$

To complete the proof we must show that there is an open set of parameters such that (S2.1) and (S2.2) and the restrictions $v > \mu$ and $\bar{x} > \delta\beta v$. To do this, fix $v > 0$ and $\delta, \beta \in (0, 1)$, and some $\varepsilon > 0$. Set $\underline{x} = (1 + \varepsilon)\delta\beta v$, $\mu = (1 + 2\varepsilon)\delta\beta v$, and $\bar{x} = (1 + 3\varepsilon)\delta\beta v$. (Note that these conditions are consistent with a range of distributions, including the uniform.) By construction this satisfies $\bar{x} > \delta\beta v$ and (S2.2).

If $\varepsilon < \frac{1 - \delta\beta}{2\delta\beta}$ then $\mu = (1 + 2\varepsilon)\delta\beta v < (1 + \frac{1 - \delta\beta}{\delta\beta})\delta\beta v = v$, and for

$\varepsilon < \frac{\delta(1 - \beta)}{(3 + 2\delta\beta)}$ we compute that

$$\begin{aligned}
 \delta\beta(v(1 + \delta) - \mu) &= \delta\beta(v(1 + \delta) - (1 + 2\varepsilon)\delta\beta v) \\
 &= \delta\beta v(1 + \delta(1 - \beta) - 2\varepsilon\delta\beta) \\
 &> \\
 &\delta\beta v(1 + 3\varepsilon) \\
 &= \bar{x}.
 \end{aligned}$$

This shows that there is a ‘2-cycle equilibrium’ whenever ε is sufficiently small. Since the inequalities in (12) and (13) hold strictly for the specified relationship between the parameters, they hold for an open set of β, δ, V, μ ; the inequalities also hold for a range of distributions with the given mean and endpoints. \square

ⁱ We do not know if the first-order condition has a unique solution, except in the logarithmic case.