# V. Supplemental Appendix

## A. Convex Cost of Storage

Assume that the cost of storage is a twice continuously differentiable function $c(S)$ with $c'(S) > 0$, $c''(S) > 0$, and $c(0) = c'(0) = 0$.

Consider any fixed sequence of prices $\{p_t\}_{t=1}^{T}$. Suppose that the buyer begins date $t$ with a stock $S_{t-1}$ of the good. Let $S_t(p_1, ..., p_T)$ be the optimal storage choice by the consumer. The following Lemma provides a simple characterization of the solution of the buyer's problem.

**Lemma 7** *Assume that $p_t \leq p_T^m$ for all $t$. Then, at date $t \leq T-1$ the consumer stores quantity $S_t$ that solves*

$$c'(S_t) = \max\{0, p_{t+1} - p_t\}, \tag{27}$$

*consumes*

$$x_t = D_t(p_t),$$

*and purchases $b_t = D_t(p_t) + S_t - S_{t-1}$ units.*

By Lemma 7, we can write the consumer's optimal storage decision at period $t$ as a function of period $t$ and period $t+1$ prices only. Denote by $S_t(p_t, p_{t+1})$ the optimal storage decisions at period $t$ as defined by equation (27).

### 1. Commitment

By Lemma 7, under commitment the monopolist chooses a sequence of prices $\{p_t\}_{t=1}^{T}$ to maximize

$$\sum_{t=1}^{T}[D_t(p_t) - S_{t-1}(p_{t-1}, p_t) + S_t(p_t, p_{t+1})]p_t \tag{28}$$

with $S_0(p_0, p_1) = S_T(p_T, p_{T+1}) = 0$ and $S_t(p_t, p_{t+1})$ defined by equation (27).

The first order conditions at period $t$ is:

(29)

$$MR_t(p_t) - S_{t-1}(p_{t-1}, p_t) + S_t(p_t, p_{t+1}) - \frac{\partial S_{t-1}(p_{t-1}, p_t)}{\partial p_t}(p_t - p_{t-1}) - \frac{\partial S_t(p_t, p_{t+1})}{\partial p_t}(p_{t+1} - p_t) = 0.$$

Notice first that $S_t(p_t, p_{t+1})$ might not be differentiable in $p_t$ when $p_t = p_{t+1}$. This is because, for a fixed $p_{t+1}$, $\frac{\partial S_t(p_t,p_{t+1})}{\partial p_t} < 0$ if $p_t < p_{t+1}$ and $\frac{\partial S_t(p_t,p_{t+1})}{\partial p_t} = 0$ if $p_t > p_{t+1}$. Similarly, $S_t(p_t, p_{t+1})$ might not be differentiable in $p_{t+1}$ when $p_{t+1} = p_t$. Notice however that the limits from both the right and the left of $S_t(p_t, p_{t+1})$ exists and are both nonpositive. Finally, notice that if $p_t \neq p_{t+1}$

$$\frac{\partial S_t(p_t, p_{t+1})}{\partial p_t} = -\frac{\partial S_t(p_t, p_{t+1})}{\partial p_{t+1}}.$$

Summing (29) over $t$ and recalling that $S_0(p_0, p_1) = S_T(p_T, p_{T+1}) = 0$ we obtain

$$\sum_{t=1}^{T} MR_t(p_t^c) = 0. \quad (30)$$

This equation is the counterpart of equation (7) that we obtained in the case of linear costs of storage. Note however that equation (30) is not as informative: because prices are now not necessarily rising at a constant rate $c$, we need $T$ conditions to obtain each price.

2. No Commitment

The construction of the equilibrium absent commitment is quite similar to the analysis in Section II. The main difference is that equilibrium storage $S_t(p_t)$ at date $t$ must satisfy

$$c'(S_t(p_t)) = \max\{0, p_{t+1} - p_t\}.$$

Appropriately modifying the analysis of Section (D.) we obtain that equilibrium is characterized, at period $t$, by:

$$MR_t(p_t) = S_{t-1} - S_t^{nc}(p_t) + (p_{t+1}^{nc}(p_t) - p_t)\frac{\partial S_t^{nc}(p_t)}{\partial p_t}$$

where $S_0 = S_T^{nc}(p_T) = 0$.

Summing over $t$ we obtain

$$\sum_{t=1}^{T} MR_t(p_t^{nc}) = \sum_{t=1}^{T}\left( (p_{t+1}^{nc} - p_t^{nc}) \left.\frac{\partial S_t^{nc}(p_t)}{\partial p_t}\right|_{p_t^{nc}} \right).$$

Going through similar steps as in the proof of Lemma 4 we can show that

$$\frac{\partial S_t^{nc}(p_t)}{\partial p_t} = \frac{\frac{\partial^2 V_{t+1}}{\partial p_{t+1}^2}}{1 - c''\left(S_t^{nc}\right)\frac{\partial^2 V_{t+1}}{\partial p_{t+1}^2}} \leq 0.$$

Because, as in the previous section, we can prove that $\frac{\partial S_{T-1}^{nc}(p_{T-1})}{\partial p_{T-1}} < 0$, we can conclude that $\sum_{t=1}^{T} MR_t(p_t^{nc}) < \sum_{t=1}^{T} MR_t(p_t^{c})$. Because $MR_t$ are decreasing functions for all $t$, we can conclude that prices under commitment cannot be uniformly higher and there is also a sense in which they have to be lower "on average."

The problem of comparing prices at each period stems from the fact that both under commitment and without commitment the price sequence is determined by $T$ conditions. Comparison of prices at each period implies the comparison of $T$ conditions:

$$\begin{cases} MR_1(p_1^c) = -S_1(p_1^c, p_2^c) + (p_2^c - p_1^c)\frac{\partial S_1(p_1,p_2)}{\partial p_1} \\ \dots \\ MR_t(p_t^c) = S_{t-1}(p_{t-1}, p_t) - S_t(p_t, p_{t+1}) + \left(p_t^c - p_{t-1}^c\right)\frac{\partial S_{t-1}(p_{t-1},p_t)}{\partial p_{t-1}} + \left(p_{t+1}^c - p_t^c\right)\frac{\partial S_t(p_t,p_{t+1})}{\partial p_t} \\ \dots \\ MR_T(p_T^c) = S_{T-1}(p_{T-1}, p_T) + \left(p_T^c - p_{T-1}^c\right)\frac{\partial S_{T-1}(p_{T-1},p_T)}{\partial p_{T-1}} \end{cases}$$

for the case of commitment, and

$$\begin{cases} MR_1(p_1^{nc}) = -S_1 + (p_2(p_1^{nc}) - p_1^{nc})\,\frac{\partial S_1^{nc}(p_1)}{\partial p_1} \\ \ldots \\ MR_t(p_t^{nc}) = S_{t-1}^{nc} - S_t^{nc} + (p_{t+1}^{nc}(p_t) - p_t)\frac{\partial S_t^{nc}(p_t)}{\partial p_t} \\ \ldots \\ MR_T(p_T^{nc}) = S_{T-1}^{nc} \end{cases}$$

without commitment.

B. Solution Algorithm

In this section we provide an algorithm to solve the problem of choosing a sequence of prices $\{p_t^c\}_{t=1}^T$ to maximize

(31) $$\sum_{t=1}^{T} D_t(p_t)$$

subject to

(32) $$p_{t+1}^c \leq p_t^c + c \text{ for all } t = 1, ..., T-1.$$

The solution algorithm is based on the following intuition. Consider first the case in which there is only one period, that is $T = 1$. In this case, the problem above reduces to the maximization of static monopoly profits $D_1(p_1)p_1$. The solution is $p_1^c = p_1^m$, where $p_1^m$ is the static monopoly price. Consider now the two period problem. If $D_1(p_1)$ and $D_2(p_2)$ are such that $p_2^m > p_1^m + c$, the constraint (32) becomes binding and $p_1^c$ is found by maximizing

(33) $$D_1(p_1) + D_2(p_1 + c).$$

In particular, $p_1^m \leq p_1^c$. If instead $D_1(p_1)$ and $D_2(p_2)$ are such that $p_2^m \leq p_1^m + c$, the constraint (32) is not binding and the price that maximizes (33) is smaller than the static monopoly price $p_1^m$.

This observation can be generalized to many periods. Specifically, if constraint (32) is binding for the first $T_1$ periods, then the *argmax* of

$$\sum_{t=1}^{T_1} D_t(p + (t-1)c)$$

is greater than the *argmax* of

$$\sum_{t=1}^{\tau} D_t(p + (t-1)c)$$

for all $\tau = 1, ..., T_1$. This statement will be made more precise and proved in Lemma 8 below.

We now introduce some notation and describe the algorithm. Consider an interval of periods $t = t_1, ..., t_2$, with $t_1 \leq t_2$, and let $p(t_1, t_2)$ be the solution to the equation

$$\sum_{t=t_1}^{t_2} MR_t(p + (t - t_1)c) = 0. \tag{34}$$

Because the functions $MR_t(p_t)$ are strictly decreasing in $p$, this sum is also decreasing in $p$ and $p(t_{1,}t_2)$ is unique.

The algorithm is defined by iterating on $i$. At the first step of the algorithm consider the prices $p(1, t)$, with $t = 1, ..., T$. Let $T_1$ be

$$T_1 = \arg \max_{t=1,...,T} p(1, t). \tag{35}$$

If the argmax is not unique, let $T_1$ be the greatest. For all $t = 1, ..., T_1$, set $p_t^c$ according to:

$$p_t^c = p(1, T_1) + (t - 1)c. \tag{36}$$

At the $i^{th}+1$ step of the algorithm, consider the interval of periods $t = T_i + 1, ..., T$ and compute the prices $p(T_i + 1, t)$. Let $T_{i+1}$ be

$$T_{i+1} = \arg \max_{t=T_i+1,...,T} p(T_i + 1, t).$$

If $T_{i+1}$ is not unique, consider the greatest. For each $t = T_i + 1, ..., T$, set $p_t^c$ according to

$$p_t^c = p(T_i + 1, T_{i+1}) + (t - T_i - 1)c.$$

The algorithm proceeds until, at some iteration, $T_{i+1} = T$.

**Remark 1** *By construction, the algorithm delivers a unique solution.*

Before we prove the correspondence between the solution to the algorithm and the equilibrium we show that the price sequence obtained with the algorithm satisfies the constraint $p_{t+1}^c \leq p_t^c + c$. This allows us to draw an analogy between the $T_i$'s of this section and those of Lermma (5).

**Lemma 8** *The sequence of prices $\{p_t^c\}_{t=1}^T$ that solves the algorithm satisfies the constraint*

$$p_{t+1}^c \leq p_t^c + c$$

*for all $t = 1, ..., T$.*

**Proof.** By construction, constraints (32) are satisfied by $p_t^c$ for all $t = T_i + 1, ..., T_{i+1}$. To prove that this is true also for $p_{T_i}^c$ and $p_{T_i+1}^c$, consider for simplicity the first and the second iteration. Suppose by way of contradiction that $p(T_1 + 1, T_2) > p_{T_1}^{nc} + c$ that is:

$$p(T_1 + 1, T_2) > p(1, T_1) + T_1 c. \quad (37)$$

Because, by definition of $p(T_1+1, T_2)$,

$$\sum_{t=T_1+1}^{T_2} MR_t(p(T_1+1, T_2) + (t - T_1 - 1)c) = 0$$

and because the functions $MR_t(p)$ are strictly decreasing, inequality (37) implies that

$$\sum_{t=1}^{T_2} MR_t(p(1, T_1) + (t-1)c) > 0. \tag{38}$$

Recalling that

$$\sum_{t=1}^{T_2} MR_t(p(1, T_2) + (t-1)c) = 0$$

by definition of $p(1, T_2)$, inequality (38) implies that $p(1, T_2) > p(1, T_1)$. This contradicts the hypothesis that $T_1 = arg \max_{t=1,...,T} p(1, t)$ and concludes the proof. ■

We now prove that the solution to the algorithm and the solution to the maximization problem are the same.

**Lemma 9** *A price sequence $\{p_t^c\}_{t=1}^T$ maximizes (31) subject to (32) if and only if it is a solution of the algorithm.*

**Proof.** Let $\{p_t^c\}_{t=1}^T$ be a solution to the maximization of (31) subject to (32). Because both $\{p_t^c\}_{t=1}^T$ and the solution to the algorithm are unique, it is enough to show that $\{p_t^c\}_{t=1}^T$ solves the algorithm.

With the usual notation, let $(T_1, ..., T_m)$ be a sequence of dates, with $1 \leq T_1 \leq ... \leq T_m \leq T$, such that, for each $i$, $p_{T_i}^c + c < p_{T_i+1}^c$, i.e. storage is not binding between periods $T_i$ and period $T_{i+1}$. Without loss of generality, consider the set of periods $t = T_1+1, ..., T_2$ and corresponding prices $p_t^c$.

We first show that $p^c_{T_1+1} = p(T_1+1, T_2)$. This follows immediately from Lemma (5)

$$\sum_{t=T_1+1}^{T_2} MR_t(p^c_{T_1+1} + (t - T_1 - 1)c) = 0, \tag{39}$$

and from the definition and uniqueness of $p(T_1+1, T_2)$.

We now show that $T_2 = \arg \max_{t=T_1+1,...,T} p(T_1+1, t)$, that is the algorithm breaks the solution price vector at $T_2$. We break this step in two parts.

Assume by way of contradiction that there exists a $\tau$ with $T_2 + 1 \leq \tau$ such that $p^c_{T_1+1} < p(T_1+1, \tau)$. Because the functions $MR_t(p_t)$ are decreasing in $p_t$, equality (39) implies that

$$\sum_{t=T_1+1}^{T_2} MR_t(p(T_1+1, \tau) + (t - T_1 - 1)c) < 0. \tag{40}$$

Because, by construction,

$$\sum_{t=T_1+1}^{\tau} MR_t(p(T_1+1, \tau) + (t - T_1 - 1)c) = 0,$$

inequality (40) implies that

$$\sum_{t=T_2+1}^{\tau} MR_t(p(T_1+1, \tau) + (t - T_1 - 1)c) > 0. \tag{41}$$

Notice now that $p^c_t \leq p^c_{T_1+1} + (t - T_1 - 1)c$ for all $t = T_2+1, ..., T$. This holds because $p^c_{t+1} \leq p^c_t + c$ for all $t = T_1+1, ..., T-1$. Using $p^c_{T_1+1} < p(T_1+1, \tau)$ we have that $p^c_t < p(T_1+1, \tau) + (t - T_1 - 1)c$ for all $t = T_2+1, ..., T$. Hence, by inequality (41),

$$\sum_{t=T_2+1}^{\tau} MR_t(p^c_t) > 0. \tag{42}$$

This means that profits could be increased by marginally increasing prices for all $t = T_2 + 1, ..., \tau$. This contradicts the hypothesis that $\{p^c_t\}_{t=1}^T$ is an optimal sequence of prices.

Finally, assume by way of contradiction that there exists a $\tau$ with $\tau < T_2$ such that $p^c_{T_1+1} < p(T_1+1,\tau)$. By definition of $p(T_1+1,\tau)$, this would imply that

$$\sum_{t=T_1+1}^{\tau} MR_t(p^c_t) > 0.$$

Profits could be increased by marginally increasing prices for all $t = T_1+1, ..., \tau$. This contradicts the hypothesis that $\{p^c_t\}_{t=1}^{T}$ is an optimal sequence of prices and concludes the proof. ■