A Spatial Theory of Trade: Technical Appendix

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Proofs of Propositions 1 and 2 and Matlab Program

APPENDIX 1: PROOF OF PROPOSITION 1

LEMMA 1: *The correspondence* $\varphi^F : R_{++} \to R$ *is strictly decreasing,* $\pi' > \pi$, $H^{F\prime} \in \varphi^F(\pi')$, and $H^F \in \varphi^F(\pi)$ imply $H^{F\prime} < H^F$, *and for every* $\pi > 0$, *every point in* $\varphi^F(\pi)$ *is reached by a unique equilibrium relative price path* $p(r, \pi)$ *on* $[-S, S]$.

The correspondence $\varphi^I : R_{++} \to R$ *is strictly increasing,* $\pi' > \pi$, $H^{I\prime} \in \varphi^I(\pi')$, and $H^I \in \varphi^I(\pi)$ imply $H^{I\prime} < H^I$, *and for every* $\pi > 0$, *every point in* $\varphi^I(\pi)$ *is reached by a unique equilibrium relative price path* $p(r, \pi)$ *on* $[-S, S]$.

PROOF: It is clear from the construction described in the text that the relative price paths satisfy $p(r, \pi') > p(r, \pi)$ at all r if $\pi' > \pi$. Given the firms problems in (2.1) and (2.2), parts (i)-(iii) of Assumption A guarantee that a larger relative price strictly increases production of the intermediate good and strictly decreases the use of the intermediate good as an input in final good producing locations. Thus, by (2.3) and (2.4), if $p(r, \pi') > p(r, \pi)$ for any r, the corresponding stocks satisfy $H^F(s, \pi') < H^F(s, \pi)$ and $H^I(s, \pi') > H^I(s, \pi)$ for all $s > r$. In the same way, if $p(r, \pi)$ and $\hat{p}(r, \pi)$ are two paths both starting from π that satisfy $p(r, \pi) > \hat{p}(r, \pi)$ for any r, the corresponding stocks of excess supply satisfy $H^F(s, \pi) < \hat{H}^F(s, \pi)$ and $H^I(s, \pi) < \hat{H}^I(s, \pi)$ for all $s > r$. Conversely, no two terminal stocks can differ unless their associated relative price paths differ at some r.$\square$

Lemma 2: *For all* π, *if* $a, b \in \varphi^F(\pi)$ *and* $a < b$, *then there is a point* $c \in \varphi^F(\pi)$ *with* $a < c < b$. *For all* π, *if* $a', b' \in \varphi^I(\pi)$ *and* $a' > b'$, *then there is a point* $c' \in \varphi^I(\pi)$ *with* $a' > c' > b'$

Proof: By Lemma 1, there are distinct relative price paths, $p_a(r)$ and $p_b(r)$, say, that reach a and b for the final good and a' and b' for the intermediate good, with $p_a(-S) = p_b(-S) = \pi$. Since $a \neq b$, these paths must diverge at some point, which is to say that there is a point r_1 such that $p_a(r) = p_b(r)$ for $r \in [-S, r_1]$ and $p_a(r) > p_b(r)$ for $r \in (r_1, S]$. At this point r_1, the corresponding values of the state variables implied by these two paths are $H_a^F(r_1, \pi) = H_b^F(r_1, \pi) = H_a^I(r_1, \pi) = H_b^I(r_1, \pi) = 0$ and the relative prices satisfy $p_a(r_1) = p_b(r_1) = p_m(r_1)$. To the right of r_1, there are several ways in which $p_a(r)$ and $p_b(r)$ can diverge. We consider each in turn.

Suppose $p_a(r)$ begins to grow at the rate $\kappa^F + \kappa^I$, while $p_b(r)$ begins to decline at the rate $\kappa^F + \kappa^I$. Then define the path $p_c(r)$ by $p_c(r) = p_a(r)$ for $r \in [-S, r_1]$, $p_c(r) = p_m(r)$ for $r \in (r_1, r_1 + \varepsilon]$, and $p_c(r) = p_m(r_1 + \varepsilon)e^{(\kappa^F + \kappa^I)(r - r_1 - \varepsilon)}$ to the right of $r_1 + \varepsilon$, where $\varepsilon > 0$ is chosen small enough so that $p_m(r)$ satisfies (2.11) on $(r_1, r_1 + \varepsilon]$. Assume that the path $p_c(r)$ is then continued as described in the text. Then the terminal stock of excess supply of the final good c associated with the relative price path $p_c(r)$ is between a and b, and since $p_c(-S) = \pi$, $c \in \varphi^F(\pi)$ as was to be shown. Similarly, the terminal stock of excess supply of the intermediate good c' associated with $p_c(r)$ is between a' and b', and so $c' \in \varphi^I(\pi)$.

Suppose instead that $p_a(r)$ begins to grow at the rate $\kappa^F + \kappa^I$ while $p_b(r) = p_m(r)$ for $r \in (r_1, r_1 + \varepsilon]$, for some $\varepsilon > 0$, chosen as above. Then let $p_c(r) = p_b(r)$ for $r \in [-S, r_1 + \varepsilon/2]$ and $p_c(r) = p_m(r_1 + \varepsilon/2)e^{(\kappa^F + \kappa^I)(r - r_1 - \varepsilon/2)}$ to the right of $r_1 + \varepsilon/2$. Continue this path as described above. Then the terminal stock of excess supply of the final good c associated with $p_c(r)$ is between a and b, and $c \in \varphi^F(\pi)$. Similarly, the terminal stock of excess supply of the intermediate good c' associated with $p_c(r)$ is between a' and b', and so $c' \in \varphi^I(\pi)$.

The final possibility, that $p_a(r) = p_m(r)$ for $r \in (r_1, r_1 + \varepsilon]$ while $p_b(r)$ begins to decline at the rate $\kappa^F + \kappa^I$, can be treated in the same way.□

LEMMA 3: *For all* $\pi, \varphi^F(\pi)$ and $\varphi^I(\pi)$ *are closed.*

PROOF: We show that $\{a_n\} \in \varphi^F(\pi)$ and $a_n \to \bar{a}$ imply $\bar{a} \in \varphi^F(\pi)$. If there is some N such that $n \geq N$ implies $a_n = \bar{a}$, then $a_n \in \varphi^F(\pi)$ implies $\bar{a} \in \varphi^F(\pi)$. Then without loss of generality we can take $a_n \neq a_m$ for $n \neq m$.

Since $a_n \in \varphi^F(\pi)$ for all n, the relative price functions $p_{a_n}(r)$ and $p_{a_{n+1}}(r)$ associated with a_n and a_{n+1} must be identical on an interval $[-S, r_n]$ and different on $(r_n, S]$, where r_n has the properties $H_n^F(r_n, \pi) = 0$ and $p_{a_n}(r_n) = p_m(r_n)$. Since the sequence $\{r_n\}$ so defined lies in the compact $[-S, S]$, it has a subsequence converging to a point $\bar{r} \in [-S, S]$.

All of the relative price paths $p_{a_n}(r)$ depart from the mixed path, either growing or declining at the rate $\kappa^F + \kappa^I$. One of these two possibilities must occur infinitely often. To be specific, we take this to be growth at the rate $\kappa^F + \kappa^I$. Thus we can choose a subsequence of $\{a_n\}$ with associated relative price paths $\{p_{a_n}(r)\}$ and departure locations $\{r_n\}$ such that $r_n \to \bar{r}$ and such that every $p_{a_n}(r)$ departs from the mixed path at r_n by growing at the rate $\kappa^F + \kappa^I$. We can take $\{a_n\}$ to be this subsequence.

Now we define the relative price path $p(r)$ by $p(r) = \lim_{n\to\infty} p_{a_n}(r)$ for $r \in [-S, \bar{r})$, $p(r) = p_m(\bar{r})$, and $p(r) = p_m(\bar{r})e^{(\kappa^F+\kappa^I)(r-\bar{r})}$ to the right of $\bar{r}$. Let $\hat{a}$ be the terminal stock of excess supply of the final good associated with the path $p(r)$. Since $p(-S) = \lim_{n\to\infty} p_{a_n}(-S) = \pi$, $\hat{a} \in \varphi^F(\pi)$. We have constructed $p(r)$ in such a way that $a_n \to \hat{a}$. Hence $\bar{a} = \hat{a}$, which implies $\bar{a} \in \varphi^F(\pi)$. The same argument holds to show that $\varphi^I(\pi)$ is closed.□

LEMMA 4: *For all* $\pi, \varphi^F(\pi)$ and $\varphi^I(\pi)$ *are convex.*

PROOF: Suppose $a, b \in \varphi^F(\pi)$ and $a < c < b$. By Lemma 2, we can construct a monotonic sequence $\{a_n\}$ of points in $\varphi^F(\pi)$, either increasing toward c or decreasing toward c. Considering the first case only, suppose the limit of any such increasing sequences is less than or equal to a point $d < c$. By Lemma 3, $d \in \varphi^F(\pi)$. But then by Lemma 2 there is a point $e \in \varphi^F(\pi)$ with $e > d$: a contradiction. The same arguments can be used to show that $\varphi^I(\pi)$ is convex.□

LEMMA 5: *The correspondences* φ^F and φ^I *are compact valued and upper-hemi-continuous at all* $\pi > 0$.

PROOF: The sets $\varphi^F(\pi)$ are closed by Lemma 3 and each element of $\varphi^F(\pi)$ is associated with a relative price path contained in the interval $[\pi e^{-(\kappa^F+\kappa^I)2S}, \pi e^{(\kappa^F+\kappa^I)2S}]$. Thus φ^F is compact valued. Suppose $\{\pi_n\}$ is a sequence of positive numbers with $\pi_n \to \pi > 0$, and that $\{a_n\}$ is a sequence of numbers with $a_n \in \varphi^F(\pi_n)$. If $\pi_n = \pi$ for $n > N$, for some N, then $\{a_n\}$ has a subsequence converging to a point $a \in \varphi^F(\pi)$ by the compactness of the set $\varphi^F(\pi)$. To show that $\{a_n\}$ has a subsequence converging to a point $a \in \varphi^F(\pi)$ for all sequences $\{\pi_n\}$, then, we need only show that on the intervals on the interior of which $\varphi^F(\pi)$ is a function, it is continuous.

By Lemma 1, each point a_n is reached by a distinct relative price path $p(r, \pi_n)$, and by the construction above it is clear that $\pi_n \to \pi$ implies $p(r, \pi_n) \to p(r, \pi)$ for all r. If $\varphi^F(\pi)$ is a singleton, then the path $p(r, \pi)$ is uniquely defined by Lemma 1. If $\varphi^F(\pi)$ is not a singleton then it is an interval, and we take the path $p(r, \pi)$ to be the uniquely defined path that reaches the appropriate endpoint $a \in \varphi^F(\pi)$. The continuity of the relative price paths then ensures that $\pi_n \to \pi$ implies $H^F(S, \pi_n) \to H^F(S, \pi)$, or that $a_n \to a$. The same argument applies to show that φ^I is compact valued and upper-hemicontinuous.□

PROOF OF PROPOSITION 1: The correspondence φ^F satisfies $\varphi^F(\pi) > 0$ if π is close enough to 0, and $\varphi^F(\pi) < 0$ if π is large enough. It is upper-hemicontinuous, convex valued, and strictly decreasing in the sense of Lemma 1. Hence there is a unique π^* such that $0 \in \varphi^F(\pi^*)$. By Lemma 1, there is a unique relative price path that reaches the terminal stock of excess supply of the final good 0 from π^*. We have shown that a unique allocation satisfying (i)-(v) is consistent with a given relative price path. The same argument can be made for φ^I, so there is a unique relative price π^{**} such that $0 \in \varphi^I(\pi^{**})$, and a unique relative price path that reaches the terminal stock of excess supply of the intermediate good 0 from π^{**}. Condition (2.10) together with Lemma 1 implies that both relative price path are the same and so $\pi^* = \pi^{**}$.□

APPENDIX 2: PROOF OF PROPOSITION 2

LEMMA 6: $T^F : M_+ \times M_+ \to M_+$. and $T^I : M_+ \times M_+ \to M_+$.

PROOF: The function $e^{-\delta|r-s|}$ is continuous in r, so $T^F(z^F, z^I)$ and $T^I(z^F, z^I)$ are continuous. It is non-negative, because the functions $\theta(\cdot; z^F, z^I)$, $n^F(\cdot; z^F, z^I)$ and $n^I(\cdot; z^F, z^I)$ are non-negative. □

LEMMA 7: *For each* r, $p(r; z^I, z^F)$ *is continuous in* z^I and z^F.

PROOF: z^F and z^I determine the relative price path through the mixed path relative price function $p_m(r) \equiv \hat{p}_m(z^F(r), z^I(r))$ only. If a sequence $\{z_n^F\}$ converges to z^F in the sup norm, the sequence $\{\hat{p}_m(z_n^F(r), z^I(r))\}$ will converge to $\hat{p}_m(z^F(r), z^I(r))$ in this norm, too, since the function $\hat{p}_m(z^F, z^I)$, as are all the functions x^F, x^I, n^F, n^I, c^I, is continuous. Then we follow a construction of a relative price path from a given initial π. The same exponentials are followed, and their meeting times with the mixed path vary continuously with z^F and z^I. Hence, the implied stocks $H^F(r, \pi; z^F, z^I)$ and $H^I(r, \pi; z^F, z^I)$ vary continuously with z^F and z^I. The next argument that shows how these results in a continuous relative price function is made for the case of z^F both a parallel argument applies for the case of z^I.

Suppose that the equilibrium relative price path satisfies $p(-S) = \pi^*(z^F, z^I)$ where $\varphi^F(\pi^*(z^F, z^I))$ is a singleton. By Lemmas 1 and 5, φ^F is a continuous and decreasing function for π close to π^*. Since $\pi^*(z^F, z^I) = \left\{\pi : H^F(S, \pi; z^F, z^I) = 0\right\}$, $\pi^*(z^F, z^I)$ is continuous in z^F. Then $p(r; z^F, z^I)$ is piecewise exponential with intercept continuous in z^F, and slope sign changes at locations that vary continuously with z^F. Hence $p(r; z^F)$ is continuous in z^F.

Suppose that the equilibrium relative price path satisfies $p(-S) = \pi^*(z^F, z^I)$ where $\varphi^F(\pi^*(z^F, z^I))$ is not a singleton and $0 \in int(\varphi(\pi^*(z^F, z^I)))$. Assume further that for locations such that $p(r; z^F, z^I) = \hat{p}_m(z^F(r), z^I(r))$, $\hat{p}_m(z^F(r), z^I(r))$ satisfies the relative price arbitrage condition (2.11) with strict inequality. Then there exists an $r^m(z^F, z^I)$ such that $H^F(r^m(z^F, z^I), \pi^*(z^F, z^I); z^F, z^I) = 0$ and $p(r^m(z^F, z^I); z^F, z^I) = \hat{p}_m(z^F(r^m), z^I(r^m))$. Notice that under the above assumptions, for π close enough to π^*, $H^F(r, \pi; z^F, z^I)$ is continuous in z^F, z^I, and $H^F(r, \pi; z^F, z^I) = 0$ defines π as a continuous function of z^F, z^I. Hence $r^m(z^F, z^I)$ and $\pi^*(z^F, z^I)$ are continuous functions of z^F. $\hat{p}_m(z^F, z^I)$ continuous and the argument above then imply that $p(r; z^F, z^I)$ is continuous in z^F.

Assume that either $0 \notin int(\varphi^F(\pi^*(z^F, z^I)))$ but $0 \in \varphi^F(\pi^*(z^F, z^I))$ or $\hat{p}_m(z^F(r), z^I(r))$ satisfies condition (2.11) with equality for some location such that $p(r; z^F, z^I) = \hat{p}_m(z^F(r), z^I(r))$. It was proven in Lemma 5 that if $\varphi^F(\bar{\pi})$ is not a singleton $\lim_{\pi \to \bar{\pi}} \varphi^F(\pi) \in \varphi^F(\bar{\pi})$, where $\varphi^F(\pi)$ is a singleton for π close enough to $\bar{\pi}$. Hence since in this case $0 \in \lim_{\pi \to \pi^*(z^F, z^I)} \varphi^F(\pi)$, by both proofs above $\pi^*(z^F, z^I)$ is continuous in z^F and so $p(r; z^F, z^I)$ is continuous in z^F.□

LEMMA 8: *The operator* $T^F : M_+ \times M_+ \to M_+$ *and* $T^I : M_+ \times M_+ \to M_+$ *are continuous with respect to* z^F *and* z^I *in the sup norm* $\|\cdot\|$.

PROOF: For the continuity of T^F with respect to z^F, we need to show that for every $z^F \in M_+$ and any $\varepsilon > 0$ there is an $\eta > 0$ such that $z^{F\prime} \in M_+$ and $\left\| z^F - z^{F\prime} \right\| < \eta$ implies $\left\| T^F(z^F, z^I) - T^F(z^{F\prime}, z^I) \right\| < \varepsilon$. For any $z^F, z^{F\prime} \in M_+$ we have

$$\left\| T^F(z^F, z^I) - T^F(z^{F\prime}, z^I) \right\|$$

$$= \left\| \int_{-S}^{S} \delta^F e^{-\delta^F |r-s|} \hat{n}^F(s, p(s; z^F, z^I); z^F) \theta(s; z^F, z^I) ds - \int_{-S}^{S} \delta^F e^{-\delta^F |r-s|} \hat{n}^F(s, p(s; z^{F\prime}, z^I); z^{F\prime}) \theta(s; z^{F\prime}, z^I) ds \right\|$$

$$\leq \left\| \int_{-S}^{S} \delta^F e^{-\delta^F |r-s|} \hat{n}^F(s, p(s; z^F, z^I); z^F) \left[\theta(s; z^F, z^I) - \theta(s; z^{F\prime}, z^I) \right] ds \right\|$$

$$+ \left\| \int_{-S}^{S} \delta^F e^{-\delta^F |r-s|} \left[\hat{n}^F(s, p(s; z^F, z^I); z^F) - \hat{n}^F(s, p(s; z^{F\prime}, z^I); z^{F\prime}) \right] \theta(s; z^{F\prime}, z^I) ds \right\|$$

$$\leq \int_{-S}^{S} \delta^F e^{-\delta^F |r-s|} \hat{n}^F(s, p(s; z^F, z^I); z^F) \left| \theta(s; z^F, z^I) - \theta(s; z^{F\prime}, z^I) \right| ds$$

$$+\int_{-S}^{S}\delta^F e^{-\delta^F|r-s|}\left|\hat{n}^F(s,p(s;z^F,z^I);z^F)-\hat{n}^F(s,p(s;z^{F\prime},z^I);z^{F\prime})\right|\theta(s;z^{F\prime},z^I)ds.$$

By Lemma 7, $p(s;\cdot)$ is continuous in z^F for all $s\in[-S,S]$. Since $\hat{n}^F$ is a continuous function in both arguments, the second term can be made arbitrarily small by choice of z^F and $z^{F\prime}$.

For the first term, recall that the definition of equilibrium implies that $\theta(s;z^F,z^I)$ $=1$ if $p_m(s;z^F,z^I)>p(s;z^F,z^I)$, and $\theta(s;z^F,z^I)=0$ if $p_m(r;z^F,z^I)<p(r;z^F,z^I)$. When $p_m(r;z^F,z^I)=p(r;z^F,z^I)$, $\theta(s;z^F,z^I)\in[0,1]$ varies continuously with $p_m(\cdot)$. From Lemma 7, both $p_m(s;z^F,z^I)$ and $p(s;z^F,z^I)$ are continuous functions of z^F. Hence, the points at which $\theta(\cdot;z^F,z^I)$ jumps are also continuous in z^F. Since we are integrating over $s\in[-S,S]$ this implies that the first term can be made arbitrarily small by choice of z^F and $z^{F\prime}$.

For the continuity of T^F with respect to z^I, notice that for any $z^I,z^{I\prime}\in M_+$ we have that

$$\left\|T^F(z^F,z^I)-T^F(z^F,z^{I\prime})\right\|$$

$$=\left\|\int_{-S}^{S}\delta^F e^{-\delta^F|r-s|}\hat{n}^F(s,p(s;z^F,z^I);z^F)\theta(s;z^F,z^I)ds\right.$$

$$\left.-\int_{-S}^{S}\delta^F e^{-\delta^F|r-s|}\hat{n}^F(s,p(s;z^F,z^{I\prime});z^F)\theta(s;z^F,z^{I\prime})ds\right\|$$

$$\leq\left\|\int_{-S}^{S}\delta^F e^{-\delta^F|r-s|}\hat{n}^F(s,p(s;z^F,z^I);z^F)\left[\theta(s;z^F,z^I)-\theta(s;z^F,z^{I\prime})\right]ds\right\|$$

$$+\left\|\int_{-S}^{S}\delta^F e^{-\delta^F|r-s|}\left[\hat{n}^F(s,p(s;z^F,z^I);z^F)-\hat{n}^F(s,p(s;z^F,z^{I\prime});z^F)\right]\theta(s;z^F,z^{I\prime})ds\right\|$$

$$\leq\int_{-S}^{S}\delta^F e^{-\delta^F|r-s|}\hat{n}^F(s,p(s;z^F,z^I);z^F)\left|\theta(s;z^F,z^I)-\theta(s;z^F,z^{I\prime})\right|ds$$

$$+\int_{-S}^{S}\delta^F e^{-\delta^F|r-s|}\left|\hat{n}^F(s,p(s;z^F,z^I);z^F)-\hat{n}^F(s,p(s;z^F,z^{I\prime});z^F)\right|\theta(s;z^F,z^{I\prime})ds.$$

By Lemma 7, $p(s;\cdot)$ is continuous in z^I for all $s\in[-S,S]$. Since $\hat{n}^F$ is a continuous function of the relative price, the second term can be made arbitrarily small by choice of z^I and $z^{I\prime}$. From Lemma 7, both $p_m(s;z^F,z^I)$ and $p(s;z^F,z^I)$ are continuous functions of z^I. Hence, the points at which $\theta(\cdot; z^F, z^I)$ is discontinuous are continuous in z^F. Since we are integrating over $s \in [-S, S]$ this implies that the first term can be made arbitrarily small by choice of z^I and $z^{I\prime}$.

The argument to show that T^I is continuous in z^F and z^I parallels the argument above since, under Assumption A, $\hat{n}^I(s, p(s; z^F, z^I); z^I)$ is a continuous function in both arguments.□

LEMMA 9: *For each r, $p(r; z^F, z^I)$ is an increasing function of z^F, in the sense that $z^F(r) \geq z^{F\prime}(r)$ for all r implies $p(r; z^F, z^I) \geq p(r; z^{F\prime}, z^I)$ for all r.*

For each r, $p(r; z^F, z^I)$ is a decreasing function of z^I, in the sense that $z^I(r) \geq z^{I\prime}(r)$ for all r implies $p(r; z^F, z^I) \leq p(r; z^F, z^{I\prime})$ for all r.

PROOF: The function $p_m(r; z^F, z^I)$ is increasing in z^F. This implies that if $z^{F\prime}(r) \geq z^F(r)$ for all r, then for any given π, the associated terminal stocks of excess supply of the final good satisfy $H^F(S, \pi; z^F, z^I) \leq H^F(S, \pi; z^{F\prime}, z^I)$. Since by Lemma 1 $H^F(S, \pi; z^F, z^I)$ is strictly decreasing in π, this implies that $\pi^*(z^F, z^I) \geq \pi^*(z^{F\prime}, z^I)$. It follows that $p(r; z^F, z^I) \geq p(r; z^{F\prime}, z^I)$ for all r.

The function $p_m(r; z^F, z^I)$ is decreasing in z^I. This implies that if $z^{I\prime}(r) \geq z^I(r)$ for all r, then for any given π, the associated terminal stocks of excess supply of the intermediate good satisfy $H^I(S, \pi; z^F, z^I) \geq H^I(S, \pi; z^F, z^{I\prime})$. Since by Lemma 1 $H^I(S, \pi; z^F, z^I)$ is strictly increasing in π, this implies that $\pi^*(z^F, z^I) \leq \pi^*(z^{F\prime}, z^I)$. It follows that $p(r; z^F, z^I) \leq p(r; z^{F\prime}, z^I)$ for all r.□

LEMMA 10: *Under Assumption (A) there exists a positive number $\bar{z}^F$ such that if $z^F(r) \leq \bar{z}^F$ for all $r \in [-S, S]$, then $T^F(z^F, z^I)(r) \leq \bar{z}^F$ for all $r \in [-S, S]$. There also exists a positive number $\bar{z}^I$ such that if $z^I(r) \leq \bar{z}^I$ for all $r \in [-S, S]$, then $T^I(z^F, z^I)(r) \leq \bar{z}^I$ for all $r \in [-S, S]$.*

PROOF: We prove each part of the lemma in turn. For the first part, let z^F be any function on $[-S, S]$ that takes values $z^F(r) \in [0, \bar{z}^F]$ for some $\bar{z}^F > 0$ and let z^I be any positive and continuous function on $[-S, S]$. We find a lower bound for $p(r; z^F, z^I)$ and then an upper bound for $\hat{n}^F\left[p(r; z^F, z^I), z^F(r)\right]$.

By parts (ii) and (iii) of Assumption A and (2.6), the mixed relative price is an increasing function of $g^F(z^F)$ and a decreasing function of $g^I(z^I)$. Hence, since part (iv) of Assumption A guarantees that g^I is bounded above and g^F below, we can construct a minimum mixed relative price, $\underline{p}_m$. Since in equilibrium the relative price function has to cross the mixed relative price function for some $r \in [-S, S]$ at least once, we know that for all $r \in [-S, S]$, $p(r; z^F, z^I) > e^{-(\kappa^F+\kappa^I)2S}\underline{p}_m \equiv \underline{p}$.

Problem (2.2) and part (ii) of Assumption A imply that $\hat{n}^F\left[p(r; z^F, z^I), z^F(r)\right] \leq \hat{n}^F\left[\underline{p}, z^F(r)\right]$ for all $r \in [-S, S]$. Since $z^F(r) \leq \bar{z}^F$ for all r, part (ii) of Assumption A also implies $\hat{n}^F\left[\underline{p}, z^F(r)\right] \leq \hat{n}^F\left[\underline{p}, \bar{z}^F\right]$ for all $r \in [-S, S]$. The shares $\theta(s; z^F, z^I)$ are bounded by one, and $\delta^F e^{-\delta^F|r-s|} \in \left[0, \delta^F\right]$. Thus

$$\begin{aligned} T^F(z^F, z^I)(r) &= \int_{-S}^{S} \delta^F e^{-\delta^F|r-s|} \hat{n}^F(s, p(s; z^F, z^I); z^F)\theta(s; z^F, z^I)ds \\ &\leq \delta^F \int_{-S}^{S} \hat{n}^F(s, p(s; z^F, z^I); z^F)ds \\ &\leq \delta^F 2S\hat{n}^F\left[\underline{p}, \bar{z}^F\right]. \end{aligned}$$

By part (v) of Assumption A, $\bar{z}^F$ can be chosen so that $z^F(r) \leq \bar{z}^F$ for all r implies $\delta^F 2S\hat{n}^F\left[\underline{p}, \bar{z}^F\right] \leq \bar{z}^F$. Hence if $z^F(r) \leq \bar{z}^F$ for all r, $T^F(z^F, z^I)(r) \leq \bar{z}^F$ for all r[1].

For the second part of the proof, let z^I be any function on $[-S, S]$ that takes values $z^I(r) \in [0, \bar{z}^I]$ for some $\bar{z}^I > 0$ and let z^F be any positive and continuous function on $[-S, S]$. This time we have to find an upper bound for $p(r; z^F, z^I)$ and then an upper bound for $\hat{n}^I\left[p(r; z^F, z^I), z^I(r)\right]$.

Construct a maximum mixed relative price, $\overline{p}_m$, using the upper bound of g^F and the lower bound of g^I. Since in equilibrium the relative price function has to cross the mixed relative price function for some $r \in [-S, S]$ at least once, we know that for all $r \in [-S, S]$, $p(r; z^F, z^I) < e^{(\kappa^F+\kappa^I)2S}\overline{p}_m \equiv \overline{p}$.

Problem (2.1) and part (iii) of Assumption A imply that $\hat{n}^I$ is increasing in the relative price so $\hat{n}^I\left[p(r; z^F, z^I), z^I(r)\right] \leq \hat{n}^I\left[\overline{p}, z^I(r)\right]$ for all $r \in [-S, S]$. Since $z^I(r) \leq$

[1]Notice that it may be the case that $\bar{z}^F < inv\left[g^F(\overline{\varepsilon})\right] \leq \infty$.

$\bar{z}^I$ for all r, part (iii) of Assumption A also implies $\hat{n}^I\left[\overline{p}, z^I(r)\right] \leq \hat{n}^I\left[\overline{p}, \bar{z}^I\right]$ for all $r \in [-S, S]$. The shares $\theta(s; z^F, z^I)$ are bounded below by zero, and $\delta^I e^{-\delta^I|r-s|} \in \left[0, \delta^I\right]$. Thus,

$$\begin{aligned} T^I(z^F, z^I)(r) &= \int_{-S}^{S} \delta^I e^{-\delta^I|r-s|} \hat{n}^I(s, p(s; z^F, z^I); z^F)(1 - \theta(s; z^F, z^I))ds \\ &\leq \delta^I \int_{-S}^{S} \hat{n}^I(s, p(s; z^F, z^I); z^I)ds \\ &\leq \delta^I 2S\hat{n}^I\left[\overline{p}, \bar{z}^I\right]. \end{aligned}$$

By part (v) of Assumption A, $\bar{z}^I$ can be chosen so that $z^I(r) \leq \bar{z}^I$ for all r implies $\delta^I 2S\hat{n}^I\left[\overline{p}, \bar{z}^I\right] \leq \bar{z}^I$. Hence if $z^I(r) \leq \bar{z}^I$ for all r, $T^I(z^F, z^I)(r) \leq \bar{z}^I$ for all r[2].□

LEMMA 11: *Given z^I the operator T^F maps the set of uniformly continuous functions $z^F : [-S, S] \rightarrow \left[0, \bar{z}^F\right]$ into itself, and given z^F the operator T^I maps the set of uniformly continuous functions $z^I : [-S, S] \rightarrow \left[0, \bar{z}^I\right]$ into itself.*

PROOF: By Lemma 6, $T^F(z^F, z^I)$ is continuous in $z^F$and $T^I(z^F, z^I)$ is continuous in z^I. Since $[-S, S]$ is compact, both $T^F(z^F, z^I)$ and $T^I(z^F, z^I)$ are uniformly continuous. The result then follows by Lemma 10.□

LEMMA 12: *Let M_+^F be the set of uniformly continuous functions $z^F : [-S, S] \rightarrow \left[0, \bar{z}^F\right]$. Then $T^F(M_+^F, z^I)$ is equicontinuous.*

PROOF: By Lemma 11, for any $\varepsilon > 0$ there is an $\eta_z > 0$ such that $|r - r'| < \eta_z$ implies $\left|z^F(r) - z^{F\prime}(r)\right| < \varepsilon$. Since $z^F \in \left[0, \bar{z}^F\right]$, we can define $\bar{\eta} = \inf\left[\eta_z\right] > 0$, where the inequality comes from the fact that the space of functions with compact range and domain is closed in the sup norm. Hence $T^F(M_+^F, z^I)$ is equicontinuous. □

PROOF OF PROPOSITION 2: By Lemmas 8 and 12, T^F is a continuous operator in z^F and $T^F(M_+^F, z^I)$ is equicontinuous. Hence, given z^I, Schauder's fixed point Theorem guarantees that there exists an operator $Z^F : M_+ \rightarrow M_+^F$ such that $T^F(Z^F z^I, z^I)(r) = Z^F z^I(r)$. Let M_+^I be the set of uniformly continuous functions $z^I : [-S, S] \rightarrow \left[0, \bar{z}^I\right]$. Define the operator Z^I by $Z^I z^I(r) \equiv T^I(Z^F z^I, z^I)(r)$. By

[2] Again, notice that it may be the case that $\bar{z}^I < inv\left[g^I(\bar{\varepsilon})\right] \leq \infty$.

Lemma 11, $Z^I : M_+^I \to M_+^I$. We will use Schauder's fixed point theorem again to show that this operator has a fixed point. For this, we first need to prove that Z^I is continuous in the sup norm. The operator Z^F is defined as the fixed point of T^F given z^I. Since by Lemma 8 T^F is continuous with respect to z^I, Z^F is continuous in the sup norm. Lemma 8 also guarantees that T^I is continuous in z^I and z^F, which implies that Z^I is continuous in the sup norm.

The proof that $Z^I(M_+^I)$ is equicontinuous parallels the proof of Lemma 12. The key in the proof is that $\bar{z}^I$ is not a function of z^F and so the properties of Z^F do not play a role in the argument. By Lemma 11, for any $\varepsilon > 0$ there is an $\eta_z > 0$ such that $|r - r'| < \eta_z$ implies $\left|z^I(r) - z^{I\prime}(r)\right| < \varepsilon$. Since $z^I \in \left[0, \bar{z}^I\right]$, we can define $\bar{\eta} = \inf[\eta_z] > 0$, where the inequality comes from the fact that the space of functions with compact range and domain is closed in the sup norm. Hence $Z^I(M_+^F)$ is equicontinuous.

It follows from Schauder's fixed point theorem that there exist a function z^I such that, for all $r \in [-S, S]$, $Z^I z^I(r) = z^I(r)$. This implies that there exists a pair of functions $z^F = Z^F z^I$ and z^I such that $T^I(Z^F z^I, z^I)(r) = z^I(r)$,and $T^F(Z^F z^I, z^I)(r) = Z^F z^I(r)$. Hence, there exist a pair of functions (z^F, z^I) that satisfy condition (vi) in the definition of equilibrium. Proposition 1 then yields the result.□

APPENDIX 3: MATLAB PROGRAM

```
% MainProgram.m
% This program computes numerically the model in "A Spatial Theory of Trade"
% It computes an equilibrium given a pair of initial productivity functions
% (zfini and ziini)
clear all
% Define constants
r = 1:1000;
```

```
r = r./100;
dr = r(2)-r(1);
gammaf = 0.09;
gammai = 0.09;
alphaf = 0.8;
betaf = 0.1;
alphai = 0.7;
kappaf = 0.005;
kappai = 0.005;
deltaf = 5;
deltai = 5;
ubar = 1;
betau = 0.8;
crit = 0.001; % Convergence criterion
% Define the productivity functions
load zfini
load ziini
zf(2,:)=zfini;
zi(2,:)=ziini;
%Difference in initial iteration
zf(1,:) = zf(2,:)+ 1500*crit;
%counter of iterations
k=2;
%initial relative price
rp(1) = 1.1; % (-S relative price)
%productivity function iteration loop
    while max(abs(zf(k,:)-zf(k-1,:)))>(crit*length(r))
     %display iteration number and convergence level
```

```
disp(k)
disp(max(abs(zf(k,:)-zf(k-1,:))))
%Initialize vectors
ni = [];
 ci = [];
 nf = [];
 xi = [];
 xf = [];
 Hf = [];
 Hi = [];
 ste = crit/100; % Initial step size for relative price
 conve = crit + 1;
 ind1 = 0;
 ind2 = 0;
 %Calculate cf
 cf = ubar.^(1/betau);
 %Calculate rpm=pI/PF|m
 a1 = 1-alphai;
 ab1 = 1-alphaf-betaf;
 rpm = (zf(k,:).^(gammaf/ab1)).*(zi(k,:).^(-gammai/a1)).* (cf.^(-alphaf/ab1)).*
       (cf.^(alphai/a1)).* (alphai.^( alphai/a1)).*(alphaf.^(alphaf/ab1)).*
            (betaf.^(betaf/ab1)).*(ab1/a1);
rpm = (rpm.^((a1*ab1)/(1-alphaf+alphai*betaf)));
 %Define initial relative price (loop)
 Hf(1) = 0;
 %Mix use indicator
 mix = 0;
 con = 0.*r;
```

```
%loop to find initial relative price
while abs(conve)>crit
      ni=[];
       xi=[];
       nf=[];
       ci=[];
       xf=[];
       %Calculation of equilibrium given productivity levels and initial
       %price
       for i = 1:1000
             if rp(i) > rpm(i) + crit % Intermediate good case
                   theta(i) = 0;
                  ni(i) = (rp(i)^(1/a1))*(zi(k,i)^(gammai/a1))* (alphai^(1/a1))
                        *(cf^(1/a1));
                  xi(i) = (zi(k,i)^(gammai))*(ni(i)^(alphai));
                  Hf(i+1) = Hf(i) + (-rp(i)*xi(i)-kappaf*abs(Hf(i)))*dr;
                  Hi(i+1) = -Hf(i+1)*(1/rp(i));
                   nf(i) = 0;
                   ci(i) = 0;
                  xf(i) = 0;
             else
                  if rp(i) < rpm(i) - crit % Final good case
                        theta(i) = 1;
                        nf(i) = (rp(i)^(betaf/ab1))*(zf(k,i)^(gammaf/ab1))*
                              (alphaf^((1-betaf)/ab1))*(betaf^(betaf/ab1))*
                                    (cf^((betaf-1)/ab1));
                        ci(i) = (rp(i)^((alphaf-1)/ab1))*(zf(k,i)^
                              (gammaf/ab1))* (alphaf^((alphaf)/ab1))*(betaf^
```

```
            ((1-alphaf)/ab1))*(cf^((-alphaf)/ab1));
        xf(i) = (zf(k,i)^(gammaf))*(nf(i)^(alphaf))*
            (ci(i)^(betaf));
        Hf(i+1) = Hf(i)+(rp(i)*ci(i)-kappaf*abs(Hf(i)))*dr;
        Hi(i+1) = -Hf(i+1)*(1/rp(i));
        ni(i) = 0;
        xi(i) = 0;
    else %Mixed case
        ni(i) = (rp(i)^(1/a1))*(zi(k,i)^(gammai/a1))*
            (alphai^(1/a1))*(cf^(1/a1));
        xi(i) = (zi(k,i)^(gammai))*(ni(i)^(alphai));
        nf(i) = (rp(i)^(betaf/ab1))*(zf(k,i)^
            (gammaf/ab1))*(alphaf^((1-betaf)/ab1))*
                (betaf^(betaf/ab1))*(cf^((betaf-1)/ab1));
        ci(i) = (rp(i)^((alphaf-1)/ab1))*
            (zf(k,i)^(gammaf/ab1))*(alphaf^((alphaf)/ab1))*
                (betaf^((1-alphaf)/ab1))*(cf^((-alphaf)/ab1));
        xf(i) =(zf(k,i)^(gammaf))*(nf(i)^(alphaf))*
            (ci(i)^(betaf));
        theta(i) = (cf*ni(i))/(xf(i)-cf*nf(i)+cf*ni(i));
        Hf(i+1) = Hf(i)+rp(i)*((theta(i)*ci(i)-(1-
            theta(i))*xi(i))-kappaf*abs(Hf(i)))*dr;
        Hi(i+1) = -Hf(i+1)*(1/rp(i));
    end
end
if Hf(i+1) > 0 & con(i)==0
    rp(i+1) = rp(i)*exp((-r(2)+r(1))*(kappai+kappaf));
elseif Hf(i+1) < 0 & con(i)==0
```

```
        rp(i+1) = rp(i)*exp((-r(1)+r(2))*(kappai+kappaf));
    elseif con(i)==1
        rp(i+1) = rpm(i+1);
        Hf(i+1) = 0;
    end
    if i<length(r) & abs(Hf(i+1))<crit & abs(rp(i)-rpm(i))<crit & con(i)==0
        rp(i+1) = rpm(i+1);
        Hf(i+1) = 0;
        if mix == 1
            rp(i+1) = rp(i)*exp((r(2)-r(1))*(kappai+kappaf))+crit;
        elseif mix == 2
            rp(i+1) = rp(i)*exp((r(1)-r(2))*(kappai+kappaf))-crit;
        else
            mix = 0;
            con(i) = 1;
        end
        mix = mix+1;
    end
end
conve = Hf(length(r)+1);
disp(conve)
% Change step size if sign changes
if conve>0
    ind1 = 1;
else
    ind2 = 1;
end
if ind1+ind2 == 2
```

```
            ste = ste/10;
            ind1 = 0;
            ind2 = 0;
        end
        if Hf(length(r)+1) > crit & mix == 0
            rp(1) = rp(1)+ste;
        elseif Hf(length(r)+1) < -crit & mix ==0
            rp(1) = rp(1)-ste;
        end
        if abs(conve)>crit
            Hf=[];
            Hi=[];
        end
        Hf(1) = 0;
    end
    %Save variables in each iteration
    k = k+1;
    rpma(k,:) = rpm;
    rpa(k,:) = rp;
    Hfa(k,:) = Hf;
    nfa(k,:) = nf;
    nia(k,:) = ni;
    %Calculate new productivity functions
    for j = 1:length(r)
        zf(k,j) = deltaf*sum(exp(-deltaf.*abs(r-r(j))).*nf.*theta);
        zi(k,j) = deltai*sum(exp(-deltai.*abs(r-r(j))).*ni.*(1-theta));
    end
end
```