

Online Appendix for “Asset Prices and Institutional Investors”

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Appendix A: Proofs

Proof of Lemma 1. Since the securities market in our setup is dynamically complete, it is well known that there exists a state price density process, ξ , such that the time- t value of a payoff C_T at time T is given by $E_t[\xi_T C_T]/\xi_t$. In our setting, the state price density is a martingale and follows the dynamics

$$d\xi_t = -\xi_t \kappa_t d\omega_t, \quad (\text{A1})$$

where $\kappa_t \equiv \mu_{S_t}/\sigma_{S_t}$ is the Sharpe ratio process. Accordingly, investor i 's dynamic budget constraint (3) can be restated as

$$E_t[\xi_T W_{iT}] = \xi_t W_{it}. \quad (\text{A2})$$

Maximizing the institutional investor's expected objective function (5) subject to (A2) evaluated at time $t = 0$ leads to the institution's optimal terminal wealth as

$$W_{iT} = \frac{1 + bD_T}{y_I \xi_T},$$

where $1/y_I$ solves (A2) evaluated at $t = 0$. Using the fact that D_t is lognormally distributed for all t , we obtain

$$\frac{1}{y_I} = \frac{\lambda \xi_0 S_0}{1 + bD_0}.$$

Consequently, the institution's optimal terminal wealth is given by

$$W_{iT} = \frac{\lambda \xi_0 S_0}{\xi_T} \frac{1 + bD_T}{1 + bD_0}, \quad (\text{A3})$$

and from (A2) its optimal time- t wealth by

$$\xi_t W_{it} = \lambda \xi_0 S_0 \frac{1 + bD_t}{1 + bD_0}. \quad (\text{A4})$$

Applying Itô's lemma to both sides of (A4), and using (3) and (A1), leads to

$$\xi_t W_{it} (\phi_{It} \sigma_{S_t} - \kappa_t) d\omega_t = \lambda \xi_0 S_0 \frac{bD_t}{1 + bD_0} \sigma d\omega_t,$$

which after matching the diffusion terms and rearranging gives the institutional investor's optimal portfolio (6). Similarly, we obtain the retail investor's optimal terminal and time- t wealth as

$$W_{\mathcal{R}T} = \frac{(1 - \lambda)\xi_0 S_0}{\xi_T}, \quad (\text{A5})$$

$$\xi_t W_{\mathcal{R}t} = (1 - \lambda)\xi_0 S_0. \quad (\text{A6})$$

Application of Itô's lemma leads to the standard retail investor's optimal portfolio in (7). *Q.E.D.*

Proof of Proposition 1. By no arbitrage, the stock market price in this complete market setup is given by

$$S_t = \frac{E_t[\xi_T D_T]}{\xi_t}. \quad (\text{A7})$$

We proceed by first determining the equilibrium state price density process ξ . Imposing the market clearing condition $W_{\mathcal{R}T} + W_{\mathcal{I}T} = D_T$, and substituting (A3) and (A5) yields

$$\left(\lambda \frac{1 + bD_T}{1 + bD_0} + (1 - \lambda) \right) \frac{\xi_0 S_0}{\xi_T} = D_T, \quad (\text{A8})$$

which after rearranging leads to the equilibrium terminal state price density:

$$\xi_T = \frac{\xi_0 S_0}{1 + bD_0} \frac{1}{D_T} (1 + bD_0 + \lambda b(D_T - D_0)). \quad (\text{A9})$$

Consequently, the equilibrium state price density at time t is given by

$$\begin{aligned} \xi_t &= E_t[\xi_T] \\ &= \frac{\xi_0 S_0}{1 + bD_0} E_t[1/D_T] (1 + bD_0 + \lambda b(1/E_t[1/D_T] - D_0)) \\ &= \frac{\xi_0 S_0}{1 + bD_0} \frac{e^{\sigma^2(T-t)}}{D_t} \left(1 + bD_0 + \lambda b \left(e^{-\sigma^2(T-t)} D_t - D_0 \right) \right), \end{aligned} \quad (\text{A10})$$

where the last equality employs the fact that D_t is lognormally distributed.

Finally, to determine the equilibrium stock market level, we substitute (A9)–(A10) into (A7) and manipulate to obtain the stated expression (8). The stock market level \bar{S} in the benchmark economy with no institutions (9) follows by considering the special case of $b = 0$ in (8). The property that the stock market is higher in the presence of institutions follows from the fact that the factor multiplying \bar{S}_t in expression (8) is strictly greater than one,

and being increasing in λ from the fact that the numerator in that factor is increasing at a faster rate than the denominator does in λ . *Q.E.D.*

Proof of Proposition 2. We write the equilibrium stock price in (8) as

$$S_t = \bar{S}_t \frac{X_t}{Z_t}, \quad (\text{A11})$$

where

$$\begin{aligned} \bar{S}_t &= e^{-\sigma^2(T-t)} D_t, \\ X_t &= 1 + b D_0 + \lambda b (D_t - D_0), \\ Z_t &= 1 + b D_0 + \lambda b \left(e^{-\sigma^2(T-t)} D_t - D_0 \right). \end{aligned}$$

Applying Itô's lemma to (A11) we obtain

$$\sigma_{S_t} = \sigma + \sigma_{X_t} - \sigma_{Z_t}, \quad (\text{A12})$$

where

$$\begin{aligned} \sigma_{X_t} &= \frac{\lambda b D_t}{1 + b D_0 + \lambda b (D_t - D_0)} \sigma, \\ \sigma_{Z_t} &= \frac{\lambda b e^{-\sigma^2(T-t)} D_t}{1 + b D_0 + \lambda b (e^{-\sigma^2(T-t)} D_t - D_0)} \sigma. \end{aligned}$$

We note that $X_t \sigma_{X_t} = \lambda b D_t \sigma$, $Z_t \sigma_{Z_t} = X_t \sigma_{X_t} e^{-\sigma^2(T-t)}$, and so $X_t \sigma_{X_t} = Z_t \sigma_{Z_t} e^{-\sigma^2(T-t)}$. Hence, we have

$$X_t \sigma_{X_t} Z_t - Z_t \sigma_{Z_t} X_t = X_t \sigma_{X_t} (1 - e^{-\sigma^2(T-t)}) (1 + (1 - \lambda) b D_0). \quad (\text{A13})$$

Substituting (A13) into the expression $\sigma_{X_t} - \sigma_{Z_t} = (X_t \sigma_{X_t} Z_t - Z_t \sigma_{Z_t} X_t) / X_t Z_t$, and then into (A12) leads to the equilibrium stock index volatility expression in (10). The property that the stock volatility is higher than the volatility in the benchmark with no institutions is immediate since $\sigma_{X_t} - \sigma_{Z_t} > 0$. *Q.E.D.*

Proof of Proposition 3. We first determine the investors' equilibrium fractions of wealth invested in the stock index, ϕ_{it} , $i = \mathcal{I}, \mathcal{R}$. From (A7) and (A9) we have

$$\begin{aligned} \xi_t S_t &= E_t [\xi_T D_T] \\ &= \frac{\xi_0 S_0}{1 + b D_0} (1 + (1 - \lambda) b D_0 + \lambda b D_t). \end{aligned} \quad (\text{A14})$$

Applying Itô's lemma to both sides of (A14), we obtain

$$\sigma_{S_t} - \kappa_t = \frac{\lambda b D_t}{1 + (1 - \lambda) b D_0 + \lambda b D_t} \sigma,$$

or

$$\frac{\kappa_t}{\sigma_{S_t}} = 1 - \frac{\lambda b D_t}{1 + (1 - \lambda) b D_0 + \lambda b D_t} \frac{\sigma}{\sigma_{S_t}}, \quad (\text{A15})$$

where σ_{S_t} is as given in Proposition 2. Substituting (A15) into the investors' optimal portfolios (6)–(7) in Lemma 1 yields their equilibrium portfolios ϕ_{it} .

Next, we determine the investors' wealth per unit of the stock market level, W_{it}/S_t , in equilibrium. Substituting the deflated time- t wealth of investors, (A4) and (A6), along with the deflated stock market level (A14), we obtain

$$\frac{W_{\mathcal{I}t}}{S_t} = \frac{\xi_t W_{\mathcal{I}t}}{\xi_t S_t} = \lambda \frac{1 + b D_t}{1 + b D_0 + \lambda b (D_t - D_0)}, \quad (\text{A16})$$

$$\frac{W_{\mathcal{R}t}}{S_t} = \frac{\xi_t W_{\mathcal{R}t}}{\xi_t S_t} = (1 - \lambda) \frac{1 + b D_0}{1 + b D_0 + \lambda b (D_t - D_0)}. \quad (\text{A17})$$

As a remark, we here note that the ratio of the two investors' wealth in equilibrium is given by substituting (A16) in (A17):

$$\frac{W_{\mathcal{I}t}}{W_{\mathcal{R}t}} = \frac{\lambda}{1 - \lambda} \frac{1 + b D_t}{1 + b D_0}, \quad (\text{A18})$$

as highlighted in footnote (6). Finally, the investors' equilibrium portfolio weights ϕ_{it} above, along with their equilibrium per unit of stock index level leads to their equilibrium strategies in units of shares π_{it} , as given by (11)–(12) in Proposition 3.

The leverage property follows by substituting (A15) into (6) and rearranging to get the fraction of wealth invested in the riskless bond as

$$1 - \phi_{\mathcal{I}t} = \frac{\lambda b D_t}{1 + (1 - \lambda) b D_0 + \lambda b D_t} \frac{\sigma}{\sigma_{S_t}} - \frac{b D_t}{1 + b D_t} \frac{\sigma}{\sigma_{S_t}} < 0.$$

Q.E.D.

Proof of Proposition 4. Applying Itô's Lemma to both sides of (A10) and manipulating leads to the equilibrium Sharpe ratio expression (13). The benchmark Sharpe ratio with no institutions is obtained by considering the special case of $b = 0$ in (13). The properties reported are straightforward to derive from the expression in (13). *Q.E.D.*

Proof of Proposition 5. We first consider the investors' optimal choices in partial equilibrium. The retail investor's optimal terminal wealth and time- t wealth are as in the proof

of Lemma 1, given by (A5)–(A6). The “leveraged” institutional investor with initial wealth $W_{I_0} = \theta\lambda S_0$ now chooses its optimal terminal wealth and time- t wealth as

$$W_{IT} = \frac{\theta\lambda\xi_0 S_0}{\xi_T} \frac{1+bD_T}{1+bD_0} \quad \text{and} \quad \xi_t W_{It} = \theta\lambda\xi_0 S_0 \frac{1+bD_t}{1+bD_0}, \quad (\text{A19})$$

which are the same as in the baseline economy but with $\theta\lambda$ replacing λ . Both the levered institutional and retail investors’ optimal portfolios are as before, given by (6)–(7) in Lemma 1.

Moving to general equilibrium, we first note that in the presence of the additional buy-and-hold institutional investor with initial assets $W_{L_0} = (1-\theta)\lambda S_0$, the market clearing condition is now:

$$W_{IT} + W_{RT} = (1 - (1 - \theta)\lambda)D_T. \quad (\text{A20})$$

Substituting the investors’ optimal terminal wealth (A5) and (A19) into (A20), we have

$$\left(\lambda\theta \frac{1+bD_T}{1+bD_0} + (1-\lambda) \right) \frac{\xi_0 S_0}{\xi_T} = (1 - (1 - \theta)\lambda)D_T.$$

Manipulating, we obtain the equilibrium terminal state price density as

$$\xi_T = \frac{\xi_0 S_0}{1+bD_0} \frac{1}{(1 - (1 - \theta)\lambda)D_T} \left((1-\lambda)(1+bD_0) + \theta\lambda(1+bD_T) \right). \quad (\text{A21})$$

and the time- t state price density as

$$\xi_t = E_t[\xi_T] = \frac{\xi_0 S_0}{1+bD_0} \frac{e^{\sigma^2(T-t)}}{(1 - (1 - \theta)\lambda)D_t} \left((1-\lambda)(1+bD_0) + \theta\lambda(1+be^{-\sigma^2(T-t)}D_t) \right). \quad (\text{A22})$$

The stock price is then given by

$$S_t = \frac{E_t[\xi_T D_T]}{\xi_t} = e^{-\sigma^2(T-t)} D_t \frac{1+bD_0 + \frac{\theta\lambda}{1-\lambda}(1+bD_t)}{1+bD_0 + \frac{\theta\lambda}{1-\lambda}(1+be^{-\sigma^2(T-t)}D_t)}, \quad (\text{A23})$$

which is the same formula as in the baseline economy (Proposition 1) but with $\lambda/(1-\lambda)$ replaced by $\lambda'/(1-\lambda')$. It then follows immediately that the volatility σ_{S_t} is the same as in Proposition 2, but with λ replaced by λ' . The same transformation also applies to the investors’ portfolios. One can see this from the proof of Proposition 3, which gets modified analogously. The comparative statics results parallel those in Propositions 1-3 because λ' increases in θ . *Q.E.D.*

Proof of Lemma 2. The securities market is still dynamically complete in this multi-stock setup with N risky stocks and N sources of risk. Hence, there exists a state price density process, ξ , which is a martingale and follows the dynamics

$$d\xi_t = -\xi_t \boldsymbol{\kappa}_t^T d\boldsymbol{\omega}_t, \quad (\text{A24})$$

where $\boldsymbol{\kappa}_t \equiv \boldsymbol{\sigma}_{S_t}^{-1} \boldsymbol{\mu}_{S_t}$ is the N -dimensional Sharpe ratio process.

Following the same steps as in the proof of Lemma 1, the single-stock case, and using the fact that the index cash flow news I is lognormally distributed, we obtain the institutional investor's optimal terminal wealth and time- t wealth as

$$W_{IT} = \frac{\lambda \xi_0 S_{MKT0}}{\xi_T} \frac{1 + b I_T}{1 + b I_0}, \quad (\text{A25})$$

$$\xi_t W_{It} = \lambda \xi_0 S_{MKT0} \frac{1 + b I_t}{1 + b I_0}. \quad (\text{A26})$$

Applying Itô's lemma to (A26) leads to

$$\xi_t W_{It} (\boldsymbol{\phi}_{It}^T \boldsymbol{\sigma}_{S_t} - \boldsymbol{\kappa}_t^T) d\boldsymbol{\omega}_t = \lambda \xi_0 S_{MKT0} \frac{b I_t}{1 + b I_0} \boldsymbol{\sigma}_I d\boldsymbol{\omega}_t, \quad (\text{A27})$$

which after matching coefficients yields the institutional optimal portfolio as reported in (23). The retail investor's optimal terminal wealth and time- t wealth are as in the single-stock case given by (A5) and (A6), which leads to the optimal portfolio in (24).

To prove property (i), we first note from (A27) that

$$\boldsymbol{\sigma}_{S_t}^T \boldsymbol{\phi}_{It} = \boldsymbol{\kappa}_t + A_t \boldsymbol{\sigma}_I^T, \quad (\text{A28})$$

where the scalar $A_t \equiv b I_t / (1 + b I_t)$. This is a system of N equations in N unknowns ($\boldsymbol{\phi}_{It}$). We represent its solution in the form

$$\boldsymbol{\phi}_{It} \equiv \boldsymbol{\phi}_{\mathcal{R}t} + \boldsymbol{\phi}_{Ht},$$

where $\boldsymbol{\phi}_{\mathcal{R}t}$ is the mean-variance portfolio and $\boldsymbol{\phi}_{Ht}$ denotes the hedging portfolio. The mean-variance portfolio is given by (24), and together with (A24) satisfies

$$\boldsymbol{\sigma}_{S_t}^T \boldsymbol{\phi}_{\mathcal{R}t} = \boldsymbol{\kappa}_t. \quad (\text{A29})$$

The hedging portfolio is well-known to be a portfolio that has the maximal correlation with the state variable I_t (e.g., Ingersoll, 1987). Here the securities market is dynamically complete, and so the perfect correlation of 1 can be achieved. Let us now consider an auxiliary securities market in which we replace any of the stocks, say the first stock, by the index S_I itself. In equilibrium, S_I and $\boldsymbol{\sigma}_{S_I}$ are given by (26) and (30) in Propositions 6 and 7, respectively, with the subscript j replaced by the subscript I . The index value S_I is therefore driven by a single state variable I_t , and hence by investing in the index S_I one can achieve a unit correlation with I_t . So, we can conclude that the hedging portfolio $\boldsymbol{\phi}_H$ is of the form $\boldsymbol{\phi}_{Ht} = C_t(1, 0, \dots, 0)^T$, where C_t is a scalar, satisfying

$$\boldsymbol{\sigma}_{S_t}^T \boldsymbol{\phi}_{Ht} = A_t \boldsymbol{\sigma}_{I_t}^T, \quad (\text{A30})$$

which together with (A29) satisfy (A28). The relation (A30) holds because the first row of $\boldsymbol{\sigma}_{S_t}$ in the auxiliary economy, $\boldsymbol{\sigma}_{S_{I_t}}$, and $\boldsymbol{\sigma}_I$ are collinear. This is nothing else but the three-fund separation property, with the funds being the mean-variance efficient portfolio, the index, and the riskless bond. Moreover, $\boldsymbol{\phi}_{H_t} > 0$ since $\boldsymbol{\sigma}_{S_I} > 0$ and $A_t, C_t > 0$. In this auxiliary economy, therefore, the optimal hedging portfolio puts zero weights on all securities but the index. Mapping the auxiliary economy back into the original economy and recognizing that the index S_I is a portfolio of one share in each of the index stocks, we arrive at property (i). Property (ii) then follows immediately. *Q.E.D.*

Proof of Proposition 6. We first determine the equilibrium state price density process. Imposing the market clearing condition $W_{TT} + W_{RT} = D_T$, substituting (A25) and (A5), and manipulating yields the terminal equilibrium state price density as

$$\xi_T = \frac{\xi_0 S_{MKT0}}{1 + bI_0} \frac{1}{D_T} (1 + bI_0 + \lambda b(I_T - I_0)). \quad (\text{A31})$$

To obtain the time- t equilibrium state price density, we use the properties of lognormal distribution $E_t[1/D_T] = e^{\|\boldsymbol{\sigma}\|^2(T-t)}/D_t$, $E_t[I_T/D_T] = e^{(\|\boldsymbol{\sigma}\|^2 - \boldsymbol{\sigma}_I^T \boldsymbol{\sigma})(T-t)} I_t/D_t$, which along with (A31) and some manipulations we get

$$\xi_t = \frac{\xi_0 S_{MKT0}}{1 + bI_0} \frac{e^{\|\boldsymbol{\sigma}\|^2(T-t)}}{D_t} \left(1 + bI_0 + \lambda b(e^{-\boldsymbol{\sigma}_I^T \boldsymbol{\sigma}(T-t)} I_t - I_0)\right). \quad (\text{A32})$$

To determine the equilibrium market portfolio price, we first compute its deflated process from (A31) as, after some manipulation

$$\begin{aligned} \xi_t S_{MKTt} &= E_t[\xi_T D_T] \\ &= \frac{\xi_0 S_{MKT0}}{1 + bI_0} (1 + bI_0 + \lambda b(I_t - I_0)). \end{aligned} \quad (\text{A33})$$

Substituting (A32) into (A33) yields the market portfolio level as reported in (25). The price in the benchmark economy with no institution is obtained as a special case by setting $b = 0$.

To determine the equilibrium price of an index stock $j = 1, \dots, M - 1$, we first find its deflated process:

$$\xi_t S_{jt} = E_t[\xi_T D_{jT}]. \quad (\text{A34})$$

From (A31), we have

$$\xi_T D_{jT} = \frac{\xi_0 S_{MKT0}}{1 + bI_0} \frac{D_{jT}}{D_T} (1 + bI_0 + \lambda b(I_T - I_0)). \quad (\text{A35})$$

After some manipulations and substitution of the properties of the properties of lognormally distributed processes

$$E_t \left[\frac{D_{jT}}{D_T} \right] = e^{(\|\boldsymbol{\sigma}\|^2 - \boldsymbol{\sigma}_j^T \boldsymbol{\sigma})(T-t)} \frac{D_{jt}}{D_t},$$

$$E_t \left[\frac{D_{jT} I_T}{D_T} \right] = e^{(\sigma_j^T \sigma_I + \|\sigma\|^2 - \sigma_I^T \sigma - \sigma_j^T \sigma)(T-t)} \frac{D_{jt} I_t}{D_t},$$

we obtain

$$\begin{aligned} E_t [\xi_T D_{jT}] &= \frac{\xi_0 S_{MKT0}}{1 + b I_0} e^{(\|\sigma\|^2 - \sigma_j^T \sigma)(T-t)} \frac{D_{jt}}{D_t} \\ &\quad \times \left(1 + b I_0 + \lambda b (e^{(-\sigma_I^T \sigma + \sigma_j^T \sigma_I)(T-t)} I_t - I_0) \right). \end{aligned} \quad (\text{A36})$$

Finally, substituting (A32) and (A36) into (A34), we obtain the equilibrium price of an index stock as reported in (26) of Proposition 5. The index stock price in the benchmark economy is obtained as a special case by setting $b = 0$.

To determine the equilibrium price of a nonindex stock $k = M + 1, \dots, N - 1$, we proceed as in the index stock case and obtain the same stock price equation (26) but now with the correlation with the index $\sigma_k^T \sigma_I = 0$ substituted in. With this zero correlation, the nonindex stock price collapses to its value in the benchmark economy with no institutions. The stated property of higher market portfolio and index stock prices is immediate from the expressions (25)–(26). *Q.E.D.*

Proof of Proposition 7. To determine the equilibrium volatilities in this multi-stock case, we proceed as in Proposition 3. For the market portfolio, we express its equilibrium price as $S_{MKT,t} \equiv \bar{S}_{MKT,t} X_t / Z_t$ and apply Itô's lemma to obtain

$$\sigma_{MKT,t} = \sigma + \sigma_{X_t} - \sigma_{Z_t},$$

where

$$\begin{aligned} \sigma_{X_t} &= \frac{\lambda b I_t}{1 + b I_0 + \lambda b (I_t - I_0)} \sigma_I, \\ \sigma_{Z_t} &= \frac{\lambda b e^{-\sigma_I^T \sigma (T-t)} I_t}{1 + b I_0 + \lambda b (e^{-\sigma_I^T \sigma (T-t)} I_t - I_0)} \sigma_I. \end{aligned}$$

So we have $\sigma_{Z_t} Z_t = \sigma_{X_t} X_t e^{-\sigma_I^T \sigma (T-t)}$, implying after some manipulation that

$$(\sigma_{X_t} - \sigma_{Z_t}) X_t Z_t = \lambda b (1 - e^{-\sigma_I^T \sigma (T-t)}) (1 + (1 - \lambda) b I_0) I_t \sigma_I,$$

leading to the market portfolio volatility as reported in (29).

For the index stock volatility, analogously we express the equilibrium price of an index stock $j = 1, \dots, M - 1$ as $S_{jt} \equiv \bar{S}_{jt} X_{jt} / Z_{jt}$. Applying Itô's lemma we obtain

$$\sigma_{S_{jt}} = \sigma_j + \sigma_{X_{jt}} - \sigma_{Z_{jt}},$$

where

$$\begin{aligned}\boldsymbol{\sigma}_{X_{jt}} &= \frac{\lambda b e^{(-\boldsymbol{\sigma}_I^T \boldsymbol{\sigma} + \boldsymbol{\sigma}_j^T \boldsymbol{\sigma}_I)(T-t)} I_t}{1 + (1 - \lambda) b I_0 + \lambda b e^{(-\boldsymbol{\sigma}_I^T \boldsymbol{\sigma} + \boldsymbol{\sigma}_j^T \boldsymbol{\sigma}_I)(T-t)} I_t} \boldsymbol{\sigma}_I, \\ \boldsymbol{\sigma}_{Z_{jt}} &= \frac{\lambda b e^{-\boldsymbol{\sigma}_I^T \boldsymbol{\sigma}(T-t)} I_t}{1 + (1 - \lambda) b I_0 + \lambda b e^{-\boldsymbol{\sigma}_I^T \boldsymbol{\sigma}(T-t)} I_t} \boldsymbol{\sigma}_I.\end{aligned}$$

hence, we have $\boldsymbol{\sigma}_{Z_{jt}} Z_{jt} = \boldsymbol{\sigma}_{X_{jt}} X_{jt} e^{-\boldsymbol{\sigma}_j^T \boldsymbol{\sigma}_I (T-t)}$, implying

$$(\boldsymbol{\sigma}_{X_{jt}} - \boldsymbol{\sigma}_{Z_{jt}}) X_{jt} Z_{jt} = \lambda b (1 - e^{-\boldsymbol{\sigma}_j^T \boldsymbol{\sigma}_I (T-t)}) (1 + (1 - \lambda) b I_0) e^{(-\boldsymbol{\sigma}_I^T \boldsymbol{\sigma} + \boldsymbol{\sigma}_j^T \boldsymbol{\sigma}_I)(T-t)} I_t \boldsymbol{\sigma}_I,$$

leading to the market portfolio volatility as reported in (30).

The implications that the market portfolio and index stock volatilities are higher follow immediately from the expressions (29)–(30). As for the higher correlation property (ii) amongst index stocks, we need to show that for two index stocks j and l

$$\frac{\boldsymbol{\sigma}_{S_{jt}}^T \boldsymbol{\sigma}_{S_{lt}}}{\sqrt{\|\boldsymbol{\sigma}_{S_{jt}}\|^2 \|\boldsymbol{\sigma}_{S_{lt}}\|^2}} > \frac{\bar{\boldsymbol{\sigma}}_{S_{jt}}^T \bar{\boldsymbol{\sigma}}_{S_{lt}}}{\sqrt{\|\bar{\boldsymbol{\sigma}}_{S_{jt}}\|^2 \|\bar{\boldsymbol{\sigma}}_{S_{lt}}\|^2}}.$$

Since $\bar{\boldsymbol{\sigma}}_{S_{jt}}^T \bar{\boldsymbol{\sigma}}_{S_{lt}} = \bar{\boldsymbol{\sigma}}_j^T \bar{\boldsymbol{\sigma}}_l = 0$, above is equivalent to showing $\boldsymbol{\sigma}_{S_{jt}}^T \boldsymbol{\sigma}_{S_{lt}} > 0$. From (25), for an index stock we have

$$\boldsymbol{\sigma}_{S_{jt}} = \boldsymbol{\sigma}_j + f_j(I_t) \boldsymbol{\sigma}_I,$$

where f_j is some strictly positive function of I_t specific to stock j . Consequently, we have

$$\boldsymbol{\sigma}_{S_{jt}}^T \boldsymbol{\sigma}_{S_{lt}} = \boldsymbol{\sigma}_j^T \boldsymbol{\sigma}_l + f_j \boldsymbol{\sigma}_j^T \boldsymbol{\sigma}_I + f_l \boldsymbol{\sigma}_l^T \boldsymbol{\sigma}_I + f_j f_l \|\boldsymbol{\sigma}_I\|^2 > 0, \quad (\text{A37})$$

proving the desired result. The correlation property regarding the nonindex stocks is obvious.

Q.E.D.

Appendix B: Generalization to Nonzero Dividend Growth and Interest Rate

In this appendix, we generalize our setup to additionally feature a nonzero growth rate for the stock dividend and a nonzero riskless rate for the bond. This setting turns out to be equally tractable, leading to closed-form expressions for all quantities, as demonstrated below. Importantly, however, all our previous conclusions and intuitions remain robust to this generalization.

The economic setup is as in Section 2.1, but now the stock market payoff (the “dividend”) D_T is the terminal value of the process D_t with dynamics

$$dD_t = D_t[\mu dt + \sigma d\omega_t], \quad (\text{B1})$$

where the growth rate μ and $\sigma > 0$ are constant. Consequently, D_t is lognormally distributed, as before. Moreover, the zero-net supply bond now pays a nonzero, riskless interest at a constant rate r . As becomes evident from the analysis below, when $\mu \neq 0$, the expressions in the text and the appendices remain the same, replacing D_t by $D'_t = e^{\mu(T-t)}D_t$ (and, consequently, replacing D_0 by $D'_0 = e^{\mu T}D_0$).

Given the dynamically complete market, there exists a state price density process, ξ , which is no longer a martingale and follows the modified dynamics

$$d\xi_t = -\xi_t r dt - \xi_t \kappa_t d\omega_t, \quad (\text{B2})$$

where $\kappa_t \equiv (\mu_{S_t} - r)/\sigma_{S_t}$ is the modified Sharpe ratio process. Accordingly, investor i 's dynamic budget constraint (3) can again be restated as

$$E_t[\xi_T W_{iT}] = \xi_t W_{it}. \quad (\text{B3})$$

We first determine the investors' optimal portfolios. Maximizing the institutional investor's objective function (5) subject to (B3) evaluated at time $t = 0$ leads to the institution's optimal terminal wealth as

$$W_{iT} = \frac{1 + bD_T}{y_I \xi_T},$$

where $1/y_I$ solves (B3) evaluated at $t = 0$, and with D_t lognormally distributed, we obtain

$$\frac{1}{y_I} = \frac{\lambda \xi_0 S_0}{1 + b e^{\mu T} D_0}.$$

Consequently, the institution's optimal terminal wealth is given by

$$W_{iT} = \frac{\lambda \xi_0 S_0}{\xi_T} \frac{1 + bD_T}{1 + b e^{\mu T} D_0}, \quad (\text{B4})$$

and from (B3) its optimal time- t wealth by

$$\xi_t W_{it} = \lambda \xi_0 S_0 \frac{1 + b e^{\mu(T-t)} D_t}{1 + b e^{\mu T} D_0}. \quad (\text{B5})$$

Applying Itô's lemma to both sides of (B5), and using (3) and (B2), leads to

$$\xi_t W_{it} (\phi_{I_t} \sigma_{S_t} - \kappa_t) d\omega_t = \lambda \xi_0 S_0 \frac{b e^{\mu(T-t)} D_t}{1 + b e^{\mu T} D_0} \sigma d\omega_t,$$

which after matching the diffusion terms and rearranging gives the institutional investor's optimal portfolio below. The retail investor's optimal portfolio is obtained similarly.

Lemma B1. *The institutional and retail investors' portfolios are given by*

$$\phi_{\mathcal{I}t} = \frac{\mu_{S_t} - r}{\sigma_{S_t}^2} + \frac{b e^{\mu(T-t)} D_t}{1 + b e^{\mu(T-t)} D_t} \frac{\sigma}{\sigma_{S_t}}, \quad (\text{B6})$$

$$\phi_{\mathcal{R}t} = \frac{\mu_{S_t} - r}{\sigma_{S_t}^2}. \quad (\text{B7})$$

As in Lemma 1, the institution demands a higher fraction of wealth in the stock market index than the retail investor, due to the hedging portfolio in (B6), and the same intuition holds.

We next turn to the equilibrium asset pricing implications of the presence of institutional investors. To determine the equilibrium state price density process ξ , we impose the market clearing condition $W_{\mathcal{R}T} + W_{\mathcal{I}T} = D_T$, and substitute (B4) and (A5) to obtain

$$\left(\lambda \frac{1 + b D_T}{1 + b e^{\mu T} D_0} + (1 - \lambda) \right) \frac{\xi_0 S_0}{\xi_T} = D_T, \quad (\text{B8})$$

which after rearranging leads to the equilibrium terminal state price density:

$$\xi_T = \frac{\xi_0 S_0}{1 + b e^{\mu T} D_0} \frac{1}{D_T} (1 + b e^{\mu T} D_0 + \lambda b (D_T - e^{\mu T} D_0)). \quad (\text{B9})$$

From (B2) we have

$$\begin{aligned} \xi_T &= \xi_t e^{-r(T-t) - \frac{1}{2} \int_t^T \kappa_s^2 ds - \int_t^T \kappa_s d\omega_s} \\ &= \xi_t e^{-r(T-t)} \eta_T / \eta_t, \end{aligned} \quad (\text{B10})$$

where η is the exponential martingale defined by $\eta_t = e^{-\frac{1}{2} \int_0^t \kappa_s^2 ds - \int_0^t \kappa_s d\omega_s}$. Taking expectations on both sides of (B10) leads to

$$E_t[\xi_T] = \xi_t e^{-r(T-t)}.$$

Consequently, the equilibrium state price density at time t is given by

$$\begin{aligned} \xi_t &= e^{r(T-t)} E_t[\xi_T] \\ &= \frac{\xi_0 S_0 e^{r(T-t)}}{1 + b e^{\mu T} D_0} E_t[1/D_T] (1 + b e^{\mu T} D_0 + \lambda b (1/E_t[1/D_T] - e^{\mu T} D_0)) \\ &= \frac{\xi_0 S_0}{1 + b e^{\mu T} D_0} \frac{e^{(r-\mu+\sigma^2)(T-t)}}{D_t} \left(1 + b e^{\mu T} D_0 + \lambda b \left(e^{(\mu-\sigma^2)(T-t)} D_t - e^{\mu T} D_0 \right) \right), \end{aligned} \quad (\text{B11})$$

where the last equality employs the fact that D_t is lognormally distributed. To determine the equilibrium stock market level, we substitute (B9)–(B11) into (A7) and manipulate to obtain

the stated expression (B12) below. The stock market level \bar{S} in the benchmark economy with no institutions follows by considering the special case of $b = 0$ in (B12).

Proposition B1. *In the economy with institutional investors, the equilibrium level of the stock market index is given by*

$$S_t = \bar{S}_t \frac{1 + b e^{\mu T} D_0 + \lambda b (e^{\mu(T-t)} D_t - e^{\mu T} D_0)}{1 + b e^{\mu T} D_0 + \lambda b (e^{(\mu - \sigma^2)(T-t)} D_t - e^{\mu T} D_0)}, \quad (\text{B12})$$

where \bar{S}_t is the equilibrium index level in the benchmark economy with no institutional investors given by

$$\bar{S}_t = e^{(\mu - r - \sigma^2)(T-t)} D_t.$$

As in Section 3.1, the stock market index level is increased in the presence of institutional investors, $S_t > \bar{S}_t$, with identical price pressure intuition.

To derive the stock market volatility, we write the equilibrium stock price in (B12) as

$$S_t = \bar{S}_t \frac{X_t}{Z_t}, \quad (\text{B13})$$

where

$$\begin{aligned} X_t &= 1 + b e^{\mu T} D_0 + \lambda b (e^{\mu(T-t)} D_t - e^{\mu T} D_0), \\ Z_t &= 1 + b e^{\mu T} D_0 + \lambda b (e^{(\mu - \sigma^2)(T-t)} D_t - e^{\mu T} D_0). \end{aligned}$$

Applying Itô's lemma to (B13) and following the same steps as in the proof of Proposition 2 in Appendix A we obtain the following.

Proposition B2. *In the equilibrium with institutional investors, the volatility of the stock market index returns is given by*

$$\sigma_{st} = \bar{\sigma}_{st} + \lambda b \sigma \frac{(1 - e^{-\sigma^2(T-t)}) (1 + (1 - \lambda) b e^{\mu T} D_0) e^{\mu(T-t)} D_t}{(1 + (1 - \lambda) b e^{\mu T} D_0 + \lambda b e^{(\mu - \sigma^2)(T-t)} D_t) (1 + (1 - \lambda) b e^{\mu T} D_0 + \lambda b e^{\mu(T-t)} D_t)},$$

where $\bar{\sigma}_{st}$ is the equilibrium index volatility in the benchmark economy with no institutions, given by

$$\bar{\sigma}_{st} = \sigma.$$

Consequently, the index volatility is increased in the presence of institutions, $\sigma_{st} > \bar{\sigma}_{st}$, as in the analysis of Section 3.1.

Finally, we determine the investors' equilibrium portfolios following identical steps as in the proof of Proposition 3 in Appendix A and obtain the following, with the same implications as in the analysis of Section 3.2.

Proposition B3. *The institutional and retail investors' portfolios in equilibrium in terms of shares in the stock index are given by*

$$\begin{aligned}\pi_{\mathcal{I}t} &= \lambda \frac{1 + b e^{\mu(T-t)} D_t}{1 + (1 - \lambda) b e^{\mu T} D_0 + \lambda b e^{\mu(T-t)} D_t} \\ &\quad \times \left(1 - \frac{\lambda b e^{\mu(T-t)} D_t}{1 + (1 - \lambda) b e^{\mu T} D_0 + \lambda b e^{\mu(T-t)} D_t} \frac{\sigma}{\sigma_{St}} + \frac{b e^{\mu(T-t)} D_t}{1 + b e^{\mu(T-t)} D_t} \frac{\sigma}{\sigma_{St}} \right), \\ \pi_{\mathcal{R}t} &= (1 - \lambda) \frac{1 + b e^{\mu T} D_0}{1 + (1 - \lambda) b e^{\mu T} D_0 + \lambda b e^{\mu(T-t)} D_t} \left(1 - \frac{\lambda b e^{\mu(T-t)} D_t}{1 + (1 - \lambda) b e^{\mu T} D_0 + \lambda b e^{\mu(T-t)} D_t} \frac{\sigma}{\sigma_{St}} \right),\end{aligned}$$

where σ_{St} is as in Proposition B2.

Consequently, the institutional investor is always levered, $W_{\mathcal{I}t}(1 - \phi_{\mathcal{I}t}) < 0$.

The results of Section 4 with multiple stocks generalize analogously, and all our economic insights obtained in that section remain exactly the same.

Appendix C: Stocks-Only Economy

This appendix presents a variant of our multi-stock economy in Section 3 in which there are only risky stocks available for trading and there is no riskless bond. Such a variant is perhaps more appropriate for modeling institutional investors for whom portfolios are typically restricted to a single asset class, e.g., equities, and do not normally involve leverage. We first extend our earlier analysis to such a stocks-only setting and show that our main implications presented in Section 3 remain valid. The main difference here is that instead of borrowing through the riskless bond to finance the additional demand for index stocks, the institutional investors reduce their positions in nonindex stocks to fund this additional demand. This model, however, is less tractable because unlike in Section 3, we have only been able to demonstrate such portfolio implications numerically.

The economic setup is as follows. The securities market is driven by N sources of risk represented by the N -dimensional Brownian motion $\boldsymbol{\omega} = (\omega_1, \dots, \omega_N)^T$, but now features $N + 1$ risky stocks and no riskless bond. As in Section 3, each stock is in positive net supply of one share and is a claim against a terminal payoff D_{jT} at time T . Each stock price, S_j , $j = 1, \dots, N + 1$, is then posited to have dynamics

$$dS_{jt} = S_{jt}[\mu_{S_{jt}} dt + \boldsymbol{\sigma}_{S_{jt}} d\boldsymbol{\omega}_t],$$

where the vector of stock mean returns $\boldsymbol{\mu}_S \equiv (\mu_{S_1}, \dots, \mu_{S_{N+1}})^T$ and the stock volatility matrix $\boldsymbol{\sigma}_S \equiv \{\sigma_{S_{j\ell}}, j = 1, \dots, N + 1, \ell = 1, \dots, N\}$, now with dimensions $(N + 1) \times 1$ and $(N + 1) \times N$, respectively, are determined in equilibrium. The stock market is again the sum of all the stocks in the economy with the terminal payoff $S_{MKT T} = D_T$, while the stock index is made up of the first M stocks with the terminal payoff $S_{IT} = I_T$. The primary difference here from the setup in Section 3 is the presence of the additional stock, $N + 1$, and the absence of the

riskless bond. By dropping the riskless bond, we are departing from the typical investment opportunity set featured in the canonical asset pricing model. We note, however, that in this dynamically complete-markets setting such a bond can be synthetically replicated using the $N + 1$ risky stocks.

We first examine the investors' optimal portfolios. In this stocks-only economy, each investor type $i = \mathcal{I}, \mathcal{R}$ now chooses an $(N+1)$ -dimensional portfolio process $\boldsymbol{\phi}_i \equiv (\phi_{i1}, \dots, \phi_{iN+1})^T$, where $\boldsymbol{\phi}_i$ denotes the portfolio weights in each risky stock. The investor's investment portfolio value W_i then follows the dynamics

$$dW_{it} = W_{it} \boldsymbol{\phi}_{it}^\top [\boldsymbol{\mu}_{st} dt + \boldsymbol{\sigma}_{st} d\boldsymbol{\omega}_t].$$

Following the analysis of Section 3, and particularly the same steps as in the proof of Lemma 2, we obtain the same equations (A24)–(A28) in determining the institution's optimal portfolio. In particular, we still have that

$$\boldsymbol{\sigma}_{st}^T \boldsymbol{\phi}_{\mathcal{I}t} = \boldsymbol{\kappa}_t + A_t \boldsymbol{\sigma}_I^T, \quad (\text{C1})$$

where $A_t \equiv bI_t/(1 + bI_t)$. The only difference now is that this is a system of N equations in $N + 1$ unknowns ($\boldsymbol{\phi}_{\mathcal{I}t}$). The last equation that is needed to determine the optimal portfolio process is that the portfolio weights add up to one:

$$\mathbf{1}^T \boldsymbol{\phi}_{\mathcal{I}t} = 1, \quad (\text{C2})$$

where $\mathbf{1}$ is an $(N + 1) \times 1$ vector of 1's. Equations (C1) and (C2) together fully determine the institution's portfolio.

To derive the analogue of Lemma 2, we define the following augmented volatility matrix and vectors:

$$\tilde{\boldsymbol{\sigma}}_{st} \equiv \begin{bmatrix} 1 \\ \boldsymbol{\sigma}_{st} \\ \vdots \\ 1 \end{bmatrix}, \quad \tilde{\boldsymbol{\kappa}}_t \equiv \begin{bmatrix} \boldsymbol{\kappa}_t \\ 1 \end{bmatrix}, \quad \tilde{\boldsymbol{\sigma}}_I \equiv \begin{bmatrix} \boldsymbol{\sigma}_I \\ 0 \end{bmatrix},$$

where we have added the $N + 1$ st column of $\mathbf{1}$ to the $\boldsymbol{\sigma}_{st}$ matrix, appended "1" to the Sharpe ratio vector $\boldsymbol{\kappa}$ and "0" to the $\boldsymbol{\sigma}_I$ vector. We then obtain from (C1) and (C2) that

$$\boldsymbol{\phi}_{\mathcal{I}t} = (\tilde{\boldsymbol{\sigma}}_{st}^T)^{-1} \tilde{\boldsymbol{\kappa}}_t + \frac{bI_t}{1 + bI_t} (\tilde{\boldsymbol{\sigma}}_{st}^T)^{-1} \tilde{\boldsymbol{\sigma}}_I. \quad (\text{C3})$$

The retail investor's optimal portfolio is determined by setting $b = 0$:

$$\boldsymbol{\phi}_{\mathcal{R}t} = (\tilde{\boldsymbol{\sigma}}_{st}^T)^{-1} \tilde{\boldsymbol{\kappa}}_t. \quad (\text{C4})$$

Substituting $\tilde{\boldsymbol{\mu}}_{st} \equiv (\tilde{\boldsymbol{\sigma}}_{st}^T)^{-1} \tilde{\boldsymbol{\kappa}}_t = \boldsymbol{\mu}_{st} + \mathbf{1}$, we obtain the following.

Lemma 2'. *The institutional and retail investors' optimal portfolio processes in the stocks-only economy are given by*

$$\begin{aligned}\Phi_{I_t} &= (\tilde{\sigma}_{St} \tilde{\sigma}_{St}^T)^{-1} \tilde{\kappa}_t + \frac{bI_t}{1 + bI_t} (\tilde{\sigma}_{St}^T)^{-1} \tilde{\sigma}_I, \\ \Phi_{R_t} &= (\tilde{\sigma}_{St} \tilde{\sigma}_{St}^T)^{-1} \tilde{\mu}_{St}.\end{aligned}$$

The structure of the optimal portfolios here closely resembles that presented in Lemma 2. The main difference is that the hedging portfolio of the institutional investor is not collinear with the index because of the last element of $\tilde{\sigma}_I$. This breaks the simple three-fund separation property that we have relied on in deriving our implications in Lemma 2. In particular, it is no longer the case that the hedging portfolio consists of index stocks only. It turns out that this portfolio has positive portfolio weights in index stocks and negative portfolio weights nonindex stocks. That is, the institution has a positive tilt in the index stocks and a negative tilt in nonindex stocks.¹⁰ We have not, however, been able to prove this implication analytically, unlike the implications in Lemma 2. Our numerical analysis, consistent with our intuitions, confirms that this implication is true for a wide range of parameters. We depict the typical institutional portfolio in Figure 7. We further note that the counterparty to the institutional investor, the retail investor, ends up tilting his portfolio in the opposite direction. One may reinterpret the retail investor in our model as another institution, but one that is not benchmarked to the same index—for example, a hedge fund, whose performance is evaluated relative to a different benchmark, can be the counterparty to our institutional investor.

The implications reported in Propositions 6 and 7 go through in the stocks-only setting.

Proposition 6'. *In the stocks-only economy with $N + 1$ risky stocks, the equilibrium prices of the market portfolio, index stocks $j = 1, \dots, M - 1$ and nonindex stocks $k = M + 1, \dots, N$ are the same as those reported in Proposition 6 (in equations (26)–(27)). Consequently, all properties reported in Proposition 6 remain valid.*

Proposition 7'. *In the stocks-only economy with $N + 1$ risky stocks, the equilibrium volatilities of the market portfolio, index stocks $j = 1, \dots, M - 1$, and nonindex stocks $k = M + 1, \dots, N$ are the same as those reported in Proposition 7 (in equations (29)–(31)). Consequently, properties (i) and (ii) of the equilibrium stock volatilities and correlations remain as in Proposition 7.*

We find differences in portfolio holdings from the analysis in Section 3, due to the fact that the market structure of the available securities has changed. However, all other primitives of the model, including the objective functions and the terminal payoffs of the stock market D_T and the stock index I_T , have remained the same. Therefore, the equilibrium valuation of index and nonindex stocks, as presented in Proposition 6, and their ensuing equilibrium volatilities and correlations, as presented in Proposition 7, remain exactly the same. Hence, our asset-pricing implications are unchanged in the stocks-only economy.

¹⁰The additional demand for index and nonindex relative to our earlier analysis is effectively demand for a portfolio replicating a riskless bond.

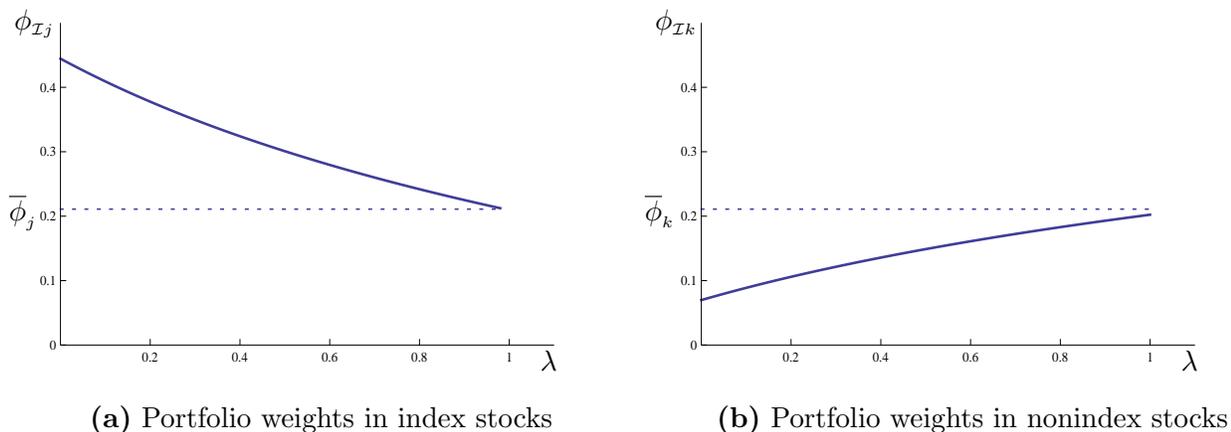


Figure 7: **The institutional investor’s portfolio weights.** Panels (a) and (b) of this figure plot the institution’s portfolio weights in an index stock j and a nonindex stock k , respectively, against the size of the institution λ . The lines for $\bar{\phi}$ correspond to the portfolio weights of an otherwise identical investor in the benchmark economy. The plots are typical. The remaining parameter values are as in Figure 6.

In the typical case of our numerical analysis, no investor in the model takes on leverage. In rare cases one can obtain a scenario in which the negative tilt in the nonindex stocks is so large that it counterbalances the positive weights of these stocks in the mean-variance portfolio and the overall portfolio weights of nonindex stocks are negative. Such portfolios, featuring leverage, resemble portfolios held by 130/30 mutual funds who are short in some stocks to finance the increase in their exposure to other stocks in their portfolio. While this is theoretically possible in our model, this situation rarely occurs in our numerical analysis.

Appendix D: An Agency Justification for Benchmarking

In this appendix, we provide possible microfoundations for the two key properties of our institutional investor’s objective function, namely that (i) it depends on the index level I_T and that (ii) the marginal utility of wealth is increasing in the index level I_T . The institutional investor can be thought of as an agent working for a principal, whom we have not explicitly specified in the body of the paper but will specify in this appendix. The agency problem is due to moral hazard.¹¹ The value of the managed portfolio is not observed perfectly by the principal, and the agent may take an unobservable action that reduces the portfolio value. For the former, see e.g., Getmansky, Lo and Makarov (2004) for evidence that hedge funds report returns that are smoother than true returns. For the latter see e.g., Lakonishok et al. (1991) for evidence on “window dressing” by money managers—managers engage in

¹¹A related recent work modelling portfolio delegation under moral hazard is Dybvig, Farnsworth and Carpenter (2010) who show that under certain conditions, benchmarking the manager against the index emerges as an optimal compensation contract.

unnecessary trades from the viewpoint of fund investors and incur unwarranted trading costs. See also Zitzewitz (2006) for evidence on value-destroying late trading activity by mutual funds, and Mahoney (2004) for an overview article on the hidden costs of investor-manager conflicts in mutual funds. The key property (i) of the institutional objective is relatively straightforward to obtain in this context. The argument that it may be beneficial for the principal to make the agent’s contract depend on an index dates back to Holmström (1979). Since the index I_T is correlated with the (unobserved) portfolio value, it adds valuable information. Therefore for different contingencies signalled by I_T , the agent should receive different remunerations. To support key property (ii), one needs to characterize the optimal contract. For a dynamic model with a continuous state space like ours, it is well-known that this task is highly complex. To make it manageable, we adopt the tractable contracting setup recently proposed by Edmans and Gabaix (2011), which imposes few restrictions on preferences and distributions.¹²

Following Edmans and Gabaix, we specify the institutional investor’s objective as

$$E [u(v(c_T) + g(p_T))], \tag{D1}$$

where $u(\cdot)$ is his utility function and $v(\cdot)$ is the felicity function that denotes the agent’s utility from the cash compensation c_T . The agent, the “manager,” reports at time T the value of his portfolio $\hat{W}_T \equiv (1 - p_T)W_T$, where $p_T \in (0, 1)$ is the fraction of portfolio value that gets diverted by the manager at time T (by engaging in value-destroying activities, an extreme version of which is stealing). The manager derives a private benefit $g(p_T)$ from this diversion activity. Only the agent observes the true state of the world, which in our setting is captured by the state price density ξ_T . The principal can thus offer the manager a compensation contract contingent on \hat{W}_T but not on ξ_T . We also allow the manager to include the index level I_T in the compensation contract, noting that I_T is correlated with ξ_T (although not perfectly, as in our setup of Section 3). The principal knows the distribution of ξ_T conditional on I_T .

Borrowing from Edmans and Gabaix, we make two important assumptions. First, the manager “takes action after noise”—i.e., decides on the diversion policy p_T after observing the state of the world ξ_T . Second, the principal wishes to implement the action $p_T = p^*$ —i.e., we do not solve for the optimal action but specify it exogenously and solve for the

¹²An alternative setup that one may employ is the moral-hazard-based relative performance model of Bolton and Dewatripont (2005, Chapter 8). Bolton and Dewatripont consider a setting with CARA agents and normally distributed shocks and focus on linear contracts. “Output” (portfolio value, in our context) produced by each agent is affected by a common shock. In that model, if individual outputs are not independent, it is optimal to make the contract contingent on the other agent’s output. Moreover, if the two outputs are positively correlated, the other agent’s output should enter negatively, so as to filter out the common shock. As also stressed by Holmström (1982), such a contract exposes the agent to less risk. It is easy to see from the analysis of Bolton and Dewatripont that the agent’s marginal utility (under the optimal contract) increases in the output of his rival. Bolton and Dewatripont’s model simplifies in our setting because instead of the rival’s output we have an exogenous benchmark. The contract obtained in that setting will satisfy all three properties stated in Proposition D1 below, providing a valuable robustness check for our key insights in this appendix.

contract that implements it. This assumption can be justified within a setting in which the marginal benefit to the principal of reducing diversion far exceeds the benefit of diversion to the manager, and so the level of diversion should be set equal to its lower bound p^* , as specified, e.g., by the trustees of the fund.

The manager chooses his optimal terminal wealth W_T and the action p_T to maximize (D1) subject to the following constraint:

$$E[\xi_T W_T] = W_0, \quad (\text{D2})$$

where $c_T = c(\hat{W}_T, I_T)$. This constraint is the budget constraint written in a static form (see equation (A2)), which allows us replace the problem of solving for the optimal portfolio ϕ by the (simpler) problem of solving for the optimal W_T as a function of the state variable ξ_T (Cox and Huang, 1989). It is then easy to recover the portfolio ϕ that implements the optimal W_T .

The formal solution to this problem can be found in Edmans and Gabaix. Here we provide a heuristic derivation assuming that the functions u , v and g satisfy all necessary regularity conditions. The first-order conditions to the manager's problem with respect to W_T and p_T , respectively, are:

$$u'(v(c_T) + g(p_T))v'(c_T)c_{\hat{W}}(\hat{W}_T, I_T)(1 - p_T) = y_M \xi_T, \quad (\text{D3})$$

$$u'(v(c_T) + g(p_T))(v'(c_T)c_{\hat{W}}(\hat{W}_T, I_T)(-W_T) + g'(p_T)) = 0, \quad (\text{D4})$$

where y_M is the Lagrange multiplier on the manager's static budget constraint (D2). From (D4) evaluated at $p_T = p^*$, we derive that

$$v'(c_T)c_{\hat{W}}(\hat{W}_T, I_T) = \frac{1 - p^*}{\hat{W}_T} g'(p^*),$$

which after integrating over \hat{W}_T yields the optimal contract of the form

$$c(\hat{W}_T, I_T) = v^{-1} \left((1 - p^*)g'(p^*) \log(\hat{W}_T) + K(I_T) \right). \quad (\text{D5})$$

The function $K(I_T)$ is chosen by the principal so as to maximize his expected utility subject to the manager's participation constraint $E[u(v(c_T) + g(p_T))] \geq \underline{u}$, where \underline{u} is the manager's reservation utility. This is the same contract as in Edmans and Gabaix (see especially their Appendix C).

The principal is a fund investor, who is left unmodelled in the body of the paper. This investor delegates all his money to the manager. For simplicity, we assume that this investor is risk-neutral (this assumption can be relaxed in future work). He chooses the function $K(I_T)$ to minimize the expected cost of the contract in (D5) subject to the manager's participation constraint:

$$\min_{K(\cdot)} E v^{-1} \left((1 - p^*)g'(p^*) \log(\hat{W}_T) + K(I_T) \right) \quad (\text{D6})$$

$$\text{s.t. } E \left[u \left((1 - p^*)g'(p^*) \log(\hat{W}_T) + K(I_T) + g(p^*) \right) \right] \geq \underline{u}. \quad (\text{D7})$$

We now specialize the manager's preferences to $u(x) = e^{(1-\gamma)x}/(1-\gamma)$, $\gamma > 1$, and $v(x) = \log x$. This specification has been adopted by, e.g., Edmans and Gabaix (2011) and Edmans et al. (2012). Under this specification, one can use equations (D3) and (D2) to compute the optimal terminal wealth of the agent (after diversion) in closed form:

$$\hat{W}_T = \xi_T^{-\frac{1}{1-Z_2}} \frac{W_0(1-p^*)e^{M_2 K(I_T)}}{E[\xi_T^R e^{M_2 K(I_T)}]}$$

and reduce the principal's problem (D6)–(D7) to

$$\min_{K(\cdot)} E \left[e^{M_1 K(I_T)} \xi_T^{R/(1-\gamma)} \right] \left(E \left[e^{M_2 K(I_T)} \xi_T^R \right] \right)^{Z_1} \quad (\text{D8})$$

$$\text{s.t. } E \left[e^{M_1 K(I_T)} \xi_T^R \right] \geq Q \left(E \left[e^{M_2 K(I_T)} \xi_T^R \right] \right)^{Z_2}, \quad (\text{D9})$$

where

$$Z_1 = -(1-p^*)g'(p^*), \quad Z_2 = (1-\gamma)(1-p^*)g'(p^*), \quad Q = \frac{(1-\gamma)u}{(W_0(1-p^*))^{Z_2} e^{g(p^*)}},$$

$$M_1 = \frac{1-\gamma(1-\gamma)(1-p^*)g'(p^*)}{1-(1-\gamma)(1-p^*)g'(p^*)}, \quad M_2 = \frac{1-\gamma}{1-(1-\gamma)(1-p^*)g'(p^*)}, \quad R = Z_1 M_2.$$

Denote the optimal function $K(I_T)$ by $\bar{K}(I_T)$ and consider a small perturbation $\varepsilon L(I_T)$, where ε is a scalar and $L(I_T)$ is a function of I_T . Replacing $K(I_T)$ by $\bar{K}(I_T) + \varepsilon L(I_T)$ in the principal's problem (D8)–(D9) and considering only terms of up to order 1 in ε , we arrive at

$$\min_{L(I_T)} E \left[e^{M_1 \bar{K}(I_T)} \xi_T^{R/(1-\gamma)} \right] \left(E \left[e^{M_2 \bar{K}(I_T)} \xi_T^R \right] \right)^{Z_1} + E \left[\left(Y_1 e^{M_1 \bar{K}(I_T)} \xi_T^{R/(1-\gamma)} + Y_2 e^{M_2 \bar{K}(I_T)} \xi_T^R \right) \varepsilon L(I_T) \right] \quad (\text{D10})$$

s.t.

$$E \left[e^{M_1 \bar{K}(I_T)} \xi_T^R \right] - Q \left(E \left[e^{M_2 \bar{K}(I_T)} \xi_T^R \right] \right)^{Z_2} + E \left[M_1 e^{M_1 \bar{K}(I_T)} \xi_T^R \varepsilon L(I_T) \right] \geq E \left[Y_3 e^{M_2 \bar{K}(I_T)} \xi_T^R \varepsilon L(I_T) \right], \quad (\text{D11})$$

where

$$Y_1 = \left(E \left[e^{M_2 \bar{K}(I_T)} \xi_T^R \right] \right)^{Z_1} M_1, \quad Y_2 = Z_1 M_2 E \left[e^{M_1 \bar{K}(I_T)} \xi_T^{R/(1-\gamma)} \right] \left(E \left[e^{M_2 \bar{K}(I_T)} \xi_T^R \right] \right)^{Z_1-1},$$

$$Y_3 = Q Z_2 M_2 \left(E \left[e^{M_2 \bar{K}(I_T)} \xi_T^R \right] \right)^{Z_2-1}.$$

The terms in $\bar{K}(I_T)$ can be dropped from the objective because the maximization is with respect to L_T . They can be dropped from the constraint because it is satisfied for $\bar{K}(I_T)$. Hence, the principal's problem reduces to

$$\min_{L(I_T)} E \left[\left(Y_1 e^{M_1 \bar{K}(I_T)} \xi_T^{R/(1-\gamma)} + Y_2 e^{M_2 \bar{K}(I_T)} \xi_T^R \right) L(I_T) \right] \quad (\text{D12})$$

$$\text{s.t. } E \left[\left(M_1 e^{M_1 \bar{K}(I_T)} \xi_T^R - Y_3 e^{M_2 \bar{K}(I_T)} \xi_T^R \right) L(I_T) \right] \geq 0. \quad (\text{D13})$$

Since $\bar{K}(I_T)$ is optimal, this minimum must be zero. Therefore, the terms in the parentheses of the objective (D12) and of the constraint (D13) have to be the same, state-by-state, up to a multiplicative constant. Otherwise, it would be possible to find a function $L(I_T)$ which renders the objective negative without violating the constraint. We therefore have

$$\begin{aligned} Y_1 e^{M_1 \bar{K}(I_T)} E \left[\xi_T^{R/(1-\gamma)} \middle| I_T \right] + Y_2 e^{M_2 \bar{K}(I_T)} E \left[\xi_T^R \middle| I_T \right] \\ = \psi \left(M_1 e^{M_1 \bar{K}(I_T)} E \left[\xi_T^R \middle| I_T \right] - Y_3 e^{M_2 \bar{K}(I_T)} E \left[\xi_T^R \middle| I_T \right] \right), \end{aligned}$$

where the expectations are conditional on I_T because the principal observes I_T but not ξ_T , and ψ is a constant. Solving this equation for $\bar{K}(I_T)$ and simplifying, we obtain

$$\bar{K}(I_T) = \frac{1}{\gamma} \log \left(\frac{E \left[\xi_T^R \middle| I_T \right] (\psi Y_3 - Y_2)}{\psi M_1 E \left[\xi_T^R \middle| I_T \right] + Y_1 E \left[\xi_T^{R/(1-\gamma)} \middle| I_T \right]} \right), \quad (\text{D14})$$

where the constant ψ is such that the participation constraint (D9) binds with equality. Equation (D14) reveals that if the index I_T and the state price density ξ_T are independent, $\bar{K}(I_T)$ is a constant and hence the optimal contract does not depend on the index I_T . In the language of Holmström (1979), in that case the signal I_T does not carry any valuable information about the state of the world ξ_T . Only if the two random variables are correlated, the signal becomes valuable, and so it is beneficial for the principal to include I_T in the compensation contract.

To derive the relevant properties of the optimal contract, we need to evaluate the conditional expectations in (D14). Towards this, we impose distributional assumptions on the processes ξ and I . We note that the results derived below hold under milder assumptions, but at the expense of expositional clarity. We assume that

$$dI_t = \sigma_I I_t d\omega_{1t}, \quad (\text{D15})$$

$$d\xi_t = -\kappa_1 \xi_t d\omega_{1t} - \kappa_2 \xi_t d\omega_{2t}, \quad (\text{D16})$$

where the Brownian motions ω_1 and ω_2 are independent and where $\kappa_1, \kappa_2 \geq 0$ are constant. As in Section 3, the index cash flow news process I_t loads on a subset of Brownian motions driving the economy, while the state price density process ξ_t loads on all of them. By observing I_T the principal learns of the realization of ω_{1T} but not of ω_{2T} . We can now compute the conditional expectations in (D14) as follows:

$$\begin{aligned} E \left[\xi_T^R \middle| I_T \right] &= \xi_0^R e^{\left(\frac{(R\kappa_2)^2}{2} - R \frac{\|\kappa\|^2}{2} \right) T} e^{-R\kappa_1 f(I_T)} \\ E \left[\xi_T^{R/(1-\gamma)} \middle| I_T \right] &= \xi_0^{R/(1-\gamma)} e^{\left(\frac{(R/(1-\gamma)\kappa_2)^2}{2} - \frac{R}{1-\gamma} \frac{\|\kappa\|^2}{2} \right) T} e^{-\frac{R}{1-\gamma} \kappa_1 f(I_T)}, \end{aligned}$$

where $f(I_T) = (\log(I_T/I_0) + \sigma_I^2 T/2) / \sigma_I$ and $\kappa = (\kappa_1, \kappa_2)$. It follows from our definitions that $R < 0$ and $\gamma > 1$. It is then straightforward to show that

$$\frac{\partial E[\xi_T^R | I_T]}{\partial I_T} < 0 \quad \text{and} \quad \frac{\partial E[\xi_T^{R/(1-\gamma)} | I_T]}{\partial I_T} > 0.$$

The economic intuition for these results is that good states of the world (low ξ_T states) are more likely to occur when the index I_T is high.

Under our assumptions, the (indirect) utility of the manager is given by

$$u_{\mathcal{I}}(\hat{W}_T, I_T) = \frac{e^{(1-\gamma)((1-p^*)g'(p^*) \log \hat{W}_T + \bar{K}(I_T) + g(p^*))}}{1 - \gamma},$$

where we have substituted the optimal contract. Taking the pertinent derivatives and signing them, one can prove the following result.

Proposition D1. *As long as the index I_T and the state price density ξ_T are correlated ($\kappa_1 \neq 0$),*

- (i) *the manager's optimal compensation contract is contingent on I_T ;*
- (ii) *the manager's compensation decreases in the level of the index $\left(\frac{\partial c(\hat{W}_T, I_T)}{\partial I_T} < 0\right)$;*
- (iii) *the marginal utility of wealth of the manager increases in the level of the index $\left(\frac{\partial^2 u_{\mathcal{I}}(\hat{W}_T, I_T)}{\partial \hat{W}_T \partial I_T} > 0\right)$.*

In our discussion of the optimal contract we have already highlighted property (i) of Proposition D1. This property does not rely on any distributional assumptions on ξ_T and I_T . The intuition for property (ii) can be adapted from Holmström (1979): the manager should not be excessively penalized for poor performance (beyond optimal risk sharing) if his index has also performed poorly; on the contrary, if the index has done well, the manager's poor performance could be an indication of a high level of cash flow diversion, and so the manager should be penalized. Note that the objective function in the body of the paper does not satisfy property (ii). However, as discussed in Remark 1, the institutional investor's objective function can be made decreasing in I_T without any change to our results (by, e.g., subtracting from it a sufficiently increasing function of I_T). Since the contract explicitly penalizes the manager for underperformance relative to his index, his marginal utility of wealth is especially high in the states in which the index has done well. This intuition is formalized in property (iii) of Proposition D1. Property (iii) plays an important role in our results reported in the body of the paper. As our analysis in this appendix demonstrates, an agency problem in money management is a channel through which it may arise.

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