BARGAINING IN STATIONARY NETWORKS
ONLINE APPENDIX

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1. Stable Networks

In our model the network structure is exogenously given and we do not study network formation. Nonetheless, we may ask whether patient players can benefit in the bargaining game from forming new links or severing existing ones. The algorithm $\mathcal{A}(G)$ can be used to address this question. Fix the set of players N, and let $\mathcal{G}$ be the set of networks G with vertex set equal to N. A **payoff function** u assigns to each player $i \in N$ a payoff, denoted $u_i(G)$, for every network $G \in \mathcal{G}$. If $v_i^{*\delta}(G)$ and $v_i^*(G)$ denote the equilibrium payoff of player i in the bargaining game on the network G (with the uniform matching technology) for the common discount factor δ, and respectively its limit as $\delta \to 1$, then the profiles $(v_i^{*\delta}(G))_{i \in N, G \in \mathcal{G}}$ and $(v_i^*(G))_{i \in N, G \in \mathcal{G}}$ define payoff functions. For every network G and any $i \neq j \in N$, let $G + ij$ ($G - ij$) denote the network obtained by adding (deleting) the link ij to (from) G.

Definition 1 (Stability)**.** A network G is **unilaterally stable** with respect to the payoff function u if $u_i(G) \geq u_i(G - ij)$ for all $ij \in G$. A network G is **pairwise stable** with respect to the payoff function u if it is unilaterally stable with respect to u, and for all $ij \notin G$, $u_i(G + ij) > u_i(G)$ only if $u_j(G + ij) < u_j(G)$.

To rephrase, a network is unilaterally stable if no player benefits from severing one of his links. Pairwise stability requires additionally that no pair of players benefit from forming a new link. Matthew O. Jackson and Asher Wolinsky (1996) motivate the definitions by the fact that the formation of the link ij necessitates the consent of both players i and j, but its severance can be done unilaterally by either i or j.

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Theorem 1. *(i) Every network is unilaterally stable with respect to* $(v_i^*(G))_{i\in N, G\in\mathcal{G}}$. *(ii) A network is pairwise stable with respect to* $(v_i^*(G))_{i\in N, G\in\mathcal{G}}$ *if and only if it is equitable.*

Part (i) of the statement is not surprising, but its proof is involved, as the removal of a single link may create a chain effect in the procedure for determining limit equilibrium payoffs and completely change the decomposition of the network into oligopoly subnetworks.

Proof of Theorem 1. (i) Let $\tilde{G}$ be a network with $ij \in \tilde{G}$, and let $G = \tilde{G} - ij$ be the network obtained by deleting the link ij from $\tilde{G}$. Denote by $\mathcal{A}(G) = (r_s, x_s, M_s, L_s, N_s, G_s)_{s=1,2,\ldots,\overline{s}}$ and $\mathcal{A}(\tilde{G}) = (\tilde{r}_{\tilde{s}}, \tilde{M}_{\tilde{s}}, \tilde{L}_{\tilde{s}}, \tilde{N}_{\tilde{s}}, \tilde{G}_{\tilde{s}})_{\tilde{s}=1,2,\ldots,\tilde{\overline{s}}}$ the outcomes of the algorithm for computing the limit equilibrium payoffs for the bargaining games on the networks G and $\tilde{G}$, respectively. Let $s(k)$ and $\tilde{s}(k)$ denote the steps at which player k is removed in the algorithms $\mathcal{A}(G)$ and respectively $\mathcal{A}(\tilde{G})$, i.e., $s(k) = \max\{s | k \in N_s\}, \tilde{s}(k) = \max\{\tilde{s} | k \in \tilde{N}_{\tilde{s}}\}$.

Without loss of generality, we may assume that $s(i) \leq s(j)$ and set out to prove both of the following inequalities,

$$v_i^*(\tilde{G}) \geq v_i^*(G) \text{ and } v_j^*(\tilde{G}) \geq v_j^*(G).$$

It can be easily shown that if $i \in L_{s(i)}$ or $s(i) = \overline{s}$ then $\mathcal{A}(G)$ and $\mathcal{A}(\tilde{G})$ lead to identical outcomes. Therefore, we may assume that $i \in M_{s(i)}$ and $s(i) < \overline{s}$. In particular, $r_{s(i)} < 1$.

Note that the outcomes of the algorithms $\mathcal{A}(G)$ and $\mathcal{A}(\tilde{G})$ are identical for steps $1, \ldots, s(i)-1$ and $\tilde{r}_{s(i)} \geq r_{s(i)}$. Since $i \in M_{s(i)}, r_{s(i)} < 1$ and $\tilde{r}_{\tilde{s}} \geq r_{s(i)}$ for $\tilde{s} \geq s(i)$, it must be that $v_k^*(\tilde{G}) \geq r_{s(i)}/(1 + r_{s(i)}) = v_i^*(G)$ for all $k \in N_{s(i)} = \tilde{N}_{s(i)}$. Hence $v_i^*(\tilde{G}) \geq v_i^*(G)$.

We next show that $v_j^*(\tilde{G}) \geq v_j^*(G)$. There are three cases to consider: $j \in L_{s(j)}, j \in M_{s(j)}$, and $s(j) = \overline{s}$. We only solve the former case; the other two can be handled by similar methods.

Henceforth we focus on the case $i \in M_{s(i)}, j \in L_{s(j)}$. Then $s(j) < \overline{s}$ and $r_{s(j)} < 1$. The following lemma will be used repeatedly.

Lemma 1. *Suppose that* $(r_s, x_s, M_s, L_s, N_s, G_s)_{s=1,2,\ldots,\overline{s}}$ *is the outcome of the algorithm* $\mathcal{A}(G)$. *For any* $s < \overline{s}$, *and any non-empty* $L' \subset L_s$,

$$\frac{|L'|}{|L^{G_s}(L') \cap M_s|} \leq r_s.$$

Proof. Let $M' = L^{G_s}(L') \cap M_s$. Then $L_s = L^{G_s}(M_s)$ and $L^{G_s}(L') \cap (M_s \setminus M') = \emptyset$ imply that $L^{G_s}(M_s \setminus M') \subset L_s \setminus L'$. Since

$$r_s = \min_{M \subset N_s,\ M \in \mathcal{I}(G)} |L^{G_s}(M)|/|M|,$$

it follows that $|L_s \setminus L'|/|M_s \setminus M'| \geq r_s = |L_s|/|M_s|$. Hence $|L'|/|M'| \leq |L_s|/|M_s| = r_s$. □

For $\tilde{s} = s(i) - 1, s(i), \ldots, \tilde{s}(j)$ we show that

(1) $\tilde{r}_{\tilde{s}} \leq r_{s(j)}$
(2) $\tilde{M}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} L_s) = \emptyset$
(3) $\tilde{L}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} M_s) = \emptyset$

by induction on $\tilde{s}$. The induction base case, $\tilde{s} = s(i) - 1$, follows trivially. Suppose we proved the three assertions for all lower values, and we proceed to proving them for $\tilde{s}$. Each of the parts below establishes the corresponding assertion.

Part 1. We prove the first part of the induction step, $\tilde{r}_{\tilde{s}} \leq r_{s(j)}$. Let $\tilde{M} = \tilde{M}_{s(i)} \cup \ldots \cup \tilde{M}_{\tilde{s}-1}$ and $\tilde{L} = L^{\tilde{G}_{s(i)}}(\tilde{M}) = \tilde{L}_{s(i)} \cup \ldots \cup \tilde{L}_{\tilde{s}-1}$. Note that $\tilde{L} = L^{\tilde{G}_{s(i)}}(\tilde{M}) = L^{G_{s(i)}}(\tilde{M})$ since $i, j \notin \tilde{M}$ (if $i \in \tilde{M}$ then $j \in \tilde{L}$, a contradiction with $\tilde{s} - 1 < \tilde{s}(j)$; $j \in \tilde{M}$ leads to a similar contradiction).

Note that $\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}$ is non-empty as it contains i, and is contained in $\tilde{N}_{\tilde{s}}$ since by the induction hypothesis, $\tilde{L}_{\tilde{s}'} \cap (\cup_{s=s(i)}^{s(j)} M_s) = \emptyset$ for $\tilde{s}' < \tilde{s}$. Thus it is sufficient to prove that

$$\frac{|L^{\tilde{G}_{\tilde{s}}}(\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M})|}{|\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}|} \leq r_{s(j)}.$$

Indeed, $\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}$ is G-independent (Lemma 5), and also $\tilde{G}$-independent ($\tilde{G} = G + ij, i \in M_{s(i)}, j \in L_{s(j)}$), so the inequality above implies that $\tilde{r}_{\tilde{s}} \leq r_{s(j)}$.

Fix $s \in \overline{s(i), s(j)}$. Let $L' = L_s \setminus L^{G_s}(M_s \cap \tilde{M})$. Note that $L^{G_s}(L') \cap M_s \subset M_s \setminus \tilde{M}$. Lemma 1 applied to step s of $\mathcal{A}(G)$ with L' defined above implies that[1]

$$\frac{|L_s| - |L^{G_s}(M_s \cap \tilde{M})|}{|M_s \setminus \tilde{M}|} \leq r_s.$$

As $L^{G_s}(M_s \cap \tilde{M}) \subset L^{G_{s(i)}}(\tilde{M}) \cap L_s$, it follows that

$$\frac{|L_s| - |L^{G_{s(i)}}(\tilde{M}) \cap L_s|}{|M_s \setminus \tilde{M}|} \leq r_s.$$

[1]The argument is only necessary and relevant when $M_s \cap \tilde{M} \neq \emptyset, M_s$.

Since $r_s \le r_{s(j)}$ for all $s \in \overline{s(i), s(j)}$, the set of inequalities above imply that

$$\frac{\sum_{s=s(i)}^{s(j)}(|L_s| - |L^{G_{s(i)}}(\tilde{M}) \cap L_s|)}{\sum_{s=s(i)}^{s(j)} |M_s \setminus \tilde{M}|} \le r_{s(j)},$$

or equivalently,

$$\frac{|\cup_{s=s(i)}^{s(j)} L_s| - |L^{G_{s(i)}}(\tilde{M}) \cap (\cup_{s=s(i)}^{s(j)} L_s)|}{|\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}|} \le r_{s(j)}.$$

The letter inequality can be rewritten as

$$\frac{|\cup_{s=s(i)}^{s(j)} L_s \setminus L^{G_{s(i)}}(\tilde{M})|}{|\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}|} \le r_{s(j)}.$$

But $L^{\tilde{G}_{\tilde{s}}}(\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}) \subset L^{\tilde{G}_{s(i)}}(\cup_{s=s(i)}^{s(j)} M_s) = L^{G_{s(i)}}(\cup_{s=s(i)}^{s(j)} M_s) = \cup_{s=s(i)}^{s(j)} L_s$ (the first equality follows from $\tilde{G} = G + ij, i \in M_{s(i)}, j \in L_{s(j)}$), and $\tilde{G}_{\tilde{s}}$ does not contain any players in $\tilde{L} = L^{G_{s(i)}}(\tilde{M})$. Then $L^{\tilde{G}_{\tilde{s}}}(\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}) \subset \cup_{s=s(i)}^{s(j)} L_s \setminus L^{G_{s(i)}}(\tilde{M})$, and the inequality above implies that

$$\frac{|L^{\tilde{G}_{\tilde{s}}}(\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M})|}{|\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}|} \le r_{s(j)},$$

as desired.

Part 2. We prove the second part of the induction step, $\tilde{M}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} L_s) = \emptyset$, by contradiction. Suppose that $\tilde{M}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} L_s) \neq \emptyset$, and let s_0 be the smallest index $s \in \overline{s(i), s(j)}$ for which $\tilde{M}_{\tilde{s}} \cap L_s \neq \emptyset$. Define $B = \tilde{M}_{\tilde{s}} \cap L_{s_0}$ and $A = L^{G_{s_0}}(B) \cap M_{s_0}$.

We argue that $A \subset \tilde{N}_{\tilde{s}}$. Fix $k \in A$. Player k has a G-link to a player $l \in B$. If k is removed at step $\tilde{s}' < \tilde{s}$ in the algorithm $\mathcal{A}(\tilde{G})$ then $k \in \tilde{M}_{\tilde{s}'}$ by the induction hypothesis ($\tilde{L}_{\tilde{s}'} \cap (\cup_{s=s(i)}^{s(j)} M_s) = \emptyset$). Then $l \in \tilde{L}_{\tilde{s}'}$ or $l \notin \tilde{N}_{\tilde{s}'}$, contradicting that $l \in \tilde{M}_{\tilde{s}}$. Therefore, $k \in \tilde{N}_{\tilde{s}}$.

Note that $L^{\tilde{G}_{\tilde{s}}}(A) \cap \tilde{M}_{\tilde{s}} \subset B \cup \{j\}$ since players in $A \subset M_{s_0}$ may only have G-links to players in $L_1 \cup L_2 \cup \ldots \cup L_{s_0}$ (Lemma 5), and $\tilde{M}_{\tilde{s}} \cap (L_1 \cup L_2 \cup \ldots \cup L_{s_0}) = B$ by the definition of s_0. If $i \in A$ then we could have $j \in L^{\tilde{G}_{\tilde{s}}}(A) \cap \tilde{M}_{\tilde{s}}$. Lemma 1 applied for step $\tilde{s}$ of $\mathcal{A}(\tilde{G})$ with $L' = A$ and Part 1 imply that

$$\frac{|A|}{|B| + 1} \le \tilde{r}_{\tilde{s}} \le r_{s(j)} < 1.$$

Hence $|A| < |B| + 1$, or $|A| \le |B|$.

Since $A = L^{G_{s_0}}(B) \cap M_{s_0}$, Lemma 1 applied to step s_0 of $\mathcal{A}(G)$ with $L' = B$ implies that

$$\frac{|B|}{|A|} \leq r_{s_0} \leq r_{s(j)} < 1.$$

Hence $|A| > |B|$, a contradiction with $|A| \leq |B|$. Therefore $\tilde{M}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} L_s) = \emptyset$.

Part 3. To establish the third part of the induction hypothesis, $\tilde{L}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} M_s) = \emptyset$, we proceed by contradiction. Suppose that $k \in \tilde{L}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} M_s)$. It should be that k has a $\tilde{G}_{\tilde{s}}$ link to a player $l \in \tilde{M}_{\tilde{s}}$. By Lemma 5, since $\tilde{G}_{\tilde{s}}$ is a subnetwork of $G_{s(i)}$, $k \in \cup_{s=s(i)}^{s(j)} M_s$ may only have $\tilde{G}_{\tilde{s}}$ links to players in $\cup_{s=s(i)}^{s(j)} L_s$, so $l \in \cup_{s=s(i)}^{s(j)} L_s$. Therefore, $l \in \tilde{M}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} L_s)$, a contradiction with Part 2.

In particular, for $\tilde{s} = \tilde{s}(j)$ the induction hypothesis implies that $j \in \tilde{L}_{\tilde{s}(j)}$ and $\tilde{r}_{\tilde{s}(j)} \leq r_{s(j)}$. Then $v_j^*(\tilde{G}) = 1/(1 + \tilde{r}_{\tilde{s}(j)}) \geq 1/(1 + r_{s(j)}) = v_j^*(G)$.

(ii) To prove the "if" part of the statement, let $\tilde{G}$ be an equitable network. Part (i) shows that $\tilde{G}$ is unilaterally stable with respect to $(v_i^*(G))_{i \in N, G \in \mathcal{G}}$. Note that by Theorem 5 when a link is added to an equitable network another equitable network obtains. Hence $\tilde{G} + ij$ is equitable, and $v_i^*(\tilde{G} + ij) = v_i^*(\tilde{G}) = 1/2$ for all $i \neq j \in N$. Therefore, $\tilde{G}$ is pairwise stable with respect to $(v_i^*(G))_{i \in N, G \in \mathcal{G}}$.

To prove the "only if" part of the statement, let $\tilde{G}$ be a network that is pairwise stable with respect to $(v_i^*(G))_{i \in N, G \in \mathcal{G}}$. Suppose that $\tilde{G}$ is not equitable. Let $(\tilde{r}_{\tilde{s}}, \tilde{M}_{\tilde{s}}, \tilde{L}_{\tilde{s}}, \tilde{N}_{\tilde{s}}, \tilde{G}_{\tilde{s}})_{\tilde{s}}$ denote the outcome of the algorithm $\mathcal{A}(\tilde{G})$. Then there exist $i, j \in \tilde{M}_1$ such that $v_i^*(\tilde{G}) = v_j^*(\tilde{G}) < 1/2$ ($|\tilde{M}_1| \geq 2$). The limit equilibrium payoffs of players i and j in the game on the network $\tilde{G} + ij$ satisfy $v_i^*(\tilde{G} + ij) \geq v_i^*(\tilde{G})$ and $v_j^*(\tilde{G} + ij) \geq v_j^*(\tilde{G})$ by part (i) of the theorem. By Proposition 2, $v_i^*(\tilde{G} + ij) + v_j^*(\tilde{G} + ij) \geq 1$. Hence, $v_i^*(\tilde{G} + ij) + v_j^*(\tilde{G} + ij) > v_i^*(\tilde{G}) + v_j^*(\tilde{G})$, which together with $v_i^*(\tilde{G} + ij) \geq v_i^*(\tilde{G})$ and $v_j^*(\tilde{G} + ij) \geq v_j^*(\tilde{G})$, leads to a violation of the pairwise stability of $\tilde{G}$.[2] The contradiction proves that $\tilde{G}$ is equitable. □

While every network is unilaterally stabile with respect to the equilibrium payoffs in the limit as players become patient, the conclusion does not necessarily apply before taking the limit. Indeed, not every network is unilaterally stable with respect to $(v_i^{*\delta}(G))_{i \in N, G \in \mathcal{G}}$ for $\delta < 1$. Consider the network G_2 from Figure 2 in the paper. Note that G_2^* is obtained

[2] It can be shown that the latter two inequalities hold strictly, which is necessary for the proof of the subsequent Corollary 1.

from G_2 by removing the link $(1,5)$. For every discount factor $\delta \in (10(9-\sqrt{2})/79, 1)$, the network G_2^* leads to higher equilibrium payoffs for players 1 and 5 than the network G_2 in the bargaining game,

$$v_1^{*\delta}(G_2) = \frac{2}{10-7\delta} < \frac{2}{8-5\delta} = v_1^{*\delta}(G_2^*) \text{ and } v_5^{*\delta}(G_2) = \frac{1}{2(5-4\delta)} < \frac{1}{2(4-3\delta)} = v_5^{*\delta}(G_2^*).$$

Thus both players 1 and 5 benefit from removing the link connecting them and prefer playing the game on G_2^* rather than G_2.

The intuition for this observation is simple. For the range of discount factors considered, $(1,5)$ is an equilibrium disagreement link in the bargaining game on G_2, whence it becomes a source of delay for the possible agreements and deflates the equilibrium payoffs of all players. However, the gains to players 1 and 5 from severing the link $(1,5)$ vanish as δ approaches 1. If players only consider deleting or adding links when the ensuing gains are significant, we need to focus on **approximate stability**.

Definition 2 (ε-Stability)**.** A network G is **unilaterally ε-stable** with respect to the payoff function u if $u_i(G) + \varepsilon \geq u_i(G - ij)$ for all $ij \in G$. A network G is **pairwise ε-stable** with respect to the payoff function u if it is unilaterally ε-stable with respect to u, and for all $ij \notin G$, $u_i(G+ij) > u_i(G) + \varepsilon$ only if $u_j(G+ij) < u_j(G) + \varepsilon$.

For any sufficiently low $\varepsilon > 0$, there exists a discount factor threshold $\underline{\delta} < 1$ such that the two statements of Theorem 1 also hold for ε-stability with respect to the equilibrium payoffs for any $\delta > \underline{\delta}$. The next result is based on ideas from the proofs of Theorem 1 above and Theorem 4 from the paper.

Corollary 1. *There exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon < \bar{\varepsilon}$ there exists $\underline{\delta} < 1$ such that the following statements are true for all $\delta > \underline{\delta}$. (i) Every network is unilaterally ε-stable with respect to $(v_i^{*\delta}(G))_{i \in N, G \in \mathcal{G}}$. (ii) A network is pairwise ε-stable with respect to $(v_i^{*\delta}(G))_{i \in N, G \in \mathcal{G}}$ if and only if it is equitable.*

In the context of buyer-seller networks the definition of pairwise stability should account for the fact that only buyer-seller pairs may consider forming new links.

Definition 3 (Buyer-seller stability)**.** A buyer-seller network G is **two-sided pairwise stable** with respect to the payoff function u if it is unilaterally stable with respect to u, and for all $(i,j) \in (B \times S) \cup (S \times B)$, $u_i(G+ij) > u_i(G)$ only if $u_j(G+ij) < u_j(G)$.

The next result is the analogue of Theorem 1.ii.

Theorem 1.ii$^{\mathbf{BS}}$. *A buyer-seller network is two-sided pairwise stable with respect to $(v_i^*(G))_{i\in N, G\in\mathcal{G}}$ if and only if it is non-discriminatory.*

2. Heterogeneous Discount Factors

In the paper all players are assumed to have the same discount factor. We can extend the results to the case of heterogeneous discount factors, where the players of type i share a discount factor δ_i. The accumulation points of the equilibrium payoffs and agreement networks along a $(\delta_1, \delta_2, \ldots, \delta_n)$ sequence that converges to $(1, 1, \ldots, 1)$ depend on the choice of the sequence.[3] One condition that guarantees convergence of the equilibrium payoffs and agreement network is that the relative rates of convergence of δ_i and δ_j to 1 be constant along the sequence of discount factors. That is, there exists $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\delta_i = \delta^{\alpha_i}$ for $\delta \in (0, 1)$. Denote by $\Gamma^{\delta,\alpha}$ the bargaining game with payoffs modified by the assumption that i_τ has discount factor δ^{α_i} for all $i \in N$ and $\tau \geq 0$.

For a fixed α, we are interested in the asymptotic equilibrium behavior in $\Gamma^{\delta,\alpha}$ as $\delta \to 1$. Theorems 1 and 2 generalize verbatim. The notation for $\Gamma^\delta, v^{*\delta}, \underline{\delta}, G^{*\delta}, G^*, v^*, \ldots$ needs to be replaced by $\Gamma^{\delta,\alpha}, v^{*\delta}(\alpha), \underline{\delta}(\alpha), G^{*\delta}(\alpha), G^*(\alpha), v^*(\alpha), \ldots$ to reflect the dependence of the variables on α.

Remark 1. For a subnetwork H of G, the analogue of the linear system 3.3 used in the proofs of Proposition 1 and Theorem 2 is

$$v_i = \left(1 - \sum_{\{j|ij\in H\}} \frac{p_{ij}}{2}\right)\delta^{\alpha_i} v_i + \sum_{\{j|ij\in H\}} \frac{p_{ij}}{2}(1 - \delta^{\alpha_j} v_j), \forall i = \overline{1, n}.$$

As in the proof of Proposition 1 the unique solution $v_i^{\delta,H}(\alpha)$ is given by Cramer's rule, as the ratio of two determinants that are finite sums of positive real powers (which are not necessarily polynomials) of δ. In order to show that for fixed i, j, H there exists a finite number of δ's solving the equation $\delta(v_i^{\delta,H}(\alpha)+v_j^{\delta,H}(\alpha)) = 1$ we invoke a result due to Edmond

[3]For example, in the game for the two player network, if discount factors are given by the pair (δ^a, δ^b) $(a, b > 0)$, then as $\delta \to 1$ the limit equilibrium payoffs are $(b/(a+b), a/(a+b))$. For different choices of (a, b) the limit equilibrium payoffs for the corresponding sequence of discount factors vary accordingly. For the sequence of discount factors indexed by n given by $(1-1/n, 1-1/n)$ for odd n and $(1-1/n, (1-1/n)^2)$ for even n, the set of equilibrium payoffs has two accumulation points, $(1/2, 1/2)$ and $(2/3, 1/3)$. Similarly, for more complicated network structures, the limit equilibrium agreement network depends on the sequence of discount factors, and convergence does not always obtain.

N. Laguerre (1883). The result extends Descartes' rule of signs, which provides a bound for the number of positive real roots of polynomials in δ, to the case of linear combinations of powers of δ. A corollary of Laguerre's result is that every finite linear combination of positive powers of δ which does not vanish everywhere has a finite number of solutions.[4]

Theorem 3^α. *For every $G^*(\alpha)$-independent set M, with partner set $L = L^{G^*(\alpha)}(M)$, the following bounds on limit equilibrium payoffs hold*

$$\min_{i\in M} v_i^*(\alpha) \le \frac{\sum_{j\in L}\alpha_j}{\sum_{i\in M}\alpha_i + \sum_{j\in L}\alpha_j}$$
$$\max_{j\in L} v_j^*(\alpha) \ge \frac{\sum_{i\in M}\alpha_i}{\sum_{i\in M}\alpha_i + \sum_{j\in L}\alpha_j}.$$

Proof. We follow similar steps to the proof of Theorem 3 in the paper. The only innovation is that we rewrite the conclusion of Lemma 2 as

$$\frac{1-\delta^{\alpha_i}}{1-\delta} v_i^{*\delta}(\alpha) = \frac{1}{1-\delta}\sum_{\{j|ij\in G\}} \frac{p_{ij}}{2}\max(1-\delta^{\alpha_i}v_i^{*\delta}(\alpha) - \delta^{\alpha_j}v_j^{*\delta}(\alpha), 0), \forall i \in N, \forall \delta \in (0,1).$$

Then

$$\sum_{j\in L}\frac{1-\delta^{\alpha_j}}{1-\delta} v_j^{*\delta}(\alpha) \ge \sum_{i\in M}\frac{1-\delta^{\alpha_i}}{1-\delta} v_i^{*\delta}(\alpha),$$

which after taking the limit $\delta \to 1$ becomes

$$\sum_{j\in L}\alpha_j v_j^*(\alpha) \ge \sum_{i\in M}\alpha_i v_i^*(\alpha).$$

The conclusion is reached as in the proof of Theorem 3. □

If we replace formula 5.1 in the algorithm $\mathcal{A}(G)$ with

$$r_s(\alpha) = \min_{M\subset N_s,\ M\in\mathcal{I}(G)} \frac{\sum_{j\in L^{G^*(\alpha)}(M)}\alpha_j}{\sum_{i\in M}\alpha_i},$$

and leave the definitions of the other variables unchanged, the new procedure delivers the limit equilibrium payoffs of $\Gamma^{\delta,\alpha}$ when $\delta \to 1$ as detailed in Theorem 4.[5]

[4]If all components of α are rational numbers we can avoid non-polynomial functions by using the substitution $\delta \to \delta^c$, where c is the least common multiple of the denominators of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

[5]The extension of the proofs of Lemma 3 and Theorem 4 virtually consists in replacing everywhere the cardinality set operator $|\cdot|$ by the α-weight operator $|\cdot|_\alpha$, defined by $|M|_\alpha = \sum_{i\in M}\alpha_i$ for every $M \subset N$.

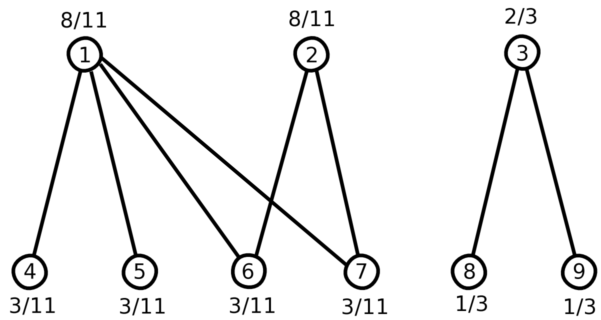

FIGURE 1. Network $G_3^*(p)$

3. An Example with Asymmetric Bargaining Protocols

The conclusions of Theorems 3 and 4 do not immediately extend to more general bargaining protocols. The following example illustrates some of the difficulties.

Example 1. Consider the network G_3 with 9 players illustrated in Figure 3 of the paper. The following procedure determines the probability $p(i \to j)$ with which player i is chosen to make an offer to player j for any link ij in G_3. Every link is selected with equal probability, and for each selection except $(2,6)$ and $(2,7)$, each of the two matched players is equally likely to be the proposer. If the link $(2,6)$ ($(2,7)$) is selected, then player 2 is twice as likely as player 6 (7) to be the proposer. Mathematically, $p(2 \to 6) = p(2 \to 7) = 1/18, p(6 \to 2) = p(7 \to 2) = 1/36$ and $p(i \to j) = 1/24$ for all other links ij in G_3. We similarly define the probability distribution of moves by nature p' to give player 2 asymmetric bargaining power in encounters with players 8 and 9, by $p'(2 \to 8) = p'(2 \to 9) = 1/18, p'(8 \to 2) = p'(9 \to 2) = 1/36$ and $p'(i \to j) = 1/24$ for all other links ij in G_3.

Consider first the game induced by the probability distribution p. By arguments similar to those from Example 2 in the paper, we obtain that the limit equilibrium agreement network is the subnetwork $G_3^*(p)$ illustrated in Figure 1. The limit equilibrium payoffs are $v_1^*(p) = v_2^*(p) = 8/11, v_3^*(p) = 2/3, v_4^*(p) = v_5^*(p) = v_6^*(p) = v_7^*(p) = 3/11$ and $v_8^*(p) = v_9^*(p) = 1/3$. The intuition is that player 2 can extort players 6 and 7 for more than 2/3, since he enjoys increased bargaining power in pairwise interactions with each of these players. Players 6 and 7 have to reach agreements when matched to bargain with 1, since they would receive limit equilibrium payoffs of at most 1/5 if they were monopolized by 2. Then player 1 will be able to take advantage of the weakness of 6 and 7, and also reach equally favorable agreements with 4 and 5. Player 1 can extort 4 and 5 because these two players do not have other bargaining partners. In the limit, since players 1 and 2 reach agreements on very favorable terms with 4, 5, 6 and 7, they are not attractive bargaining partners for 8 and 9. Players 8 and 9 have monopsony power over 3, and thus can secure limit equilibrium payoffs of 1/3. Hence, as players become patient, 8 and 9 do not have incentives to reach agreements with 1 or 2.

Consider next the game induced by p'. The limit equilibrium agreement network is identical to G_3^* (consisting of the set of thick links in Figure 3). The limit equilibrium payoffs are $v_1^* = v_2^* = v_3^* = 2/3$ and $v_4^* = v_5^* = v_6^* = v_7^* = v_8^* = v_9^* = 1/3$. Player 2 cannot use his stronger bargaining power in pairwise interactions with 8 and 9 to obtain a limit equilibrium payoff larger than 2/3. The intuition is that 8 and 9 can secure limit equilibrium payoffs of 1/3 in pairwise agreements with 3, since they constitute the only bargaining partners for 3. Hence 8 and 9 cannot be pressured to surrender more than 2/3 to player 2 in the limit, despite their relatively smaller chance of proposing when matched to bargain with 2. Equilibrium agreements do not arise across the links $(1, 8)$ and $(1, 9)$ when players are sufficiently patient for the reasons outlined in Example 2.

References

1. Jackson, Matthew O. and Asher Wolinsky (1996) A Strategic Model of Social and Economic Networks, *J. Econ. Theory*, **71**, 1, 44-74.
2. Laguerre, Edmond N. (1883) Memoire sur la theorie des equations numeriques, *J. Math. Pures Appl.*, **9**, 99-146 (translation by S.A. Levin, http://sepwww.stanford.edu/oldsep/stew/laguerre.pdf).