Intermediary Asset Pricing: Online Appendix

By Zhiguo He and Arvind Krishnamurthy *

I. Verification of optimality

In this section we take the equilibrium Price/Dividend ratio $F(y)$ as given, and verify that the specialist's consumption policy $c = D_t(1+l-y_t)$ is optimal subject to his budget constraint. Our argument is a variant of the standard one: it uses the strict concavity of $u(\cdot)$ and the specialist's budget constraint to show that the specialist's Euler equation is necessary and sufficient for the optimality of his consumption plan.

Specifically, fixing $t=0$ and the starting state (y_0, D_0), define the pricing kernel as

$$\xi_t \equiv e^{-\rho t} c_t^{-\gamma} = e^{-\rho t} D_t^{-\gamma} (1+l-\rho y_t)^{-\gamma}.$$

Consider another consumption profile $\widehat{c}$ which satisfies the budget constraint $E\left[\int_0^\infty \widehat{c}_t \xi_t dt\right] \leq \xi_0 D_0 (F_0 - y_0)$ (recall that the specialist's wealth is $D_0(F_0-y_0)$; here we require that the specialist's feasible trading strategies be well-behaved, e.g., his wealth process remains non-negative). Then we have

$$\begin{aligned} E\left[\int_0^\infty e^{-\rho t} u(c_t)\, dt\right] &\geq E\left[\int_0^\infty e^{-\rho t} u(\widehat{c}_t)\, dt\right] + E\left[\int_0^\infty e^{-\rho t} u'(c_t)(c_t - \widehat{c}_t)\, dt\right] \\ &= E\left[\int_0^\infty e^{-\rho t} u(\widehat{c}_t)\, dt\right] + E\left[\int_0^\infty \xi_t c_t dt\right] - E\left[\int_0^\infty \xi_t \widehat{c}_t dt\right]. \end{aligned}$$

If the specialist's budget equation holds in equality for the equilibrium consumption process c, i.e., if

$$E\left[\int_0^\infty \xi_t c_t dt\right] = \xi_0 D_0 (F_0 - y_0),$$

then the result follows. Somewhat surprisingly, for our model this seemingly obvious claim requires an involved argument because of the singularity at $y^b = \frac{1+l}{\rho}$.

* Respectively: University of Chicago, Booth School of Business and NBER; Northwestern University, Kellogg School of Management and NBER. E-mails: zhiguo-he@chicagobooth.edu, a-krishnamurthy@northwestern.edu. Address: 2001 Sheridan Road, Evanston, IL 60208.

One can easily check that, for $\forall T > 0$, we have

$$\xi_0 D_0 \left(F_0 - y_0\right) = \int_0^T c_t \xi_t dt + \int_0^T \sigma\left(D_t, y_t\right) dZ_t + \xi_T D_T \left(F_T - y_T\right), \tag{1}$$

where $\sigma(D_t, y_t)$ corresponds to the specialist's equilibrium trading strategy (which involves terms such as $(1 + l - \rho y)^{-\gamma-1}$ and is NOT uniformly bounded as $y \to y^b$). Our goal in the following steps is to show that in expectation, the latter two terms vanishes when $T \to \infty$.

Step 1: Limiting Behavior of y at y^b

The critical observation regarding the evolution of y is that when y approaches y^b, it approximately follows a Bessel process with a dimension $\delta = \gamma + 2 > 2$. (Given a δ-dimensional Brownian motion Z, a Bessel process with a dimension δ is the evolution of $\|Z\| = \sqrt{\sum_{i=1}^{\delta} Z_i^2}$, which is the Euclidean distance between Z and the origin.) According to standard results on Bessel processes, y^b is an entrance-no-exit point, and is not reachable if the starting value $y_0 < y^b$ (if $\delta > 2$). Intuitively, when y is close to y^b, the dominating part of μ_y is proportional to $\frac{1}{y - y^b} < 0$, while the volatility σ_y is bounded— therefore a drift that diverges to negative infinity keeps y away from the singular point y^b. This result implies that our economy never hits y^b.

To show that for y close to y^b, y's evolution can be approximated by a Bessel Process, one can easily check that when $y \to y^b$,

$$r \simeq -\frac{(\gamma+1)\sigma^2}{2} \frac{\rho^h \hat{\theta}_b G}{1 + l - \rho^h y}, \mu_y \simeq -\frac{(\gamma+1)\sigma^2}{2} \frac{\rho^h \hat{\theta}_b^2 G^2}{1 + l - \rho^h y}, \sigma_y = -G\sigma\hat{\theta}_b;$$

and therefore

$$dy = -\frac{(\gamma+1)\sigma^2}{2} \frac{\rho \hat{\theta}_b^2 G^2}{1 + l - \rho y} dt - G\sigma\hat{\theta}_b dZ_t.$$

Utilizing the result $F'\left(y^b\right) = 1$ established in Section **??**, we know that when $y \to y^b$, $\hat{\theta}_b \simeq F - \theta_s y \simeq \frac{1}{1+m} y^b = \frac{1}{1+m}\frac{1+l}{\rho}$, and $G \simeq 1 + m$. Let

$$z_t = 1 + l - \rho y_t;$$

then it is easy to show that $q = \frac{z}{G\sigma\hat{\theta}_b\rho} = \frac{z}{\sigma(1+l)}$ evolves approximately according to

$$dq_t = -\frac{1}{G\sigma\hat{\theta}_b} dy_t = \frac{(\gamma+1)}{2q_t} dt + dZ_t,$$

which is a standard Bessel process with a dimension $\delta = \gamma + 2$. Therefore, z is also a scaled version of a Bessel process, and can never reach 0 (or, y cannot reach y^b). In the following analysis, we focus on the limiting behavior of z.

Step 2: Localization

Note that in (1), due to the singularity at $y = y^b$, both the local martingale part $\int_0^T \sigma(D_t, y_t)\, dZ_t$ and the terminal wealth part $\xi_T D_T (F_T - y_T)$ are not well-behaved. To show our claim, we have to localize our economy, i.e., stop the economy once either y is sufficiently close to y^b or D is sufficiently close to 0. Specifically, we define

$$T_n = \inf\left\{t : \text{either } z_t = n^{-1} \text{ or } D_t = n^{-h}\right\}$$

where h is a positive constant (as we will see, the choice of h, which is around 1, gives some flexibility for γ other than 2). Here, because y and z have a one-to-one relation ($z = 1 + l - \rho y$), for simplicity we localize z instead.

Clearly this localization technique ensures that the local martingale part $\int_0^{T_n} \sigma(D_t, y_t)\, dZ_t$ is a martingale (one can check that $\sigma(D_t, y_t)$ is continuous in D_t and y_t, in turn D_t and x_t; therefore $\sigma(D_t, y_t)$ is bounded). As $T_n \to \infty$ when $n \to \infty$, for our claim we need to show

$$\lim_{n\to\infty} E\left[\xi_{T_n} D_{T_n}\left(F_{T_n} - y_{T_n}\right)\right] = 0$$

We substitute from the definition of ξ:

$$E\left[e^{-\rho T_n} D_{T_n}^{1-\gamma} z_{T_n}^{-\gamma}\left(F\left(y_{T_n}\right) - y_{T_n}\right)\right] \leq E\left[e^{-\rho T_n} n^{h(\gamma-1)} z_{T_n}^{-\gamma}\left(F\left(y_{T_n}\right) - y_{T_n}\right)\right].$$

Since the analysis will be obvious if $z^{-\gamma}(F(y) - y)$ is uniformly bounded (notice here $z = 1 + l - \rho y$), it is sufficient to consider $z_{T_n} = \frac{1}{n}$ (or, $y_{T_n} = y^b - \frac{1}{n\rho}$). Because $F\left(y^b\right) = y^b$ and $F'\left(y^b\right) = 1$, by Taylor expansion we know that $F\left(y^b - \frac{1}{n\rho}\right) - \left(y^b - \frac{1}{n\rho}\right)$ can be written as $\psi(n)\frac{1}{n}$ when n is sufficiently large, and $\psi(n) \to 0$ as $n \to \infty$. Therefore we have to show that, as $n \to \infty$,

$$E\left[e^{-\rho T_n} n^{(\gamma-1)(1+h)}\right]\psi(n) \to 0$$

and a sufficient condition is that there exists some constant M so that

$$E\left[e^{-\rho T_n}\right] n^{(\gamma-1)(1+h)} \to M.$$

We apply existing analytical results in the literature to show our claim. To do so, we have to separate our two state variables. We define

$$T_n^D = \inf\left\{t : D_t = n^{-h}\right\}, T_n^x = \inf\left\{t : z_t = n^{-1}\right\}.$$

We want to bound $E\left[e^{-\rho T_n}\right]$ by the sum of $E\left[e^{-\rho T_n^D}\right]$ and $E\left[e^{-\rho T_n^z}\right]$. The Laplace transform of T_n is simply

$$E\left[e^{-\rho T_n}\right] = \int_0^\infty e^{-\rho T} d\mathbf{F}\left(T\right) = \rho \int_0^\infty e^{-\rho T}\mathbf{F}\left(T\right) dT,$$

where the bold $\mathbf{F}$ denotes the distribution function of T_n. The similar relation also holds for T_n^D or T_n^z. Denote $\mathbf{F}^D\left(\cdot\right)$ (or $\mathbf{F}^z\left(\cdot\right)$) as the distribution function for T_n^D (or T_n^z), and notice that

$$\begin{aligned} 1-\mathbf{F}\left(T\right) &= \Pr\left(T_n > T\right) = \Pr\left(T_n^D > T, T_n^z > T\right) > \Pr\left(T_n^D > T\right)\Pr\left(T_n^z > T\right) \\ &= 1-\mathbf{F}^D\left(T\right) - \mathbf{F}^z\left(T\right) + \mathbf{F}^D\left(T\right)\mathbf{F}^z\left(T\right), \end{aligned}$$

because $\mathbf{1}_{\{T_n^D > T\}}$ and $\mathbf{1}_{\{T_n^z > T\}}$ are positively correlated (both take the value 1 when the Brownian Z is high).[1] Therefore $\mathbf{F}\left(T\right) < \mathbf{F}^D\left(T\right) + \mathbf{F}^z\left(T\right)$, or

$$E\left[e^{-\rho T_n}\right] n^{(\gamma-1)(1+h)} < E\left[e^{-\rho T_n^D}\right] n^{(\gamma-1)(1+h)} + E\left[e^{-\rho T_n^z}\right] n^{(\gamma-1)(1+h)}.$$

Our goal is to show the right hand side of the above inequality goes to zero when $n \to \infty$.

There are two terms in the right hand side of the above inequality. For the first term, we can use the standard result of the Laplace transform of the first-hitting time distribution for a GBM process (e.g., Borodin and Salminen (2002), page 622):

$$E\left[e^{-\rho T_n^D}\right] = n^{-\frac{h}{\sigma^2}\left(\sqrt{2\rho\sigma^2 + (g-0.5\sigma^2)^2} + g - 0.5\sigma^2\right)}.$$

Thus, by choosing some appropriate h so that

$$\frac{h}{\sigma^2}\left(\sqrt{2\rho\sigma^2 + \left(g-0.5\sigma^2\right)^2} + g - 0.5\sigma^2\right) > \left(\gamma-1\right)\left(1+h\right),$$

the first term $E\left[e^{-\rho T_n^D}\right] n^{(\gamma-1)(1+h)}$ vanishes as $n \to \infty$. For instance, this condition holds when $h = 0.9$ under our parameterization. The next step is for the second term.

Step 3: Regulated Bessel Process

For the second term $E\left[e^{-\rho T_n^z}\right] n^{(\gamma-1)(1+h)}$, because our economy (i.e., evolution

[1] Technically, using the technique of Malliavian derivatives, we can show that both z_s and D_s have positive diffusions in the martingale representations for all s. Then, the running minimum $\underline{z}_T = \min\left\{z_t : 0 < t < T\right\}$ and $\underline{D}_T = \min\left\{D_t : 0 < t < T\right\}$ have positive loadings always on the martingale representations (using the technique in *Methods of Mathematical Finance*, Karatzas and Shreve (1998), Page 367). The same technique can be applied to $\mathbf{1}_{\{T_n^z > T\}} = \mathbf{1}_{\{\underline{z}_T > T\}}$ and $\mathbf{1}_{\{T_n^D > T\}} = \mathbf{1}_{\{\underline{D}_T > T\}}$, as an indicator function can be approximated by a sequence of differentiable increasing functions.

of z) differs from the evolution of a Bessel process when z is far away from 0, an extra care needs to be taken. We consider a regulated Bessel process which is reflected at some positive constant $\overline{z}$. Intuitively, by doing so, we are putting an upper bound for $E\left[e^{-\rho T_n^z}\right]$, as the reflection makes z_t to hit n^{-1} more likely (therefore, a larger $\mathbf{F}^z$). Also, for a sufficiently small $\overline{z} > 0$, when $z \in (0, \overline{z}]$, z follows a Bessel process with a dimension $\gamma+2-\varepsilon$. Therefore, $\mathbf{F}^z$ must be bounded by the first-hitting time distribution of a Bessel process with a dimension δ, where δ takes value from $\gamma+2-\epsilon$ to $\gamma+2$, where ϵ is sufficiently small. Finally, note that by considering a Bessel process we are neglecting certain drift for z. However, one can easily check that when z is close to 0, the adjustment term for μ_y is $-\frac{1+l}{\rho}\gamma\sigma^2 < 0$. This implies that we are neglecting a positive drift for z—which potentially makes hitting less likely—thereby yielding an upper-bound estimate.

We have the following Lemma from the Bessel process.

LEMMA 1: *Consider a Bessel process x with $\delta > 2$ which is reflected at $\overline{x} > 0$. Let $\nu = \frac{\delta}{2} - 1$. Starting from $x_0 \leq \overline{x}$, we consider the hitting time $T_n^x = \inf\left\{t : x_t = \frac{1}{n}\right\}$. Then we have*

$$E\left[e^{-\rho T_n^x}\right] \propto n^{-2\nu} \text{ as } n \to \infty$$

PROOF:

Due to the standard results in Bessel process and the Laplace transform of the hitting time (e.g., see Borodin and Salminen (1996), Chapter 2), we have

$$E\left[e^{-\rho T_n^z}\right] = \varphi\left(z_0\right)/\varphi\left(n^{-1}\right),$$

where

$$\varphi\left(u\right) = c_1 u^{-\nu} I_v\left(\sqrt{2\rho} u\right) + c_2 u^{-\nu} K_v\left(\sqrt{2\rho} u\right),$$

and $I_v\left(\cdot\right)$ (and $K_v\left(\cdot\right)$) is modified Bessel function of the first (and second) kind of order v. Because R is a reflecting barrier, the boundary condition is

$$\varphi'\left(\overline{z}\right) = 0,$$

which pins down the constants c_1 and c_2 (up to a constant multiplication; notice that this does not affect the value of $E\left[e^{-\rho T_n^z}\right]$). Therefore the growth rate of $E\left[e^{-\rho T_n^z}\right]$ is determined by $n^\nu K_v\left(\sqrt{2\rho} n^{-1}\right)$ as K_v dominates I_v near 0. Since $K_v\left(z\right)$ has a growth rate $z^{-\nu}$ when $z \to 0$, the result is established.

For any y_0, redefine starting point as $z_0 = \min\left(1 + l - y_0, \overline{z}\right)$; clearly this leads to an upper-bound estimate for $E\left[e^{-\rho T_n^z}\right]$. However, since for all $\delta \in [\gamma + 2 - \epsilon, \gamma + 2]$, the above Lemma tells us that for any $\varepsilon \in [0, \epsilon]$, given $\gamma = 2$ and $h = 0.9$, when $n \to \infty$, for some sufficiently small $\epsilon > 0$ we have

$$n^{(\gamma-1)(1+h)} E\left[e^{-\rho T_n^z}\right] \propto n^{(\gamma-1)(1+h)} n^{-2\nu} = n^{(\gamma-1)(1+h)-\gamma+\varepsilon} \to 0 \text{ uniformly.}$$

Therefore we obtain our desirable result.

Finally $c_t\xi_t > 0$ implies that $\int_0^\infty c_t\xi_t dt$ converges monotonically, and therefore the specialist's budget equation $\lim_{T\to\infty} E\left[\int_0^T \xi_t c_t dt\right] = \xi_0 D_0 (F_0 - y_0)$ holds for all stopping times that converge to infinity. Q.E.D.

II. Appendix for Section 6

A. Borrowing Subsidy

We have the same ODE as in Appendix A. The only difference is that

$$\mu_y = \frac{1}{1-\theta_s F'}\left(\theta_s + l + (r+\sigma^2 - g)\hat{\theta}_b - \hat{\theta}_b \Delta r - \rho y + \frac{1}{2}\theta_s F'' \sigma_y^2\right).$$

B. Direct Asset Purchase

In this case, the intermediary holds $1-s$ of the risky asset (where s is a function of (y, D)). In the unconstrained region, $\alpha^h = 1$, and

$$\frac{\alpha^I\left(w + \alpha^h (1-\lambda) w^h\right)}{P} = 1 - s$$

which implies that $\alpha^I = \frac{(1-s)F}{F-\lambda y}$. Therefore the households' holding of the risky asset through intermediaries is

$$\theta_s^I = \frac{(1-s)(1-\lambda) y}{F - \lambda y},$$

and the total holding is $\theta_s = \theta_s^I + s = \frac{(1-s)(1-\lambda)y}{F-\lambda y} + s(y, D)$.

In the constrained region, $\alpha^h = \frac{m(F-y)}{(1-\lambda)y}$ and $\alpha^I = \frac{1}{1+m}\frac{(1-s)F}{F-y}$. So

$$\theta_s^I = \frac{m(F-y)}{(1-\lambda)y}\frac{1}{1+m}\frac{(1-s)F}{F-y}(1-\lambda)\frac{y}{F} = \frac{m}{1+m}(1-s)$$

and the total holding is

$$\theta_s = \frac{m}{1+m}(1-s) + s = \frac{m+s}{1+m}.$$

The same constraint cutoff applies $y^c = \frac{m}{1-\lambda+m}F^c$.

Finally, the expressions for the case of capital infusion (i.e., changing m) is isomorphic to the case of $s > 0$. This is because given s we can find some appropriate $m'(s) = \frac{s+m}{1-s}$ such that $\frac{m'}{1+m'} = \frac{m+s}{1+m}$.