Appendix to "Sales and Monetary Policy"

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A Solving the model

Steady state

Finding the steady state of the model characterized in section 3.4 requires solving only one equation numerically. For parameters ϵ and η satisfying condition [16], the markup ratio μ is a root of the equation $\mathfrak{R}(\mu;\epsilon,\eta)=0$, where $\mathfrak{R}(\mu;\epsilon,\eta)$ is the determinant

$$\mathfrak{R}(\mu;\epsilon,\eta)\equiv\begin{vmatrix}\mathfrak{a}_0(\mu;\epsilon,\eta) & \mathfrak{a}_1(\mu;\epsilon,\eta) & \mathfrak{a}_2(\eta) & 0 & 0\\ 0 & \mathfrak{a}_0(\mu;\epsilon,\eta) & \mathfrak{a}_1(\mu;\epsilon,\eta) & \mathfrak{a}_2(\eta) & 0\\ 0 & 0 & \mathfrak{a}_0(\mu;\epsilon,\eta) & \mathfrak{a}_1(\mu;\epsilon,\eta) & \mathfrak{a}_2(\eta)\\ \mathfrak{b}_0(\mu;\epsilon,\eta) & \mathfrak{b}_1(\mu;\epsilon,\eta) & \mathfrak{b}_2(\mu;\epsilon,\eta) & \mathfrak{b}_3(\eta) & 0\\ 0 & \mathfrak{b}_0(\mu;\epsilon,\eta) & \mathfrak{b}_1(\mu;\epsilon,\eta) & \mathfrak{b}_2(\mu;\epsilon,\eta) & \mathfrak{b}_3(\eta)\end{vmatrix}, \qquad \text{[A.1]}$$

and where the functions in the matrix are given by:

$$\mathfrak{a}_0(\mu;\epsilon,\eta)\equiv\epsilon(\epsilon-1)\mu^{\eta-\epsilon}; \qquad \text{[A.2a]}$$

$$\mathfrak{a}_1(\mu;\epsilon,\eta)\equiv\eta(\epsilon-1)\left(\frac{1-\mu^{\eta-\epsilon+1}}{1-\mu}\right)+\epsilon(\eta-1)\left(\frac{\mu^{\eta-\epsilon}-\mu}{1-\mu}\right); \qquad \text{[A.2b]}$$

$$\mathfrak{a}_2(\eta)\equiv\eta(\eta-1); \qquad \text{[A.2c]}$$

$$\mathfrak{b}_0(\mu;\epsilon,\eta)\equiv(\epsilon-1)\left(\frac{\mu^{2(\eta-\epsilon)}-\mu^{2\eta-\epsilon}}{1-\mu^{\eta}}\right); \qquad \text{[A.2d]}$$

$$\mathfrak{b}_1(\mu;\epsilon,\eta)\equiv(\eta-1)\left(\frac{\mu^{2(\eta-\epsilon)}-\mu^{\eta}}{1-\mu^{\eta}}\right)+2(\epsilon-1)\left(\frac{\mu^{\eta-\epsilon}-\mu^{2\eta-\epsilon}}{1-\mu^{\eta}}\right); \qquad \text{[A.2e]}$$

$$\mathfrak{b}_2(\mu;\epsilon,\eta)\equiv(\epsilon-1)\left(\frac{1-\mu^{2\eta-\epsilon}}{1-\mu^{\eta}}\right)+2(\eta-1)\left(\frac{\mu^{\eta-\epsilon}-\mu^{\eta}}{1-\mu^{\eta}}\right); \qquad \text{[A.2f]}$$

$$\mathfrak{b}_3(\eta)\equiv(\eta-1). \qquad \text{[A.2g]}$$

When searching for a root, it is necessary to restrict attention to economically meaningful solutions. These correspond to positive real values of the function

$$\mathfrak{z}(\mu;\epsilon,\eta)\equiv\frac{-\mathfrak{a}_1(\mu;\epsilon,\eta)-\sqrt{\mathfrak{a}_1(\mu;\epsilon,\eta)^2-4\mathfrak{a}_2(\eta)\mathfrak{a}_0(\mu;\epsilon,\eta)}}{2\mathfrak{a}_2(\eta)}. \qquad \text{[A.3]}$$

Under the conditions stated in Theorem 1, there exists a unique economically meaningful solution of the equation $\mathfrak{R}(\mu;\epsilon,\eta)=0$.

Having obtained the markup ratio μ, the quantity ratio χ is

$$\chi=\mu^{-\epsilon}\left(\frac{1+\mu^{-(\eta-\epsilon)}\mathfrak{z}(\mu;\epsilon,\eta)}{1+\mathfrak{z}(\mu;\epsilon,\eta)}\right), \qquad \text{[A.4]}$$

and the sales frequency s is

$$s=\frac{\left(\frac{\lambda}{1-\lambda}\mathfrak{z}(\mu;\epsilon,\eta)\right)^{-\left(\frac{\eta-1}{\eta-\epsilon}\right)}-1}{\mu^{-(\eta-1)}-1}. \qquad \text{[A.5]}$$

This expression for the sales frequency is economically meaningful when λ lies between the bounds $\underline{\lambda}(\epsilon,\eta)$ and $\overline{\lambda}(\epsilon,\eta)$ referred to in Theorem 1, which are given by:

$$\underline{\lambda}(\epsilon,\eta)\equiv\frac{1}{1+\mu^{-(\eta-\epsilon)}\mathfrak{z}(\mu;\epsilon,\eta)}, \quad \text{and } \overline{\lambda}(\epsilon,\eta)\equiv\frac{1}{1+\mathfrak{z}(\mu;\epsilon,\eta)}. \qquad \text{[A.6]}$$

An expression for real marginal cost x (the reciprocal of the average markup) is

$$x = \left(\lambda\left((1+\mathfrak{z}(\mu;\epsilon,\eta)) + \left((\mu^{1-\epsilon}-1) + (\mu^{1-\eta}-1)\mathfrak{z}(\mu;\epsilon,\eta)\right)s\right)\right)^{\frac{1}{\epsilon-1}}\left(\frac{(\epsilon-1)+(\eta-1)\mathfrak{z}(\mu;\epsilon,\eta)}{\epsilon+\eta\mathfrak{z}(\mu;\epsilon,\eta)}\right), \quad \text{[A.7]}$$

and the degree of price distortion $\Delta = Y/Q$ is given by:

$$\Delta = \frac{\left(\lambda\left((1+\mathfrak{z}(\mu;\epsilon,\eta)) + \left((\mu^{1-\epsilon}-1)+(\mu^{1-\eta}-1)\mathfrak{z}(\mu;\epsilon,\eta)\right)s\right)\right)^{\frac{\epsilon}{\epsilon-1}}}{\lambda\left((1+\mathfrak{z}(\mu;\epsilon,\eta)) + ((\mu^{-\epsilon}-1)+(\mu^{-\eta}-1)\mathfrak{z}(\mu;\epsilon,\eta))s\right)}. \quad \text{[A.8]}$$

DSGE model

The system of log-linearized equations of the model from section 4 is

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \frac{1}{1-\psi}\left(\kappa \mathsf{x}_t + \psi\left(\boldsymbol{\Delta}\mathsf{x}_t - \beta\mathbb{E}_t\boldsymbol{\Delta}\mathsf{x}_{t+1}\right)\right); \quad \text{[A.9a]}$$

$$\mathsf{x}_t = \frac{1}{1+\gamma\delta}\mathsf{w}_t + \frac{\gamma}{1+\gamma\delta}\mathsf{Y}_t; \quad \text{[A.9b]}$$

$$\pi_{W,t} = \beta\mathbb{E}_t\pi_{W,t+1} + \frac{(1-\phi_w)(1-\beta\phi_w)}{\phi_w}\frac{1}{1+\varsigma\theta_h^{-1}}\left(\left(\theta_c^{-1} + \frac{1}{1+\gamma\delta}\frac{\theta_h^{-1}}{\alpha}\right)\mathsf{Y}_t - \left(1+\frac{\delta}{1+\gamma\delta}\frac{\theta_h^{-1}}{\alpha}\right)\mathsf{w}_t\right); \quad \text{[A.9c]}$$

$$\boldsymbol{\Delta}\mathsf{w}_t = \pi_{W,t} - \pi_t; \quad \text{[A.9d]}$$

$$\mathsf{Y}_t = \mathbb{E}_t\mathsf{Y}_{t+1} - \theta_c\left(\mathsf{i}_t - \mathbb{E}_t\pi_{t+1}\right); \quad \text{[A.9e]}$$

$$\boldsymbol{\Delta}\mathsf{Y}_t = \boldsymbol{\Delta}\mathsf{M}_t - \pi_t; \quad \text{[A.9f]}$$

$$\boldsymbol{\Delta}\mathsf{M}_t = \mathfrak{p}\boldsymbol{\Delta}\mathsf{M}_{t-1} + (1-\mathfrak{p})\mathsf{e}_t. \quad \text{[A.9g]}$$

The Phillips curve equation is from Theorem 2 and derivations of the other equations are given in appendix F. All the coefficients apart from ψ, κ and δ are as defined in Table 2. Formulæ for ψ, κ and δ are

$$\psi = 1 - \frac{(1-s)\left(1-\left(\frac{\eta-1}{\epsilon-1}\right)\left(\frac{\mu^{1-\epsilon}-1}{\mu^{1-\eta}-1}\right)\right)}{(1+\mathfrak{z}(\mu;\epsilon,\eta)) + ((\mu^{1-\epsilon}-1)+(\mu^{1-\eta}-1)\mathfrak{z}(\mu;\epsilon,\eta))s}, \quad \kappa = \frac{(1-\phi_p)(1-\beta\phi_p)}{\phi_p}, \quad \text{and} \quad \text{[A.10a]}$$

$$\delta = \frac{s\chi\epsilon(1-\mu)}{s\chi+(1-s)} \quad \text{[A.10b]}$$

$$+\frac{s\chi\mu+(1-s)}{s\chi+(1-s)}\left(\frac{\frac{1}{\epsilon-1}\left((\epsilon-1)(\mu^{-\epsilon}-1)-\epsilon(\mu^{1-\epsilon}-1)\right) + \frac{\mathfrak{z}(\mu;\epsilon,\eta)}{\eta-1}\left((\eta-1)(\mu^{-\eta}-1)-\eta(\mu^{1-\eta}-1)\right)}{\left(\frac{\mu^{1-\eta}-1}{\eta-1}\right)-\left(\frac{\mu^{1-\epsilon}-1}{\epsilon-1}\right)}\right).$$

The solution of the system [A.9] can be obtained using standard methods for solving expectational difference equations.

The standard model without sales is a special case of [A.9] with the following parameter restrictions:

$$\psi = 0, \quad \delta = 0, \quad \xi = \frac{1}{1-x}, \quad \text{and} \quad \kappa = \frac{1}{1+\xi\gamma}\frac{(1-\phi_p)(1-\beta\phi_p)}{\phi_p},$$

where the Phillips curve then reduces to the standard New Keynesian Phillips curve.

B Properties of the demand, revenue and marginal revenue functions

The structure of household consumption preferences introduced in section 2.2 implies that firms face a demand curve $q = \mathscr{D}(p; P_B, \mathcal{E})$ of the form given in equation [10] at each shopping moment. It is easier to analyse the properties of this demand function — and the associated total and marginal revenue functions — by working with what can be thought of as the corresponding "relative" demand function $\mathcal{D}(\rho)$, defined by

$$\mathcal{D}(\rho) \equiv \lambda\rho^{-\epsilon} + (1-\lambda)\rho^{-\eta}, \tag{B.1}$$

which satisfies $\mathcal{D}(1) = 1$ for all choices of parameters. The relative demand function $\mathfrak{q} = \mathcal{D}(\rho)$ gives the "relative" quantity sold $\mathfrak{q}$ as a function of the relative price ρ, where relative price here means money price p relative to P_B, the bargain hunters' price index from [7], and relative quantity means quantity q sold relative to $\mathcal{E}/P_B^\epsilon$, where $\mathcal{E} = P^\epsilon Y$ is the measure of aggregate expenditure from [10]:

$$\rho \equiv \frac{p}{P_B}, \quad \text{and } \mathfrak{q} \equiv \frac{P_B^\epsilon}{\mathcal{E}}q. \tag{B.2}$$

With these definitions, the original demand function [10] is stated in terms of the relative demand function [B.1] as follows:

$$\mathscr{D}(p; P_B, \mathcal{E}) = \frac{\mathcal{E}}{P_B^\epsilon}\mathcal{D}\left(\frac{p}{P_B}\right). \tag{B.3}$$

The relative demand function [B.1] is a continuously differentiable function of ρ for all $\rho > 0$, and is strictly decreasing everywhere. Notice also that $\mathcal{D}(\rho) \to \infty$ as $\rho \to 0$, and $\mathcal{D}(\rho) \to 0$ as $\rho \to \infty$. By continuity and monotonicity, this implies that for every $\mathfrak{q} > 0$ there is a unique $\rho > 0$ such that $\mathfrak{q} = \mathcal{D}(\rho)$. Thus the inverse demand function $\mathcal{D}^{-1}(\mathfrak{q})$ is well defined for all $\mathfrak{q} > 0$, and is itself strictly decreasing and continuously differentiable. The revenue function $\mathcal{R}(\mathfrak{q})$, defined in terms of the relative demand function, is

$$\mathcal{R}(\mathfrak{q}) \equiv \mathfrak{q}\mathcal{D}^{-1}(\mathfrak{q}). \tag{B.4}$$

Using the inverse demand function $\rho = \mathcal{D}^{-1}(\mathfrak{q})$, an equivalent expression for the revenue function is $\mathcal{R}(\mathfrak{q}) = \mathcal{D}^{-1}(\mathfrak{q})\mathcal{D}\left(\mathcal{D}^{-1}(\mathfrak{q})\right)$, and by substituting the demand function from [B.1]:

$$\mathcal{R}(\mathfrak{q}) = \lambda\left(\mathcal{D}^{-1}(\mathfrak{q})\right)^{1-\epsilon} + (1-\lambda)\left(\mathcal{D}^{-1}(\mathfrak{q})\right)^{1-\eta}.$$

Since $\epsilon > 1$ and $\eta > 1$, and given the limiting behaviour of the demand function established above, it follows that $\mathcal{R}(\mathfrak{q}) \to \infty$ as $\mathfrak{q} \to \infty$ and $\mathcal{R}(\mathfrak{q}) \to 0$ as $\mathfrak{q} \to 0$. Hence, $\mathcal{R}(0) = 0$, and $\mathcal{R}(\mathfrak{q})$ is continuously differentiable for all $\mathfrak{q} \geq 0$.

Differentiating the revenue function $\mathcal{R}(\mathfrak{q})$ from [B.4] using the inverse function theorem, and substituting demand function [B.1] yields an expression for marginal revenue:

$$\mathcal{R}'\left(\mathcal{D}(\rho)\right) = \left(\frac{(\epsilon-1)\lambda + (\eta-1)(1-\lambda)\rho^{\epsilon-\eta}}{\epsilon\lambda + \eta(1-\lambda)\rho^{\epsilon-\eta}}\right)\rho. \tag{B.5}$$

Because $\epsilon > 1$ and $\eta > 1$, it follows that $\mathcal{R}'(\mathfrak{q}) > 0$ for all $\mathfrak{q}$, so revenue $\mathcal{R}(\mathfrak{q})$ is strictly increasing in $\mathfrak{q}$. Furthermore, because $\rho \to \infty$ as $\mathfrak{q} \to 0$, and $\rho \to 0$ as $\mathfrak{q} \to \infty$, [B.5] implies $\mathcal{R}'(\mathfrak{q}) \to \infty$ as $\mathfrak{q} \to 0$ and $\mathcal{R}'(\mathfrak{q}) \to 0$ as $\mathfrak{q} \to \infty$.

Just as [B.3] establishes the original demand function $\mathscr{D}(p; P_B, \mathcal{E})$ in [10] is connected to the relative demand function $\mathcal{D}(\rho)$ in [B.1], there are similar relations between the original inverse demand function $\mathscr{D}^{-1}(\mathfrak{q}; P_B, \mathcal{E})$, original revenue $\mathscr{R}(q; P_B, \mathcal{E})$ and marginal revenue $\mathscr{R}'(q; P_B, \mathcal{E})$ functions, and their equivalents defined in terms of the relative demand function. The link between the inverse demand functions follows directly from [B.3]:

$$\mathscr{D}^{-1}(q; P_B, \mathcal{E}) = P_B\mathcal{D}^{-1}\left(\frac{qP_B^\epsilon}{\mathcal{E}}\right). \tag{B.6}$$

Equations [11], [B.4] and [B.6] justify the following connections between the revenue functions and their derivatives:

$$\mathscr{R}(q;P_B,\mathcal{E}) = P_B^{1-\epsilon}\mathcal{E}\mathcal{R}\left(\frac{qP_B^{\epsilon}}{\mathcal{E}}\right), \quad \mathscr{R}'(q;P_B,\mathcal{E}) = P_B\mathcal{R}'\left(\frac{qP_B^{\epsilon}}{\mathcal{E}}\right), \quad \text{and } \mathscr{R}''(q;P_B,\mathcal{E}) = \frac{P_B^{1+\epsilon}}{\mathcal{E}}\mathcal{R}''\left(\frac{qP_B^{\epsilon}}{\mathcal{E}}\right). \quad \text{[B.7]}$$

The next result examines the conditions under which marginal revenue $\mathcal{R}'(\mathfrak{q})$ is non-monotonic.

Lemma 1 *Consider the marginal revenue function $\mathcal{R}'(\mathfrak{q})$ obtained from* [B.4] *using the relative demand function* [B.1], *and suppose that $\eta > \epsilon > 1$.*

(i) If $\lambda = 0$ or $\lambda = 1$ or condition [16] *does not hold then marginal revenue $\mathcal{R}'(\mathfrak{q})$ is strictly decreasing for all $\mathfrak{q} \geq 0$.*

(ii) If $0 < \lambda < 1$ and ϵ and η satisfy condition [16] *then there exist $\underline{\mathfrak{q}}$ and $\overline{\mathfrak{q}}$ such that $0 < \underline{\mathfrak{q}} < \overline{\mathfrak{q}} < \infty$ and where $\mathcal{R}'(\mathfrak{q})$ is strictly decreasing between 0 and $\underline{\mathfrak{q}}$ and above $\overline{\mathfrak{q}}$, and strictly increasing between $\underline{\mathfrak{q}}$ and $\overline{\mathfrak{q}}$.*

Proof (i) If $\lambda = 0$ then it follows from [B.5] that $\mathcal{R}'(\mathfrak{q}) = ((\eta-1)/\eta)\mathcal{D}^{-1}(\mathfrak{q})$, and if $\lambda = 1$ that $\mathcal{R}'(\mathfrak{q}) = ((\epsilon-1)/\epsilon)\mathcal{D}^{-1}(\mathfrak{q})$. Since the inverse demand function $\mathcal{D}^{-1}(\mathfrak{q})$ is strictly decreasing, then marginal revenue must also be so in these cases.

(ii) In what follows, assume $0 < \lambda < 1$. Differentiate [B.5] to obtain

$$\mathcal{D}'(\rho)\mathcal{R}''(\mathcal{D}(\rho)) = \frac{\eta(\eta-1)\left(\frac{1-\lambda}{\lambda}\rho^{\epsilon-\eta}\right)^2 - \left((\eta-\epsilon)^2 - \eta(\epsilon-1) - \epsilon(\eta-1)\right)\left(\frac{(1-\lambda)}{\lambda}\rho^{\epsilon-\eta}\right) + \epsilon(\epsilon-1)}{\left(\epsilon + \eta\left(\frac{1-\lambda}{\lambda}\rho^{\epsilon-\eta}\right)\right)^2}, \quad \text{[B.8]}$$

for all $\rho > 0$, where the assumption that $\lambda \neq 0$ is used to simplify the expression by dividing through by λ^2. Define the function $\mathcal{Z}(\mathfrak{q})$ in terms of inverse demand function $\mathcal{D}^{-1}(\mathfrak{q})$:

$$\mathcal{Z}(\mathfrak{q}) \equiv \frac{1-\lambda}{\lambda}\left(\mathcal{D}^{-1}(\mathfrak{q})\right)^{\epsilon-\eta}, \quad \text{[B.9]}$$

and use this together with [B.8] to write the derivative of marginal revenue as follows:

$$\mathcal{R}''(\mathfrak{q}) = \frac{\eta(\eta-1)\left(\mathcal{Z}(\mathfrak{q})\right)^2 - \left((\eta-\epsilon)^2 - \eta(\epsilon-1) - \epsilon(\eta-1)\right)\mathcal{Z}(\mathfrak{q}) + \epsilon(\epsilon-1)}{\mathcal{D}'\left(\mathcal{D}^{-1}(\mathfrak{q})\right)\left(\epsilon + \eta\mathcal{Z}(\mathfrak{q})\right)^2}. \quad \text{[B.10]}$$

Since $\mathcal{D}'\left(\mathcal{D}^{-1}(\mathfrak{q})\right) < 0$ for all $\mathfrak{q}$, and the remaining term in the denominator of [B.10] is strictly positive, the sign of $\mathcal{R}''(\mathfrak{q})$ is the opposite of that of the quadratic function

$$\mathscr{Q}(z) \equiv \eta(\eta-1)z^2 - \left((\eta-\epsilon)^2 - \eta(\epsilon-1) - \epsilon(\eta-1)\right)z + \epsilon(\epsilon-1), \quad \text{[B.11]}$$

evaluated at $z = \mathcal{Z}(\mathfrak{q})$. The aim is to find a region where marginal revenue is upward sloping, which corresponds to $\mathscr{Q}(z)$ being negative, which is in turn equivalent to its having positive roots (it is U-shaped because $\eta > 1$).

Under the assumptions $\epsilon > 1$ and $\eta > 1$, the product of the roots of quadratic equation $\mathscr{Q}(z) = 0$ is positive, so it has either no real roots, two negative real roots, or two positive real roots (possibly including repetitions). In the first two cases, since $\mathscr{Q}(0) = \epsilon(\epsilon-1) > 0$ it then follows that $\mathscr{Q}(z) > 0$ for all $z > 0$. To see which combinations of elasticities ϵ and η lead to positive real roots, define the following two quadratic functions of the elasticity η (for a given value of the elasticity ϵ):

$$\mathcal{G}_p(\eta;\epsilon) \equiv \eta^2 - (4\epsilon-1)\eta + \epsilon(\epsilon+1), \quad \text{and } \mathcal{G}_r(\eta;\epsilon) \equiv \eta^2 - 2(3\epsilon-1)\eta + (\epsilon+1)^2. \quad \text{[B.12]}$$

By comparing $\mathcal{G}_p(\eta;\epsilon)$ to the coefficient of z in [B.11], the sum of the roots $\mathscr{Q}(z) = 0$ is positive if and only if $\mathcal{G}_p(\eta;\epsilon) > 0$ since $\eta > 1$. Then the discriminant of the quadratic $\mathscr{Q}(z)$ is factored in terms of $\mathcal{G}_r(\eta;\epsilon)$ as follows:

$$\left((\eta-\epsilon)^2 - \eta(\epsilon-1) - \epsilon(\eta-1)\right)^2 - 4\epsilon\eta(\epsilon-1)(\eta-1) = (\eta-\epsilon)^2\mathcal{G}_r(\eta;\epsilon), \quad \text{[B.13]}$$

and as $\eta \neq \epsilon$, the equation $\mathscr{Q}(z) = 0$ has two distinct real roots if and only if $\mathcal{G}_r(\eta; \epsilon) > 0$. To summarize, the quadratic $\mathscr{Q}(z)$ has two positive real roots if and only if $\mathcal{G}_p(\eta; \epsilon) > 0$ and $\mathcal{G}_r(\eta; \epsilon) > 0$. It turns out that in the relevant parameter region $\eta > \epsilon > 1$, the binding condition is $\mathcal{G}_r(\eta; \epsilon) > 0$.

Since $\epsilon > 1$, the quadratic equations $\mathcal{G}_p(\eta; \epsilon) = 0$ and $\mathcal{G}_r(\eta; \epsilon) = 0$ in η (for a given value of ϵ) both have two distinct positive real roots (this is confirmed by verifying that the discriminants and the sums and products of the roots are all positive). Let $\eta^*(\epsilon)$ be the larger of the two roots of the equation $\mathcal{G}_r(\eta; \epsilon) = 0$:

$$\eta^*(\epsilon) = (3\epsilon - 1) + 2\sqrt{2\epsilon(\epsilon - 1)},$$

and observe that $\eta^*(\epsilon) > \epsilon$ and $\eta^{*\prime}(\epsilon) > 0$ for all $\epsilon > 1$. Since both quadratics $\mathcal{G}_p(\eta; \epsilon)$ and $\mathcal{G}_r(\eta; \epsilon)$ have positive coefficients of η^2, it follows that they are negative for all η values lying strictly between their two roots.

The difference between the two quadratic functions $\mathcal{G}_p(\eta; \epsilon)$ and $\mathcal{G}_r(\eta; \epsilon)$ in [B.12] is

$$\mathcal{G}_p(\eta; \epsilon) - \mathcal{G}_r(\eta; \epsilon) = (2\epsilon - 1)\eta - (\epsilon + 1),$$

which is a linear function of η. Thus let $\hat{\eta}(\epsilon)$ be the unique solution for η of the equation $\mathcal{G}_p(\eta; \epsilon) = \mathcal{G}_r(\eta; \epsilon)$, taking ϵ as given. As $\epsilon > 1$, such a solution exists and is unique, and $\mathcal{G}_p(\eta; \epsilon) > \mathcal{G}_r(\eta; \epsilon)$ holds if and only if $\eta > \hat{\eta}(\epsilon)$. The difference between the solution $\hat{\eta}(\epsilon)$ and ϵ is given by

$$\hat{\eta}(\epsilon) - \epsilon = \frac{2\epsilon - (2\epsilon^2 - 1)}{2\epsilon - 1}. \quad \text{[B.14]}$$

Consider first the case of ϵ values where $\hat{\eta}(\epsilon) \leq \epsilon$. This means that for all $\eta > \epsilon$, $\mathcal{G}_r(\eta; \epsilon) < \mathcal{G}_p(\eta; \epsilon)$. Since $\mathcal{G}_p(\epsilon; \epsilon) = -2\epsilon(\epsilon - 1) < 0$ for all $\epsilon > 1$, it follows that $\mathcal{G}_r(\epsilon; \epsilon) < 0$. Therefore, the smaller root of $\mathcal{G}_r(\eta; \epsilon) = 0$ is less than ϵ. This establishes that the only η values for which all the inequalities $\eta > \epsilon$, $\mathcal{G}_r(\eta; \epsilon) > 0$ and $\mathcal{G}_p(\eta; \epsilon) > 0$ hold are those satisfying $\eta > \eta^*(\epsilon)$.

Now consider what happens in the remaining case where $\hat{\eta}(\epsilon) > \epsilon$. By rearranging the terms in [B.12], notice that $\mathcal{G}_p(\eta; \epsilon) = (\eta - \epsilon)^2 - 1 - ((2\epsilon - 1)\eta - (\epsilon + 1))$. Therefore, from the definition of $\hat{\eta}(\epsilon)$, it follows that $\mathcal{G}_p(\hat{\eta}(\epsilon); \epsilon) = \mathcal{G}_r(\hat{\eta}(\epsilon); \epsilon) = (\hat{\eta}(\epsilon) - \epsilon)^2 - 1$. As $\hat{\eta}(\epsilon) > \epsilon$ in this case, equation [B.14] implies that $2\epsilon - (2\epsilon^2 - 1) > 0$, and therefore $0 < \hat{\eta}(\epsilon) - \epsilon < 1$ if $2\epsilon^2 - 1 > 1$, which is equivalent to $\epsilon^2 > 1$. This must hold since $\epsilon > 1$, and hence $(\hat{\eta}(\epsilon) - \epsilon)^2 < 1$. Thus $\mathcal{G}_p(\hat{\eta}(\epsilon); \epsilon) = \mathcal{G}_r(\hat{\eta}(\epsilon); \epsilon) < 0$. As $\mathcal{G}_p(\eta; \epsilon) > \mathcal{G}_r(\eta; \epsilon)$ holds for $\eta > \hat{\eta}(\epsilon)$, the larger of the roots of $\mathcal{G}_p(\eta; \epsilon) = 0$ lies strictly between $\hat{\eta}(\epsilon)$ and $\eta^*(\epsilon)$. Therefore in this case as well, the only values of η consistent with all the inequalities $\eta > \epsilon$, $\mathcal{G}_r(\eta; \epsilon) > 0$ and $\mathcal{G}_p(\eta; \epsilon) > 0$ are those satisfying $\eta > \eta^*(\epsilon)$.

Thus for $\eta > \epsilon > 1$, if $\eta > \eta^*(\epsilon)$ then the quadratic equation $\mathscr{Q}(z) = 0$ from [B.11] has two distinct positive real roots $\underline{z}$ and $\overline{z}$ with $\underline{z} < \overline{z}$. $\mathscr{Q}(z) < 0$ must hold for all $z \in (\underline{z}, \overline{z})$ since the coefficient of z^2 is positive. For $z \in [0, \underline{z})$ or $z \in (\overline{z}, \infty)$, the quadratic satisfies $\mathscr{Q}(z) > 0$. If $\eta \leq \eta^*(\epsilon)$ then $\mathscr{Q}(z) > 0$ for all z (except at a single isolated point when $\eta = \eta^*(\epsilon)$ exactly). Therefore, in the case where $\eta \leq \eta^*(\epsilon)$, it follows from [B.10] and [B.11] that $\mathcal{R}'(\mathfrak{q})$ is strictly decreasing for all $\mathfrak{q} \geq 0$.

Now restrict attention to the case where $\eta > \eta^*(\epsilon)$. Since $0 < \lambda < 1$, $\eta > \epsilon$, and the inverse demand function $\mathcal{D}^{-1}(\mathfrak{q})$ is strictly decreasing, the function $\mathcal{Z}(\mathfrak{q})$ defined in [B.9] is strictly increasing. Its inverse is

$$\mathcal{Z}^{-1}(z) = \mathcal{D}\left(\left(\frac{\lambda}{1 - \lambda} z\right)^{\frac{1}{\epsilon - \eta}}\right), \quad \text{[B.15]}$$

which is also a strictly increasing function. Define $\underline{\mathfrak{q}} \equiv \mathcal{Z}^{-1}(\underline{z})$ and $\overline{\mathfrak{q}} \equiv \mathcal{Z}^{-1}(\overline{z})$ using the roots $\underline{z}$ and $\overline{z}$ of the quadratic equation $\mathscr{Q}(z) = 0$. From [B.10] and [B.11] it follows that $\mathcal{R}''(\underline{\mathfrak{q}}) = 0$ and $\mathcal{R}''(\overline{\mathfrak{q}}) = 0$. As $\mathcal{Z}^{-1}(z)$ is a strictly increasing function, $\mathcal{R}'(\mathfrak{q})$ must be strictly decreasing for $0 < \mathfrak{q} < \underline{\mathfrak{q}}$ and $\mathfrak{q} > \overline{\mathfrak{q}}$, and strictly increasing for $\underline{\mathfrak{q}} < \mathfrak{q} < \overline{\mathfrak{q}}$. The condition $\eta > \eta^*(\epsilon)$ is the same as that given in [16], so this completes the proof. ∎

When the marginal revenue function $\mathcal{R}'(\mathfrak{q})$ is non-monotonic, the following result provides the foundation for verifying the existence and uniqueness of the two-price equilibrium.

Lemma 2 *Given the revenue function $\mathcal{R}(\mathfrak{q})$ defined in* [B.4], *suppose that $0 < \lambda < 1$, and ϵ and η are such that non-monotonicity condition* [16] *holds.*

(i) There exist unique values $\mathfrak{q}_S$ and $\mathfrak{q}_N$ such that $0 < \mathfrak{q}_N < \mathfrak{q}_S < \infty$ which satisfy the equations

$$\mathcal{R}'(\mathfrak{q}_S) = \mathcal{R}'(\mathfrak{q}_N) = \frac{\mathcal{R}(\mathfrak{q}_S) - \mathcal{R}(\mathfrak{q}_N)}{\mathfrak{q}_S - \mathfrak{q}_N}. \tag{B.16}$$

(ii) The solutions $\mathfrak{q}_S$ and $\mathfrak{q}_N$ of the above equations must also satisfy the inequalities

$$\mathcal{R}''(\mathfrak{q}_S) < 0, \quad \mathcal{R}''(\mathfrak{q}_N) < 0, \quad \mathcal{R}(\mathfrak{q}_S)/\mathfrak{q}_S > \mathcal{R}'(\mathfrak{q}_S), \quad \text{and } \mathcal{R}(\mathfrak{q}_N)/\mathfrak{q}_N > \mathcal{R}'(\mathfrak{q}_N). \tag{B.17}$$

(iii) The following inequality holds for all $\mathfrak{q} \geq 0$:

$$\mathcal{R}(\mathfrak{q}) \leq \mathcal{R}(\mathfrak{q}_S) + \mathcal{R}'(\mathfrak{q}_S)(\mathfrak{q} - \mathfrak{q}_S) = \mathcal{R}(\mathfrak{q}_N) + \mathcal{R}'(\mathfrak{q}_N)(\mathfrak{q} - \mathfrak{q}_N). \tag{B.18}$$

PROOF (i) When $0 < \lambda < 1$ and condition [16] hold then Lemma 1 demonstrates that there exist $\underline{\mathfrak{q}}$ and $\bar{\mathfrak{q}}$ such that $0 < \underline{\mathfrak{q}} < \bar{\mathfrak{q}} < \infty$ and $\mathcal{R}''(\underline{\mathfrak{q}}) = \mathcal{R}''(\bar{\mathfrak{q}}) = 0$. Define $\underline{\mathcal{R}'} \equiv \mathcal{R}'(\underline{\mathfrak{q}})$ and $\overline{\mathcal{R}'} \equiv \mathcal{R}'(\bar{\mathfrak{q}})$. Since Lemma 1 also shows that $\mathcal{R}'(\mathfrak{q})$ is strictly increasing between $\underline{\mathfrak{q}}$ and $\bar{\mathfrak{q}}$, it follows that $\underline{\mathcal{R}'} < \overline{\mathcal{R}'}$.

The function $\mathcal{R}'(\mathfrak{q})$ is continuously differentiable for all $\mathfrak{q} > 0$ and $\lim_{\mathfrak{q}\to 0} \mathcal{R}'(\mathfrak{q}) = \infty$. Hence there must exist a value $\underline{\mathfrak{q}}_1$ such that $\mathcal{R}'(\underline{\mathfrak{q}}_1) = \overline{\mathcal{R}'}$ and $\underline{\mathfrak{q}}_1 < \underline{\mathfrak{q}}$. Define $\bar{\mathfrak{q}}_1 \equiv \underline{\mathfrak{q}}$. According to Lemma 1, the function $\mathcal{R}'(\mathfrak{q})$ is strictly decreasing on the interval $[\underline{\mathfrak{q}}_1, \bar{\mathfrak{q}}_1]$ and thus has range $[\underline{\mathcal{R}'}, \overline{\mathcal{R}'}]$.

Define $\underline{\mathfrak{q}}_2 \equiv \underline{\mathfrak{q}}$ and $\bar{\mathfrak{q}}_2 \equiv \bar{\mathfrak{q}}$. Given the construction of $\underline{\mathcal{R}'}$ and $\overline{\mathcal{R}'}$ and the fact that $\mathcal{R}'(\mathfrak{q})$ is strictly increasing on $[\underline{\mathfrak{q}}_2, \bar{\mathfrak{q}}_2]$, the range of $\mathcal{R}'(\mathfrak{q})$ is $[\underline{\mathcal{R}'}, \overline{\mathcal{R}'}]$ on this interval.

Now define $\underline{\mathfrak{q}}_3 \equiv \bar{\mathfrak{q}}$. Since $\lim_{\mathfrak{q}\to\infty} \mathcal{R}'(\mathfrak{q}) = 0$ and $\mathcal{R}'(\mathfrak{q})$ is continuously differentiable, there must exist a $\bar{\mathfrak{q}}_3$ such that $\mathcal{R}'(\bar{\mathfrak{q}}_3) = \underline{\mathcal{R}'}$ and $\bar{\mathfrak{q}}_3 > \underline{\mathfrak{q}}_3$. Lemma 1 shows that $\mathcal{R}'(\mathfrak{q})$ is strictly decreasing on $[\underline{\mathfrak{q}}_3, \bar{\mathfrak{q}}_3]$ and so has range $[\underline{\mathcal{R}'}, \overline{\mathcal{R}'}]$ on this interval.

For each $\varkappa \in [0, 1]$, define the function $\mathfrak{q}_1(\varkappa)$ as follows:

$$\mathfrak{q}_1(\varkappa) \equiv (1 - \varkappa)\underline{\mathfrak{q}}_1 + \varkappa\bar{\mathfrak{q}}_1, \tag{B.19}$$

in other words, as a convex combination of $\underline{\mathfrak{q}}_1$ and $\bar{\mathfrak{q}}_1$. Note that $\mathfrak{q}_1(\varkappa)$ is strictly increasing in $\varkappa$. The construction of this function, the monotonicity of $\mathcal{R}'(\mathfrak{q})$ on $[\underline{\mathfrak{q}}_1, \bar{\mathfrak{q}}_1]$, and the definitions of $\underline{\mathcal{R}'}$ and $\overline{\mathcal{R}'}$ guarantee that $\underline{\mathcal{R}'} \leq \mathcal{R}'(\mathfrak{q}_1(\varkappa)) \leq \overline{\mathcal{R}'}$ for all $\varkappa \in [0, 1]$. Given that the function $\mathcal{R}'(\mathfrak{q})$ is also strictly monotonic on each of the intervals $[\underline{\mathfrak{q}}_2, \bar{\mathfrak{q}}_2]$ and $[\underline{\mathfrak{q}}_3, \bar{\mathfrak{q}}_3]$, and has range $[\underline{\mathcal{R}'}, \overline{\mathcal{R}'}]$ on both, there must exist unique values $\mathfrak{q}_2 \in [\underline{\mathfrak{q}}_2, \bar{\mathfrak{q}}_2]$ and $\mathfrak{q}_3 \in [\underline{\mathfrak{q}}_3, \bar{\mathfrak{q}}_3]$ such that $\mathcal{R}'(\mathfrak{q}_2) = \mathcal{R}'(\mathfrak{q}_3) = \mathcal{R}'(\mathfrak{q}_1(\varkappa))$ for any particular $\varkappa$. Hence define the functions $\mathfrak{q}_2(\varkappa)$ and $\mathfrak{q}_3(\varkappa)$ to give these values in the two intervals for each specific $\varkappa \in [0, 1]$:

$$\mathcal{R}'(\mathfrak{q}_1(\varkappa)) \equiv \mathcal{R}'(\mathfrak{q}_2(\varkappa)) \equiv \mathcal{R}'(\mathfrak{q}_3(\varkappa)). \tag{B.20}$$

At the endpoints of the intervals (corresponding to $\varkappa = 0$ and $\varkappa = 1$) note that

$$\mathfrak{q}_2(0) = \mathfrak{q}_3(0) = \bar{\mathfrak{q}}, \quad \text{and } \mathfrak{q}_1(1) = \mathfrak{q}_2(1) = \underline{\mathfrak{q}}. \tag{B.21}$$

Continuity and differentiability of $\mathcal{R}'(\mathfrak{q})$ and of $\mathfrak{q}_1(\varkappa)$ from [B.19] guarantee that $\mathfrak{q}_2(\varkappa)$ and $\mathfrak{q}_3(\varkappa)$ are continuous for all $\varkappa \in [0, 1]$ and differentiable for all $\varkappa \in (0, 1)$. By differentiating [B.20] it follows that

$$\mathfrak{q}_2'(\varkappa) = \frac{\mathcal{R}''(\mathfrak{q}_1(\varkappa))}{\mathcal{R}''(\mathfrak{q}_2(\varkappa))}\mathfrak{q}_1'(\varkappa), \quad \text{and } \mathfrak{q}_3'(\varkappa) = \frac{\mathcal{R}''(\mathfrak{q}_1(\varkappa))}{\mathcal{R}''(\mathfrak{q}_3(\varkappa))}\mathfrak{q}_1'(\varkappa).$$

As Lemma 1 establishes $\mathcal{R}(\mathfrak{q})$ is concave on $[\underline{\mathfrak{q}}_1, \bar{\mathfrak{q}}_1]$ and $[\underline{\mathfrak{q}}_3, \bar{\mathfrak{q}}_3]$, and convex on $[\underline{\mathfrak{q}}_2, \bar{\mathfrak{q}}_2]$, the results above show that $\mathfrak{q}_2'(\varkappa) < 0$ and $\mathfrak{q}_3'(\varkappa) > 0$ for all $\varkappa \in (0, 1)$.

Existence

For each $\varkappa \in [0,1]$, define the function $\digamma(\varkappa)$ in terms of the following integrals:

$$\digamma(\varkappa) \equiv \int_{\mathfrak{q}_2(\varkappa)}^{\mathfrak{q}_3(\varkappa)} \left(\mathcal{R}'(\mathfrak{q}) - \mathcal{R}'(\mathfrak{q}_2(\varkappa))\right) d\mathfrak{q} - \int_{\mathfrak{q}_1(\varkappa)}^{\mathfrak{q}_2(\varkappa)} \left(\mathcal{R}'(\mathfrak{q}_2(\varkappa)) - \mathcal{R}'(\mathfrak{q})\right) d\mathfrak{q}. \qquad \text{[B.22]}$$

From continuity and differentiability of $\mathfrak{q}_1(\varkappa)$, $\mathfrak{q}_2(\varkappa)$ and $\mathfrak{q}_3(\varkappa)$, it follows that $\digamma(\varkappa)$ is also continuous for all $\varkappa \in [0,1]$ and differentiable for all $\varkappa \in (0,1)$. Evaluating $\digamma(\varkappa)$ at the endpoints of the interval $[0,1]$ and making use of [B.21] yields

$$\digamma(0) = -\int_{\underline{\mathfrak{q}}_1}^{\overline{\mathfrak{q}}_2} \left(\overline{\mathcal{R}'} - \mathcal{R}'(\mathfrak{q})\right) d\mathfrak{q} < 0, \quad \text{and } \digamma(1) = \int_{\underline{\mathfrak{q}}_2}^{\overline{\mathfrak{q}}_3} \left(\mathcal{R}'(\mathfrak{q}) - \underline{\mathcal{R}'}\right) d\mathfrak{q} > 0,$$

where the first inequality follows because $\mathcal{R}'(\mathfrak{q}) < \overline{\mathcal{R}'}$ for all $\underline{\mathfrak{q}}_1 < \mathfrak{q} < \overline{\mathfrak{q}}_2$, and the second because $\mathcal{R}'(\mathfrak{q}) > \underline{\mathcal{R}'}$ for all $\underline{\mathfrak{q}}_2 < \mathfrak{q} < \overline{\mathfrak{q}}_3$. Differentiating $\digamma(\varkappa)$ in [B.22] using Leibniz's rule and substituting the definitions from [B.20] leads to the following result:

$$\digamma'(\varkappa) = -(\mathfrak{q}_3(\varkappa) - \mathfrak{q}_1(\varkappa))\mathfrak{q}_2'(\varkappa)\mathcal{R}''(\mathfrak{q}_2(\varkappa)) > 0,$$

for all $\varkappa \in (0,1)$ since $\mathfrak{q}_3(\varkappa) > \mathfrak{q}_1(\varkappa)$, $\mathfrak{q}_2'(\varkappa) < 0$, and $\mathcal{R}''(\mathfrak{q}_2(\varkappa)) > 0$ from Lemma 1. Therefore, because $\digamma(0) < 0$, $\digamma(1) > 0$, and $\digamma(\varkappa)$ is continuous and strictly increasing in $\varkappa$, there exists a unique $\varkappa^* \in (0,1)$ such that $\digamma(\varkappa^*) = 0$.

The solution of the system of equations [B.16] is found by setting $\mathfrak{q}_N \equiv \mathfrak{q}_1(\varkappa^*)$ and $\mathfrak{q}_S \equiv \mathfrak{q}_3(\varkappa^*)$, using the solution $\varkappa = \varkappa^*$ of the equation $\digamma(\varkappa) = 0$ obtained above. From [B.20], it follows immediately that $\mathcal{R}'(\mathfrak{q}_N) = \mathcal{R}'(\mathfrak{q}_S)$, establishing the first equality in [B.16]. Since $\digamma(\varkappa^*) = 0$, the definition of $\digamma(\varkappa)$ in equation [B.22] implies

$$\int_{\mathfrak{q}_2(\varkappa^*)}^{\mathfrak{q}_S} \left(\mathcal{R}'(\mathfrak{q}) - \mathcal{R}'(\mathfrak{q}_2(\varkappa^*))\right) d\mathfrak{q} = \int_{\mathfrak{q}_N}^{\mathfrak{q}_2(\varkappa^*)} \left(\mathcal{R}'(\mathfrak{q}_2(\varkappa^*)) - \mathcal{R}'(\mathfrak{q})\right) d\mathfrak{q}, \qquad \text{[B.23]}$$

which is rearranged to deduce

$$\int_{\mathfrak{q}_N}^{\mathfrak{q}_S} \mathcal{R}'(\mathfrak{q}) d\mathfrak{q} = (\mathfrak{q}_S - \mathfrak{q}_N)\mathcal{R}'(\mathfrak{q}_2(\varkappa^*)). \qquad \text{[B.24]}$$

Equation [B.20] implies $\mathcal{R}'(\mathfrak{q}_2(\varkappa^*)) = \mathcal{R}'(\mathfrak{q}_N) = \mathcal{R}'(\mathfrak{q}_S)$, which together with the above establishes that

$$\mathcal{R}'(\mathfrak{q}_S) = \mathcal{R}'(\mathfrak{q}_N) = \frac{\mathcal{R}(\mathfrak{q}_S) - \mathcal{R}(\mathfrak{q}_N)}{\mathfrak{q}_S - \mathfrak{q}_N}. \qquad \text{[B.25]}$$

Thus, the values of $\mathfrak{q}_N$ and $\mathfrak{q}_S$ are indeed a solution of the system of equations in [B.16].

Uniqueness

First note that given the monotonicity of $\mathcal{R}'(\mathfrak{q})$ on the intervals $[0, \underline{\mathfrak{q}}]$ and $[\overline{\mathfrak{q}}, \infty)$, and using the fact that the range of $\mathcal{R}'(\mathfrak{q})$ is $[\underline{\mathcal{R}'}, \overline{\mathcal{R}'}]$ on $[\underline{\mathfrak{q}}_1, \overline{\mathfrak{q}}_1]$, $[\underline{\mathfrak{q}}_2, \overline{\mathfrak{q}}_2]$ and $[\underline{\mathfrak{q}}_3, \overline{\mathfrak{q}}_3]$, it follows that no solution of [B.16] can be found in either $[0, \underline{\mathfrak{q}}_1)$ or $(\overline{\mathfrak{q}}_3, \infty)$ since marginal revenue needs to be equalized at two quantities. Furthermore, as the definitions of the functions $\mathfrak{q}_1(\varkappa)$, $\mathfrak{q}_2(\varkappa)$ and $\mathfrak{q}_3(\varkappa)$ in [B.20] make clear, it is necessary that those two quantities correspond to two out of the three of $\mathfrak{q}_1(\varkappa)$, $\mathfrak{q}_2(\varkappa)$ and $\mathfrak{q}_3(\varkappa)$ for some particular $\varkappa \in [0,1]$ if marginal revenue is to be equalized at two distinct points.

In addition to equalizing marginal revenue, the solution $\mathfrak{q}_S$ and $\mathfrak{q}_N$ must satisfy the second equality in [B.16]. If $\mathfrak{q}_N$ is set equal to $\mathfrak{q}_1(\varkappa)$ and $\mathfrak{q}_S$ equal to $\mathfrak{q}_3(\varkappa)$ for the same value of $\varkappa \in [0,1]$ then equations [B.23] and [B.24] show that the second equality in [B.16] requires $\digamma(\varkappa) = 0$. But it has already been demonstrated that there is one and only one solution of this equation.

Now consider the alternative choices of setting $\mathfrak{q}_N$ to $\mathfrak{q}_1(\varkappa)$ and $\mathfrak{q}_S$ to $\mathfrak{q}_2(\varkappa)$ for some common $\varkappa \in [0,1]$, or to $\mathfrak{q}_2(\varkappa)$ and $\mathfrak{q}_3(\varkappa)$ respectively, again for some common value of $\varkappa$. Since [B.20] holds by construction, and the function $\mathcal{R}'(\mathfrak{q})$ is strictly decreasing on the intervals $[\underline{\mathfrak{q}}_1, \overline{\mathfrak{q}}_1]$ and $[\underline{\mathfrak{q}}_3, \overline{\mathfrak{q}}_3]$, and strictly increasing on $[\underline{\mathfrak{q}}_2, \overline{\mathfrak{q}}_2]$, it follows that

$$\int_{\mathfrak{q}_1(\varkappa)}^{\mathfrak{q}_2(\varkappa)} \mathcal{R}'(\mathfrak{q})d\mathfrak{q} < (\mathfrak{q}_2(\varkappa) - \mathfrak{q}_1(\varkappa))\mathcal{R}'(\mathfrak{q}_2(\varkappa)), \quad \text{and} \quad \int_{\mathfrak{q}_2(\varkappa)}^{\mathfrak{q}_3(\varkappa)} \mathcal{R}'(\mathfrak{q})d\mathfrak{q} > (\mathfrak{q}_3(\varkappa) - \mathfrak{q}_2(\varkappa))\mathcal{R}'(\mathfrak{q}_2(\varkappa)),$$

and hence both inequalities $\mathcal{R}(\mathfrak{q}_2(\varkappa)) - \mathcal{R}(\mathfrak{q}_1(\varkappa)) < (\mathfrak{q}_2(\varkappa) - \mathfrak{q}_1(\varkappa))\mathcal{R}'(\mathfrak{q}_2(\varkappa))$ and $\mathcal{R}(\mathfrak{q}_3(\varkappa)) - \mathcal{R}(\mathfrak{q}_2(\varkappa)) > (\mathfrak{q}_3(\varkappa) - \mathfrak{q}_2(\varkappa))\mathcal{R}'(\mathfrak{q}_2(\varkappa))$ must hold for every $\varkappa \in [0, 1]$. Consequently, there is no way that all three equations in [B.25] can hold except by setting $\mathfrak{q}_N = \mathfrak{q}_1(\varkappa^*)$ and $\mathfrak{q}_S = \mathfrak{q}_3(\varkappa^*)$. Therefore the solution of [B.16] constructed above is unique.

(ii) Lemma 1 shows that $\mathcal{R}(\mathfrak{q})$ is a strictly concave function on the intervals $[0, \underline{\mathfrak{q}}]$ and $[\overline{\mathfrak{q}}, \infty)$. The argument above demonstrating the existence and uniqueness of the solution establishes that $\mathfrak{q}_N$ and $\mathfrak{q}_S$ must lie respectively in the intervals $(\underline{\mathfrak{q}}_1, \overline{\mathfrak{q}}_1)$ and $(\underline{\mathfrak{q}}_3, \overline{\mathfrak{q}}_3)$, which are themselves contained in $[0, \underline{\mathfrak{q}}]$ and $[\overline{\mathfrak{q}}, \infty)$ respectively. Together these findings imply $\mathcal{R}''(\mathfrak{q}_N) < 0$ and $\mathcal{R}''(\mathfrak{q}_S) < 0$, and that the following inequalities must hold:

$$\mathcal{R}(\mathfrak{q}) \leq \mathcal{R}(\mathfrak{q}_N) + \mathcal{R}'(\mathfrak{q}_N)(\mathfrak{q} - \mathfrak{q}_N) \quad \forall \mathfrak{q} \in [0, \underline{\mathfrak{q}}], \quad \text{and} \quad \mathcal{R}(\mathfrak{q}) \leq \mathcal{R}(\mathfrak{q}_S) + \mathcal{R}'(\mathfrak{q}_S)(\mathfrak{q} - \mathfrak{q}_S) \quad \forall \mathfrak{q} \in [\overline{\mathfrak{q}}, \infty), \tag{B.26}$$

where the inequalities are strict for $\mathfrak{q} \neq \mathfrak{q}_N$ and $\mathfrak{q} \neq \mathfrak{q}_S$ respectively. Note that an implication of the equations characterizing $\mathfrak{q}_S$ and $\mathfrak{q}_N$ in [B.16] is

$$\mathcal{R}(\mathfrak{q}_S) - \mathcal{R}'(\mathfrak{q}_S)\mathfrak{q}_S = \mathcal{R}(\mathfrak{q}_N) - \mathcal{R}'(\mathfrak{q}_N)\mathfrak{q}_N. \tag{B.27}$$

By evaluating the first inequality in [B.26] at $\mathfrak{q} = 0$, where $\mathcal{R}(0) = 0$, and making use of the equation above it is deduced that

$$\mathcal{R}(\mathfrak{q}_S) - \mathcal{R}'(\mathfrak{q}_S)\mathfrak{q}_S > 0, \quad \text{and} \quad \mathcal{R}(\mathfrak{q}_N) - \mathcal{R}'(\mathfrak{q}_N)\mathfrak{q}_N > 0,$$

and thus $\mathcal{R}(\mathfrak{q}_S)/\mathfrak{q}_S > \mathcal{R}'(\mathfrak{q}_S)$ and $\mathcal{R}(\mathfrak{q}_N)/\mathfrak{q}_N > \mathcal{R}'(\mathfrak{q}_N)$. This confirms all the inequalities given in [B.17].

(iii) By applying the inequalities in [B.26] at the endpoints $\underline{\mathfrak{q}}$ and $\overline{\mathfrak{q}}$ of the intervals $[0, \underline{\mathfrak{q}}]$ and $[\overline{\mathfrak{q}}, \infty)$ it follows that:

$$\mathcal{R}(\underline{\mathfrak{q}}) \leq \mathcal{R}(\mathfrak{q}_N) + \mathcal{R}'(\mathfrak{q}_N)(\underline{\mathfrak{q}} - \mathfrak{q}_N), \quad \text{and} \quad \mathcal{R}(\overline{\mathfrak{q}}) \leq \mathcal{R}(q_N) + \mathcal{R}'(\mathfrak{q}_N)(\overline{\mathfrak{q}} - \mathfrak{q}_N). \tag{B.28}$$

Now take any $\mathfrak{q} \in (\underline{\mathfrak{q}}, \overline{\mathfrak{q}})$ and note that because Lemma 1 demonstrates $\mathcal{R}(\mathfrak{q})$ is a convex function on this interval:

$$\mathcal{R}(\mathfrak{q}) \equiv \mathcal{R}\left(\left(\frac{\overline{\mathfrak{q}} - \mathfrak{q}}{\overline{\mathfrak{q}} - \underline{\mathfrak{q}}}\right)\underline{\mathfrak{q}} + \left(\frac{\mathfrak{q} - \underline{\mathfrak{q}}}{\overline{\mathfrak{q}} - \underline{\mathfrak{q}}}\right)\overline{\mathfrak{q}}\right) \leq \left(\frac{\overline{\mathfrak{q}} - \mathfrak{q}}{\overline{\mathfrak{q}} - \underline{\mathfrak{q}}}\right)\mathcal{R}(\underline{\mathfrak{q}}) + \left(\frac{\mathfrak{q} - \underline{\mathfrak{q}}}{\overline{\mathfrak{q}} - \underline{\mathfrak{q}}}\right)\mathcal{R}(\overline{\mathfrak{q}}), \tag{B.29}$$

using the fact that the coefficients of $\mathcal{R}(\underline{\mathfrak{q}})$ and $\mathcal{R}(\overline{\mathfrak{q}})$ in the above are positive and sum to one. A weighted average of the two inequalities in [B.28] using as weights the coefficients from [B.29] yields $\mathcal{R}(\mathfrak{q}) \leq \mathcal{R}(\mathfrak{q}_N) + \mathcal{R}'(\mathfrak{q}_N)(\mathfrak{q} - \mathfrak{q}_N)$ for all $\mathfrak{q} \in (\underline{\mathfrak{q}}, \overline{\mathfrak{q}})$. This finding, together with the inequalities in [B.26] and the equations [B.25] and [B.27], implies:

$$\mathcal{R}(\mathfrak{q}) \leq \mathcal{R}(\mathfrak{q}_S) + \mathcal{R}'(\mathfrak{q}_S)(\mathfrak{q} - \mathfrak{q}_S) = \mathcal{R}(\mathfrak{q}_N) + \mathcal{R}'(\mathfrak{q}_N)(\mathfrak{q} - \mathfrak{q}_N)$$

for all $\mathfrak{q} \geq 0$. Thus [B.18] is established, which completes the proof. ■

The existence and uniqueness of the solution of equations [B.16] has been demonstrated given condition [16] for the non-monotonicity of the marginal revenue function $\mathcal{R}'(\mathfrak{q})$. A method for computing this solution and a characterization of which parameters it depends upon is provided in the following result.

Lemma 3 *Let $\mathfrak{q}_S$ and $\mathfrak{q}_N$ be the solution of equations* [B.16] *(under the conditions assumed in Lemma 2), and let $\rho_N \equiv \mathcal{D}^{-1}(\mathfrak{q}_N)$ and $\rho_S \equiv \mathcal{D}^{-1}(\mathfrak{q}_S)$ be the corresponding relative prices consistent with the demand function* [B.1]. *In addition, define the markup ratio $\mu \equiv \mu_S/\mu_N = \rho_S/\rho_N$ and the quantity ratio $\chi \equiv \mathfrak{q}_S/\mathfrak{q}_N$.*

(i) The markup ratio $\mu \equiv \rho_S/\rho_N$ *is the only solution of the equation* $\mathfrak{R}(\mu; \epsilon, \eta) = 0$ *from* [A.1] *with* $0 < \mu < 1$ *and where* $\mathfrak{z}(\mu; \epsilon, \eta)$ *in* [A.3] *is a positive real number. Thus* μ *depends only on parameters* ϵ *and* η*.*

(ii) Given the value of μ *satisfying* $\mathfrak{R}(\mu; \epsilon, \eta) = 0$*, the quantity ratio* $\chi \equiv \mathfrak{q}_S/\mathfrak{q}_N$ *is equal to the expression in equation* [A.4]*. Hence* χ *depends only on parameters* ϵ *and* η*.*

(iii) The equilibrium markups μ_S *and* μ_N *from* [18] *depend only on* ϵ *and* η *and are given by*

$$\mu_S = \frac{\epsilon + \eta\mu^{-(\eta-\epsilon)}\mathfrak{z}(\mu; \epsilon, \eta)}{(\epsilon - 1) + (\eta - 1)\mu^{-(\eta-\epsilon)}\mathfrak{z}(\mu; \epsilon, \eta)}, \quad \text{and} \quad \mu_N = \frac{\epsilon + \eta\mathfrak{z}(\mu; \epsilon, \eta)}{(\epsilon - 1) + (\eta - 1)\mathfrak{z}(\mu; \epsilon, \eta)}, \qquad \text{[B.30]}$$

where the function $\mathfrak{z}(\mu; \epsilon, \eta)$ *is given in* [A.3]*.*

(iv) The equilibrium values of ρ_N*,* ρ_S*,* $\mathfrak{q}_N$ *and* $\mathfrak{q}_S$ *depend on parameters* ϵ*,* η *and* λ *and are obtained as follows:*

$$\rho_N = \left(\frac{\lambda}{1-\lambda}\mathfrak{z}(\mu; \epsilon, \eta)\right)^{-\frac{1}{\eta-\epsilon}}, \quad \text{and} \quad \rho_S = \left(\frac{\lambda}{1-\lambda}\mathfrak{z}(\mu; \epsilon, \eta)\right)^{-\frac{1}{\eta-\epsilon}}\mu, \qquad \text{[B.31]}$$

where $\mathfrak{q}_N = \mathcal{D}(\rho_N)$ *and* $\mathfrak{q}_S = \mathcal{D}(\rho_S)$ *using the relative demand function* $\mathcal{D}(\rho)$ *from* [B.1]*.*

Proof (i) Using the expression for marginal revenue from [B.5], the first equality in [B.16] is equivalent to the requirement that

$$\left(\frac{\lambda(\epsilon - 1) + (1-\lambda)(\eta - 1)\rho_N^{\epsilon-\eta}}{\lambda\epsilon + (1-\lambda)\eta\rho_N^{\epsilon-\eta}}\right)\rho_N = \left(\frac{\lambda(\epsilon - 1) + (1-\lambda)(\eta - 1)\rho_S^{\epsilon-\eta}}{\lambda\epsilon + (1-\lambda)\eta\rho_S^{\epsilon-\eta}}\right)\rho_S.$$

By dividing numerator and denominator of the above by λ, defining $z \equiv ((1-\lambda)/\lambda)\rho_N^{\epsilon-\eta}$, and restating the resulting equation in terms of $\mu = \rho_S/\rho_N$ and z it follows that

$$\mu = \left(\frac{\epsilon + \eta\mu^{-(\eta-\epsilon)}z}{\epsilon + \eta z}\right)\left(\frac{(\epsilon - 1) + (\eta - 1)z}{(\epsilon - 1) + (\eta - 1)\mu^{-(\eta-\epsilon)}z}\right). \qquad \text{[B.32]}$$

Since $\rho_S < \rho_N$ the markup ratio satisfies $0 < \mu < 1$, and thus neither of the denominators of the fractions above can be zero. Hence for a given value of μ, equation [B.32] is rearranged to obtain a quadratic equation in z:

$$\eta(\eta - 1)\mu^{-(\eta-\epsilon)}(1-\mu)z^2 + \left(\epsilon(\eta - 1)\left(1 - \mu^{1-(\eta-\epsilon)}\right) + \eta(\epsilon - 1)\left(\mu^{-(\eta-\epsilon)} - \mu\right)\right)z + \epsilon(\epsilon - 1)(1-\mu) = 0,$$

which as $0 < \mu < 1$ is in turn multiplied on both sides by $\mu^{\eta-\epsilon}/(1-\mu)$ to obtain an equivalent quadratic:

$$\eta(\eta - 1)z^2 + \left(\eta(\epsilon - 1)\left(\frac{1 - \mu^{\eta-\epsilon+1}}{1-\mu}\right) + \epsilon(\eta - 1)\left(\frac{\mu^{\eta-\epsilon} - \mu}{1-\mu}\right)\right)z + \epsilon(\epsilon - 1)\mu^{\eta-\epsilon} = 0. \qquad \text{[B.33]}$$

This quadratic is denoted by $\mathfrak{Q}(z; \mu, \epsilon, \eta) \equiv \mathfrak{a}_0(\mu; \epsilon, \eta) + \mathfrak{a}_1(\mu; \epsilon, \eta)z + \mathfrak{a}_2(\eta)z^2$, where the coefficient functions $\mathfrak{a}_0(\mu; \epsilon, \eta)$, $\mathfrak{a}_1(\mu; \epsilon, \eta)$ and $\mathfrak{a}_2(\eta)$ listed in [A.2] are obtained directly from [B.33].

Now note that $\mathcal{R}(\mathfrak{q}_N) - \mathfrak{q}_N\mathcal{R}'(\mathfrak{q}_N) = \mathcal{R}(\mathfrak{q}_S) - \mathfrak{q}_S\mathcal{R}'(\mathfrak{q}_S)$ is deduced by rearranging the equations in [B.16]. The definition of the revenue function $\mathcal{R}(\mathfrak{q})$ in [B.4] shows that $\mathcal{R}(\mathcal{D}(\rho)) = \rho\mathcal{D}(\rho)$ is a valid alternative expression for all $\rho > 0$. By combining these two observations and substituting $\mathfrak{q}_S = \mathcal{D}(\rho_S)$ and $\mathfrak{q}_N = \mathcal{D}(\rho_N)$, the relative prices and quantities must satisfy

$$\mathfrak{q}_S\left(\rho_S - \mathcal{R}'(\mathfrak{q}_S)\right) = \mathfrak{q}_N\left(\rho_N - \mathcal{R}'(\mathfrak{q}_N)\right). \qquad \text{[B.34]}$$

After expressing this in terms of the quantity ratio $\chi \equiv \mathfrak{q}_S/\mathfrak{q}_N$ and dividing both sides by $\mathcal{R}'(\mathfrak{q}_S) = \mathcal{R}'(\mathfrak{q}_N)$

(justified by [B.16]), equation [B.34] becomes

$$\chi = \left(\frac{\rho_N}{\mathcal{R}'\left(\mathcal{D}(\rho_N)\right)} - 1\right) \Big/ \left(\frac{\rho_S}{\mathcal{R}'\left(\mathcal{D}(\rho_S)\right)} - 1\right). \tag{B.35}$$

The formula for marginal revenue $\mathcal{R}'(\mathcal{D}(\rho))$ in [B.5] is then rearranged to show

$$\frac{\rho}{\mathcal{R}'\left(\mathcal{D}(\rho)\right)} - 1 = \frac{\lambda + (1-\lambda)\rho^{\epsilon-\eta}}{\lambda(\epsilon-1) + (\eta-1)(1-\lambda)\rho^{\epsilon-\eta}},$$

which is substituted into [B.35] to obtain

$$\chi = \left(\frac{\lambda + (1-\lambda)\rho_N^{\epsilon-\eta}}{\lambda + (1-\lambda)\rho_S^{\epsilon-\eta}}\right)\left(\frac{(\epsilon-1)\lambda + (\eta-1)(1-\lambda)\rho_S^{\epsilon-\eta}}{(\epsilon-1)\lambda + (\eta-1)(1-\lambda)\rho_N^{\epsilon-\eta}}\right).$$

By dividing numerator and denominator of both fractions by λ and recalling $\mu = \rho_S/\rho_N$ and the definition $z \equiv ((1-\lambda)/\lambda)\rho_N^{\epsilon-\eta}$, this equation is equivalent to

$$\chi = \left(\frac{1+z}{1+\mu^{-(\eta-\epsilon)}z}\right)\left(\frac{(\epsilon-1) + (\eta-1)\mu^{-(\eta-\epsilon)}z}{(\epsilon-1)+(\eta-1)z}\right). \tag{B.36}$$

The quantity ratio is then written as $\chi = \mathcal{D}(\rho_S)/\mathcal{D}(\rho_N)$ using the relative demand function $\mathfrak{q} = \mathcal{R}(\rho)$ from equation [B.1], and thus

$$\chi = \frac{\lambda\rho_S^{-\epsilon} + (1-\lambda)\rho_S^{-\eta}}{\lambda\rho_N^{-\epsilon} + (1-\lambda)\rho_N^{-\eta}}.$$

By factorizing $\lambda\rho_S^{-\epsilon}$ and $\lambda\rho_N^{-\epsilon}$ from the numerator and denominator respectively, and using $\mu = \rho_S/\rho_N$ and the definition $z \equiv ((1-\lambda)/\lambda)\rho_N^{\epsilon-\eta}$, the above expression for χ becomes

$$\chi = \mu^{-\epsilon}\left(\frac{1+\mu^{-(\eta-\epsilon)}z}{1+z}\right). \tag{B.37}$$

Putting together the two expressions for the quantity ratio χ in [B.36] and [B.37], μ and z must satisfy the equation

$$\left(\frac{1+z}{1+\mu^{-(\eta-\epsilon)}z}\right)\left(\frac{(\epsilon-1) + (\eta-1)\mu^{-(\eta-\epsilon)}z}{(\epsilon-1)+(\eta-1)z}\right) = \mu^{-\epsilon}\left(\frac{1+\mu^{-(\eta-\epsilon)}z}{1+z}\right). \tag{B.38}$$

Since the quantity ratio χ is finite, none of the terms in the denominators of [B.36] or [B.37] can be zero, so [B.38] is rearranged as follows to obtain a cubic equation in z for a given value of μ:

$$\begin{aligned}(\eta-1)\mu^{-(2\eta-\epsilon)}\left(1-\mu^{\eta}\right)z^3 + \mu^{-(2\eta-\epsilon)}\left((\epsilon-1)\left(1-\mu^{2\eta-\epsilon}\right) + 2(\eta-1) + \left(\mu^{\eta-\epsilon}-\mu^{\eta}\right)\right)z^2 \\ + \mu^{-(2\eta-\epsilon)}\left((\eta-1)\left(\mu^{2(\eta-\epsilon)}-\mu^{\eta}\right) + 2(\epsilon-1)\left(\mu^{\eta-\epsilon}-\mu^{2\eta-\epsilon}\right)\right)z \\ + (\epsilon-1)\mu^{-(2\eta-\epsilon)}\left(\mu^{2(\eta-\epsilon)}-\mu^{2\eta-\epsilon}\right) = 0.\end{aligned}$$

Because $0 < \mu < 1$, both sides of the above are multiplied by $\mu^{2\eta-\epsilon}/(1-\mu^{\eta})$ to obtain an equivalent cubic equation:

$$(\eta-1)z^3+\left((\epsilon-1)\left(\frac{1-\mu^{2\eta-\epsilon}}{1-\mu^{\eta}}\right)+2(\eta-1)\left(\frac{\mu^{\eta-\epsilon}-\mu^{\eta}}{1-\mu^{\eta}}\right)\right)z^2 \\ +\left((\eta-1)\left(\frac{\mu^{2(\eta-\epsilon)}-\mu^{\eta}}{1-\mu^{\eta}}\right)+2(\epsilon-1)\left(\frac{\mu^{\eta-\epsilon}-\mu^{2\eta-\epsilon}}{1-\mu^{\eta}}\right)\right)z \\ +(\epsilon-1)\left(\frac{\mu^{2(\eta-\epsilon)}-\mu^{2\eta-\epsilon}}{1-\mu^{\eta}}\right)=0. \quad [B.39]$$

This cubic is denoted by $\mathfrak{C}(z;\mu,\epsilon,\eta)\equiv\mathfrak{b}_0(\mu;\epsilon,\eta)+\mathfrak{b}_1(\mu;\epsilon,\eta)z+\mathfrak{b}_2(\mu;\epsilon,\eta)z^2+\mathfrak{b}_3(\eta)z^3$, where the coefficient functions $\mathfrak{b}_0(\mu;\epsilon,\eta)$, $\mathfrak{b}_1(\mu;\epsilon,\eta)$, $\mathfrak{b}_2(\mu;\epsilon,\eta)$ and $\mathfrak{b}_3(\eta)$ listed in [A.2] are obtained directly from [B.39].

These steps demonstrate that starting from a solution $\mathfrak{q}_S$ and $\mathfrak{q}_N$ of [B.16], the quadratic and the cubic equations [B.33] and [B.39] in z must simultaneously hold for $z=((1-\lambda)/\lambda)\rho_N^{\epsilon-\eta}$, with $\rho_N\equiv\mathcal{D}^{-1}(\mathfrak{q}_N)$, and where the coefficient functions [A.2] are evaluated at $\mu=\rho_S/\rho_N$, with $\rho_S\equiv\mathcal{D}^{-1}(\mathfrak{q}_S)$. If the quadratic equation $\mathfrak{Q}(z;\mu,\epsilon,\eta)=0$ and cubic equation $\mathfrak{C}(z;\mu,\epsilon,\eta)=0$ share a root then it is a standard result from the theory of polynomials that the resultant $\mathfrak{R}(\mu;\epsilon,\eta)$, as defined in [A.1], is zero. Since the coefficients of the polynomials $\mathfrak{Q}(z;\mu,\epsilon,\eta)$ and $\mathfrak{C}(z;\mu,\epsilon,\eta)$ are functions only of the markup ratio μ and the parameters ϵ and η, solving the equation $\mathfrak{R}(\mu;\epsilon,\eta)=0$ provides a straightforward procedure for finding the equilibrium markup ratio μ. Furthermore, the only parameters appearing in the equation $\mathfrak{R}(\mu;\epsilon,\eta)=0$ are ϵ and η, so the equilibrium markup ratio μ depends only on these parameters.

Lemma 2 shows that the solution of [B.16] for $\mathfrak{q}_S$ and $\mathfrak{q}_N$ is unique, and therefore the solution of $\mathfrak{R}(\mu;\epsilon,\eta)=0$ for μ must also be unique, given the added condition that the shared root z of the quadratic $\mathfrak{Q}(z;\mu,\epsilon,\eta)=0$ and cubic $\mathfrak{C}(z;\mu,\epsilon,\eta)=0$ is a positive real number. This restriction is required because $z=((1-\lambda)/\lambda)\rho_N^{\epsilon-\eta}$ and ρ_N must of course be positive real numbers. Since $\eta>\epsilon>1$, the product of the roots of the quadratic $\mathfrak{Q}(z;\mu,\epsilon,\eta)=0$ is positive, so the shared root z is positive and real if and only if either branch of the quadratic root function is positive and real. Hence this condition is verified by checking whether $\mathfrak{z}(\mu;\epsilon,\eta)$ in [A.3] (the smaller of the two roots of $\mathfrak{Q}(z;\mu,\epsilon,\eta)=0$) is positive and real.

Note that the resultant $\mathfrak{R}(\mu;\epsilon,\eta)$ is always zero at $\mu=0$ and $\mu=1$ for all values of ϵ and η. This is seen by taking limits of the coefficients in [A.2] as $\mu\to0$ and $\mu\to1$ and applying L'Hôpital's rule, which yields

$$\mathfrak{C}(z;0,\epsilon,\eta)=z\mathfrak{Q}(z;0,\epsilon,\eta),\quad\text{and}\quad\mathfrak{C}(z;1,\epsilon,\eta)=(1+z)\mathfrak{Q}(z;1,\epsilon,\eta).$$

As the polynomials $\mathfrak{Q}(z;\mu,\epsilon,\eta)$ and $\mathfrak{C}(z;\mu,\epsilon,\eta)$ clearly share roots when $\mu=0$ or $\mu=1$, it follows that $\mathfrak{R}(0;\epsilon,\eta)=\mathfrak{R}(1;\epsilon,\eta)=0$. Thus these zeros of the equation $\mathfrak{R}(\mu;\epsilon,\eta)=0$ must be ignored when solving for μ.

(ii) The quadratic equation $\mathfrak{Q}(z;\mu,\epsilon,\eta)=0$ with $z=((1-\lambda)/\lambda)\rho_N^{\epsilon-\eta}$ determines a relative price ρ_N such that with $\rho_S=\mu\rho_N$, marginal revenue is equalized at both ρ_S and ρ_N. Lemma 1 demonstrates that there are two candidate solutions for ρ_N that meet this criterion under the conditions shown by Lemma 2 to be necessary for a solution $\mathfrak{q}_S$ and $\mathfrak{q}_N$ of [B.16] to exist. However, Lemma 2 shows that both ρ_N and ρ_S are on the downward-sloping sections of the marginal revenue function. To rule out a solution in the middle upward-sloping section of marginal revenue, the smaller of the two ρ_N candidate values must be discarded to select the correct solution. Since z is decreasing in ρ_N, this is equivalent to discarding the larger of the two roots of the quadratic. Given that $\mathfrak{a}_2(\eta)$ in [A.2] is positive, the smaller of the two roots of quadratic $\mathfrak{Q}(z;\mu,\epsilon,\eta)=0$ is found using the expression for $\mathfrak{z}(\mu;\epsilon,\eta)$ in [A.3].

The equilibrium quantity ratio χ is obtained by substituting $z=\mathfrak{z}(\mu;\epsilon,\eta)$ into [B.37]. This construction demonstrates that χ depends only on ϵ and η.

(iii) Since $\rho_S\equiv P_S/P_B$ and $\rho_N\equiv P_N/P_B$ according to [B.2], the formula for the purchase multipliers in [10] implies $v_N=\rho_N^{\epsilon-\eta}$ and $v_S=\mu^{\epsilon-\eta}v_N$. Using the fact that $z\equiv((1-\lambda)/\lambda)\rho_N^{\epsilon-\eta}$, and dividing numerator and denominator of the expression in [17] by λ yields [B.30].

(iv) The expressions for the relative prices ρ_S and ρ_N in [B.31] are obtained by rearranging the definition of $z \equiv ((1-\lambda)/\lambda)\rho_N^{\epsilon-\eta}$ and using $\rho_S = \mu\rho_N$. This completes the proof. ■

C Proof of Theorem 1

Non-monotonicity of the marginal revenue function

Using the relationship between the revenue function $\mathscr{R}(q; P_B, \mathcal{E})$ and its equivalent $\mathcal{R}(\mathfrak{q})$ defined in [B.4] using the relative demand function $\mathcal{D}(\rho)$ from [B.1], the corresponding two marginal revenue functions $\mathscr{R}'(q; P_B, \mathcal{E})$ and $\mathcal{R}'(\mathfrak{q})$ are proportional according to [B.7]. Lemma 1 demonstrates that $\mathcal{R}'(\mathfrak{q})$ has the described pattern of non-monotonicity under the conditions $0 < \lambda < 1$ and [16], and is otherwise a decreasing function of q.

Existence of a two-price equilibrium

For a two-price equilibrium to exist, first-order conditions [15] for profit-maximization must be satisfied at two prices p_S and p_N, with associated quantities $q_S = \mathscr{D}(p_S; P_B, \mathcal{E})$ and $q_N = \mathscr{D}(p_N; P_B, \mathcal{E})$, where P_B is the bargain hunters' price index from [7], and $\mathcal{E} = P^{\epsilon}Y$ is the measure of aggregate expenditure from [10].

The necessary conditions for the two-price equilibrium are now restated in terms of the relative demand function $\mathcal{D}(\rho)$ defined in [B.1], and its associated total and marginal revenue functions $\mathcal{R}(\mathfrak{q})$ and $\mathcal{R}'(\mathfrak{q})$, as defined in [B.4] and analysed in appendix B. The relative demand function $\mathfrak{q} = \mathcal{D}(\rho)$ is specified in terms of the relative price $\rho \equiv p/P_B$ and relative quantity $\mathfrak{q} \equiv q/(\mathcal{E}/P_B^{\epsilon})$, in accordance with [B.2]. Using the relationships in [B.3] and [B.7], the first two optimality conditions in [15] are equivalent to

$$\mathcal{R}'\left(\frac{q_S P_B^{\epsilon}}{\mathcal{E}}\right) = \mathcal{R}'\left(\frac{q_N P_B^{\epsilon}}{\mathcal{E}}\right) = \frac{\mathcal{R}\left(\frac{q_S P_B^{\epsilon}}{\mathcal{E}}\right) - \mathcal{R}\left(\frac{q_N P_B^{\epsilon}}{\mathcal{E}}\right)}{\frac{q_S P_B^{\epsilon}}{\mathcal{E}} - \frac{q_N P_B^{\epsilon}}{\mathcal{E}}}. \qquad \text{[C.1]}$$

With $\mathfrak{q}_S \equiv q_S/(\mathcal{E}/P_B^{\epsilon})$ and $\mathfrak{q}_N \equiv q_N/(\mathcal{E}/P_B^{\epsilon})$, the first-order conditions in [C.1] are identical to the equations in [B.16] studied in Lemma 2. These clearly require the equalization of marginal revenue $\mathcal{R}'(\mathfrak{q})$ at two different quantities, which means that the marginal revenue function must be non-monotonic. Lemma 1 then shows that $0 < \lambda < 1$ and parameters ϵ and η satisfying the inequality [16] are necessary and sufficient for this. If these conditions are met then Lemma 2 demonstrates the existence of a unique solution $\mathfrak{q}_S$ and $\mathfrak{q}_N$ of the equations [B.16].

The relative quantities $\mathfrak{q}_S$ and $\mathfrak{q}_N$ must also be well defined if the solution is to be economically meaningful. This means that if $\rho_S = \mathcal{D}^{-1}(\mathfrak{q}_S)$ and $\rho_N = \mathcal{D}^{-1}(\mathfrak{q}_N)$ are the corresponding prices p_S and p_N relative to P_B then $\rho_S < 1 < \rho_N$. This is a necessary requirement because the expression [20] for the bargain hunters' price index P_B implies

$$s\rho_S^{1-\eta} + (1-s)\rho_N^{1-\eta} = 1, \qquad \text{[C.2]}$$

and the equilibrium sales frequency s must satisfy $s \in (0, 1)$.

Assume the parameters are such that ϵ and η satisfy [16], and consider a given value of $\lambda \in (0, 1)$. Lemma 3 shows that the markup ratio (or price ratio) $\mu \equiv \mu_S/\mu_N = \rho_S/\rho_N$ consistent with the unique solution of [B.16] is a function only of the elasticities ϵ and η. The equilibrium relative prices ρ_S and ρ_N are functions of all three parameters ϵ, η and λ, and are obtained from equation [B.31] by substituting the equilibrium value of μ into the function $\mathfrak{z}(\mu; \epsilon, \eta)$ defined in [A.3]. Since $\rho_S = \mu\rho_N$ and $\mu < 1$, the requirement $\rho_S < 1 < \rho_N$ implies $\mu < \rho_S < 1$. By substituting for ρ_S from [B.31], this condition is equivalent to:

$$\mathfrak{z}(\mu; \epsilon, \eta) < \frac{1-\lambda}{\lambda} < \mu^{-(\eta-\epsilon)}\mathfrak{z}(\mu; \epsilon, \eta). \qquad \text{[C.3]}$$

Define lower and upper bounds for λ conditional on ϵ and η using the formulæ in [A.6] together with the equilibrium value of μ (which is a function only of ϵ and η) and the function $\mathfrak{z}(\mu; \epsilon, \eta)$ from [A.3]. Note that if $\mathfrak{z}(\mu;\epsilon,\eta) > 0$ and $0 < \mu < 1$ then $0 < \underline{\lambda}(\epsilon,\eta) < \overline{\lambda}(\epsilon,\eta) < 1$. By rearranging the inequality [C.3] and using the definitions of the bounds on λ, the inequality is equivalent to λ lying in the interval:

$$\underline{\lambda}(\epsilon,\eta) < \lambda < \overline{\lambda}(\epsilon,\eta). \tag{C.4}$$

This restriction on λ is necessary and sufficient for the existence of an equilibrium sales frequency $s \in (0,1)$ satisfying [C.2]. The equivalence is demonstrated by substituting the expressions for ρ_S and ρ_N from [B.31] into [C.2]:

$$\left(1 + s\left(\mu^{-(\eta-1)} - 1\right)\right)\left(\frac{\lambda}{1-\lambda}\mathfrak{z}(\mu;\epsilon,\eta)\right)^{\frac{\eta-1}{\eta-\epsilon}} = 1.$$

This is a linear equation in s, and has a unique solution because $\eta > 1$ and $0 < \mu < 1$. Solving explicitly for s yields:

$$s = \frac{\left(\frac{\lambda}{1-\lambda}\mathfrak{z}(\mu;\epsilon,\eta)\right)^{-\left(\frac{\eta-1}{\eta-\epsilon}\right)} - 1}{\mu^{-(\eta-1)} - 1}. \tag{C.5}$$

Recalling the equivalence of inequalities [C.3] and [C.4], it follows that $s \in (0,1)$ if and only if $\lambda \in (\underline{\lambda}(\epsilon,\eta), \overline{\lambda}(\epsilon,\eta))$. So for $\lambda \in [0, \underline{\lambda}(\epsilon,\eta)]$ or $\lambda \in [\overline{\lambda}(\epsilon,\eta), 1]$ there is no two-price equilibrium. But given elasticities ϵ and η satisfying the non-monotonicity condition [16] and a loyal fraction $\lambda \in (\underline{\lambda}(\epsilon,\eta), \overline{\lambda}(\epsilon,\eta))$, by using the arguments above there exist two distinct relative prices $\rho_S \equiv p_S/P_B$ and $\rho_N \equiv p_N/P_B$ and a sales frequency $s \in (0,1)$ consistent with the first two equalities in [15]. Lemma 3 then demonstrates that the two purchase multipliers v_S and v_N and the two optimal markups μ_S and μ_N are determined. Equations [14] and [17] show that using the optimal markups in [18] is equivalent to satisfying the remaining first-order condition involving marginal cost in [15]. The other variables relevant to the macroeconomic equilibrium are then determined as discussed in section 3.4.

Confirming that the two-price equilibrium exists then requires checking that the remaining first-order condition [13c] is satisfied and that the first-order conditions are sufficient as well as necessary to characterize the maximum of the profit function. Using the relationships in [B.7] and the results of Lemma 2 in [B.17] the following inequalities are deduced:

$$\mathscr{R}(q_S;P_B,\mathcal{E}) - \mathscr{R}'(q_S;P_B,\mathcal{E})q_S > 0, \quad \text{and } \mathscr{R}(q_N;P_B,\mathcal{E}) - \mathscr{R}'(q_N;P_B,\mathcal{E})q_N > 0. \tag{C.6}$$

Since $s \in (0,1)$, the Lagrangian multiplier $\aleph$ from first-order conditions [13b]–[13c] is determined as follows:

$$\aleph = \mathscr{R}(q_S;P_B,\mathcal{E}) - \mathscr{R}'(q_S;P_B,\mathcal{E})q_S = \mathscr{R}(q_N;P_B,\mathcal{E}) - \mathscr{R}'(q_N;P_B,\mathcal{E})q_N,$$

and hence $\aleph > 0$ because of [C.6]. By combining this expression for the Lagrangian multiplier with the first-order condition [13c]:

$$\mathscr{R}(q;P_B,\mathcal{E}) \le \mathscr{R}(q_N;P_B,\mathcal{E}) + \mathscr{R}'(q_N;P_B,\mathcal{E})(q-q_N) = \mathscr{R}(q_S;P_B,\mathcal{E}) + \mathscr{R}'(q_S;P_B,\mathcal{E})(q-q_S), \tag{C.7}$$

which is required to hold for all $q \ge 0$. Appealing to the result of Lemma 2 in [B.18] and again using [B.7] verifies the inequality.

The assumptions about the production function [8] ensure that the total cost function $\mathscr{C}(Q;W)$ in [9] is continuously differentiable and convex, so for all $q \ge 0$:

$$\mathscr{C}(q;W) \ge \mathscr{C}(Q;W) + \mathscr{C}'(Q;W)(q-Q), \tag{C.8}$$

where $Q \equiv sq_S + (1-s)q_N$ is the specific total physical quantity sold using the two-price strategy constructed earlier. Now consider a general alternative pricing strategy for a given firm, assuming that all other firms continue to use the same two-price strategy. The new strategy is specified in terms of a distribution function $F(p)$ for prices. Let $G(q) \equiv 1 - F(\mathscr{D}(p;P_B,\mathcal{E}))$ be the implied distribution function for quantities sold. The level of profits $\mathscr{P}$ from the new strategy is obtained by making a change of variable from prices to quantities in the integrals of [12]:

$$\mathscr{P} = \int_q \mathscr{R}(q; P_B, \mathcal{E}) dG(q) - \mathscr{C}\left(\int_q q dG(q); W\right).$$

Applying the inequalities involving the revenue and total cost functions from [C.7] and [C.8] to the expression for profits yields:

$$\mathscr{P} \leq \left(\mathscr{R}(q_N; P_B, \mathcal{E}) - \mathscr{R}'(q_N; P_B, \mathcal{E})q_N\right) - \left(\mathscr{C}(Q; W) - \mathscr{C}'(Q; W)Q\right) \\ + \left(\mathscr{R}'(q_N; P_B, \mathcal{E}) - \mathscr{C}'(Q; W)\right)\left(\int_q q dG(q)\right).$$

The first-order conditions [15] imply that the coefficient of the integral in the above expression is zero, and that $\mathscr{R}(q_N; P_B, \mathcal{E}) - \mathscr{R}'(q_N; P_B, \mathcal{E})q_N = \mathscr{R}(q_S; P_B, \mathcal{E}) - \mathscr{R}'(q_S; P_B, \mathcal{E})q_S$. Recalling $Q = sq_S + (1-s)q_N$, it follows that:

$$\mathscr{P} \leq s\mathscr{R}(q_S; P_B, \mathcal{E}) + (1-s)\mathscr{R}(q_N; P_B, \mathcal{E}) - \mathscr{C}\left(sq_S + (1-s)q_N; W\right),$$

for all alternative pricing strategies. Hence there is no profit-improving deviation from the two-price strategy. This establishes that a two-price equilibrium exists when [16] and $\lambda \in (\underline{\lambda}(\epsilon, \eta), \overline{\lambda}(\epsilon, \eta))$ hold, and that it is unique within the class of two-price equilibria.

Uniqueness of the two-price equilibrium

Suppose the parameters ϵ, η and λ are such that a two-price equilibrium exists. Now consider the possibility that a one-price equilibrium also exists for the same parameters. Since all firms are symmetric, the relative price found in this one-price equilibrium is necessarily equal to one. The relative prices ρ_S and ρ_N in the two-price equilibrium cannot be on the same side of one, implying $\mu < \rho_S < 1$ and thus $\rho_S < 1 < \rho_N$, where $\rho_S = \mathcal{D}^{-1}(\mathfrak{q}_S)$ and $\rho_N = \mathcal{D}^{-1}(\mathfrak{q}_N)$ using the relative quantities $\mathfrak{q}_S$ and $\mathfrak{q}_N$. Since [B.1] implies $\mathcal{D}(1) = 1$ and because the relative demand function $\mathcal{D}(\rho)$ is strictly decreasing in ρ, it follows that $\mathfrak{q}_N < 1 < \mathfrak{q}_S$.

Given that the marginal revenue function must be non-monotonic if a two-price equilibrium is to exist, it follows from Lemma 1 that $\mathcal{R}(\mathfrak{q})$ is strictly concave on the intervals $(0, \underline{\mathfrak{q}})$ and $(\overline{\mathfrak{q}}, \infty)$, strictly convex on $(\underline{\mathfrak{q}}, \overline{\mathfrak{q}})$, and from Lemma 2 that $\mathfrak{q}_N < \underline{\mathfrak{q}} < \overline{\mathfrak{q}} < \mathfrak{q}_S$.

Consider first the case where $\underline{\mathfrak{q}} < 1 < \overline{\mathfrak{q}}$. Since $\mathfrak{q}_1 = 1$ for all firms in the one-price equilibrium, the actual common quantity produced is $q_1 = \mathcal{E}/P_B^{\epsilon}$ using [B.2], where P_B and $\mathcal{E}$ are the values of these variables associated with the putative one-price equilibrium. Since $\mathcal{R}''(1) > 0$, equation [B.7] implies $\mathscr{R}''(q_1; P_B, \mathcal{E}) > 0$. Therefore, for sufficiently small $\varepsilon > 0$, the profits $\mathscr{P}$ from selling quantity $q_1 - \varepsilon$ at one half of shopping moments and $q_1 + \varepsilon$ at the other half exceed the profits from offering one price and hence one quantity at all shopping moments:

$$\frac{1}{2}\mathscr{R}(q_1 - \varepsilon; P_B, \mathcal{E}) + \frac{1}{2}\mathscr{R}(q_1 + \varepsilon; P_B, \mathcal{E}) - \mathscr{C}\left(\frac{1}{2}(q_1 - \varepsilon) + \frac{1}{2}(q_1 + \varepsilon); W\right) > \mathscr{R}(q_1; P_B, \mathcal{E}) - \mathscr{C}(q_1; W).$$

Therefore a one-price equilibrium cannot exist in this case.

Next consider the case where $\mathfrak{q}_N < 1 < \underline{\mathfrak{q}}$. Let $p_1 = P_B$ denote the price it is claimed all firms charge in a one-price equilibrium, and $q_1 = \mathcal{E}/P_B^{\epsilon}$ the associated quantity sold. Now let $q_S = \mathscr{D}(\rho_S p_1; P_B, \mathcal{E})$ be quantity sold if the sale *relative* price $\rho_S = \mathcal{D}^{-1}(\mathfrak{q}_S)$ is used when other firms are following the one-price strategy of charging p_1 at all shopping moments. Consider an alternative strategy where price $\rho_S p_1$ is offered at a fraction ε of moments and price p_1 at the remaining fraction $1 - \varepsilon$ of moments. Profits $\mathscr{P}$ from the hybrid strategy are given by:

$$\mathscr{P} = (1 - \varepsilon)\mathscr{R}(q_1; P_B, \mathcal{E}) + \varepsilon\mathscr{R}(q_S; P_B, \mathcal{E}) - \mathscr{C}\left((1 - \varepsilon)q_1 + \varepsilon q_S; W\right). \quad \text{[C.9]}$$

As the cost function $\mathscr{C}(q;W)$ is differentiable in q, the above equation implies:

$$\mathscr{P} = (\mathscr{R}(q_1;P_B,\mathcal{E}) - \mathscr{C}(q_1;W)) + \varepsilon(q_S - q_1)\left(\frac{\mathscr{R}(q_S;P_B,\mathcal{E}) - \mathscr{R}(q_1;P_B,\mathcal{E})}{q_S - q_1} - \mathscr{C}'(q_1;W)\right) + \mathscr{O}\left(\varepsilon^2\right),$$

where $\mathscr{O}\left(\varepsilon^2\right)$ denotes second- and higher-order terms in ε. A necessary condition for a one-price equilibrium to exist is that the single price p_1 is chosen optimally, in which case first-order conditions [13] reduce to the usual marginal revenue equals marginal cost condition $\mathscr{R}'(q_1;P_B,\mathcal{E}) = \mathscr{C}'(q_1;W)$. Hence the above expression for $\mathscr{P}$ becomes:

$$\mathscr{P} = (\mathscr{R}(q_1;P_B,\mathcal{E}) - \mathscr{C}(q_1;W)) + \varepsilon(q_S - q)\left(\frac{\mathscr{R}(q_S;P_B,\mathcal{E}) - \mathscr{R}(q_1;P_B,\mathcal{E})}{q_S - q_1} - \mathscr{R}'(q_1;P_B,\mathcal{E})\right) + \mathscr{O}\left(\varepsilon^2\right). \qquad \text{[C.10]}$$

Since $\mathfrak{q}_N < 1 < \mathfrak{q}_S$ in the case under consideration and $\mathfrak{q}_1 = 1$, the results from Lemma 2 in [B.16] can be expressed as follows:

$$\int_{\mathfrak{q}_N}^{1} \mathcal{R}'(\mathfrak{q})d\mathfrak{q} + \mathcal{R}(\mathfrak{q}_S) - \mathcal{R}(\mathfrak{q}_1) = (\mathfrak{q}_S - \mathfrak{q}_N)\mathcal{R}'(\mathfrak{q}_N). \qquad \text{[C.11]}$$

As $\mathfrak{q}_N < 1 < \underline{\mathfrak{q}}$ and $\mathcal{R}'(\mathfrak{q})$ is strictly decreasing for $\mathfrak{q} < \underline{\mathfrak{q}}$, the integral above satisfies:

$$\int_{\mathfrak{q}_N}^{1} \mathcal{R}'(\mathfrak{q})d\mathfrak{q} < (1 - \mathfrak{q}_N)\mathcal{R}'(\mathfrak{q}_N). \qquad \text{[C.12]}$$

Noting that $\mathcal{R}'(\mathfrak{q}_N) > \mathcal{R}'(1)$ because of $\mathfrak{q}_N < 1 < \underline{\mathfrak{q}}$, and substituting [C.12] into [C.11] and rearranging yields:

$$\frac{\mathcal{R}(\mathfrak{q}_S) - \mathcal{R}(1)}{\mathfrak{q}_S - 1} > \mathcal{R}'(\mathfrak{q}_N) > \mathcal{R}'(1), \qquad \text{[C.13]}$$

where $\mathfrak{q}_S > 1$ ensures that the direction of the inequality is preserved. Now given the fact that $q_1 = (\mathcal{E}/P_B^{\epsilon})$ and $q_S = (\mathcal{E}/P_B^{\epsilon})\mathfrak{q}_S$ from [B.2], and the links between the functions $\mathcal{R}(\mathfrak{q})$ and $\mathscr{R}(q;P_B,\mathcal{E})$ as set out in [B.7]:

$$\frac{\mathscr{R}(q_S;P_B,\mathcal{E}) - \mathscr{R}(q_1;P_B,\mathcal{E})}{q_S - q_1} > \mathscr{R}'(q_1;P_B,\mathcal{E}). \qquad \text{[C.14]}$$

Therefore, by comparing this inequality with [C.10] and noting $q_S > q_1$, it follows for sufficiently small $\varepsilon > 0$ that $\mathscr{P} > \mathscr{R}(q_1;P_B,\mathcal{E}) - \mathscr{C}(q_1;W)$, so profits from a hybrid strategy exceed those from following the strategy required for the one-price equilibrium to exist.

The remaining case to consider is $\overline{\mathfrak{q}} < 1 < \mathfrak{q}_S$. The argument here is analogous to that given above. The alternative strategy considered is offering price $p_N = \rho_N p_1$ (where $\rho_N = \mathcal{D}^{-1}(\mathfrak{q}_N)$) at a fraction ε of shopping moments and price $p_1 = P_B$ at the remaining fraction $1 - \varepsilon$, with quantities sold respectively at those moments of $q_N = \mathscr{D}(\rho_N p_1; P_B, \mathcal{E})$ and q_1. Following the steps in [C.9]–[C.10] leads to an expression for profits $\mathscr{P}$ resulting from this hybrid strategy:

$$\mathscr{P} = (\mathscr{R}(q_1;P_B,\mathcal{E}) - \mathscr{C}(q_1;W)) + \varepsilon(q_1 - q_N)\left(\mathscr{R}'(q_1;P_B,\mathcal{E}) - \frac{\mathscr{R}(q_1;P_B,\mathcal{E}) - \mathscr{R}(q_N;P_B,\mathcal{E})}{q_1 - q_N}\right) + \mathscr{O}\left(\varepsilon^2\right). \qquad \text{[C.15]}$$

Appealing to the properties of $\mathcal{R}(\mathfrak{q})$ for $\mathfrak{q} > \overline{\mathfrak{q}}$ and following similar steps to those in [C.11]–[C.13] implies $\mathcal{R}'(1) > \mathcal{R}'(\mathfrak{q}_S) > (\mathcal{R}(1) - \mathcal{R}(\mathfrak{q}_N))/(1 - \mathfrak{q}_N)$, and hence an equivalent of [C.14]:

$$\mathscr{R}'(q_1;P_B,\mathcal{E}) > \frac{\mathscr{R}(q_1;P_B,\mathcal{E}) - \mathscr{R}(q_N;P_B,\mathcal{E})}{q_1 - q_N}. \qquad \text{[C.16]}$$

Since $q_1 > q_N$, for sufficiently small $\varepsilon > 0$, [C.15] and [C.16] demonstrate that there is a hybrid strategy which delivers higher profits than the one-price strategy used by all other firms. This proves that for all parameters where the two-price equilibrium exists, a one-price equilibrium cannot exist for any of these same parameter values.

One-price equilibrium

The first point to note is that when a two-price equilibrium fails to exist owing to a violation of the non-monotonicity condition [16], Lemma 1 implies that marginal revenue $\mathscr{R}'(q; P_B, \mathcal{E})$ is strictly decreasing for all q. This is equivalent to revenue $\mathscr{R}(q; P_B, \mathcal{E})$ being a strictly concave function of quantity q. Since total cost $\mathscr{C}(q; W)$ is a convex function of the quantity produced, it follows immediately that the profit function is globally concave, and thus a one-price equilibrium always exists, and is the only possible equilibrium in the parameter range where ϵ or η fail to satisfy [16], or where $\lambda = 0$ or $\lambda = 1$.

Now suppose the parameters are such that the marginal revenue function is non-monotonic, but a two-price equilibrium fails to exist owing to λ not lying between $\underline{\lambda}(\epsilon, \eta)$ and $\overline{\lambda}(\epsilon, \eta)$. Note that [C.3] and [A.6] imply $\lambda \in [0, \underline{\lambda}(\epsilon, \eta)]$ and $\lambda \in [\overline{\lambda}(\epsilon, \eta), 1]$ are equivalent to $1 > \mathfrak{q}_S$ and $1 < \mathfrak{q}_N$ respectively.

Taking the first of these cases, Lemma 1 demonstrates the concavity of $\mathcal{R}(\mathfrak{q})$ on $[\overline{\mathfrak{q}}, \infty)$ (containing $\mathfrak{q}_S$), which establishes that $\mathcal{R}(\mathfrak{q}) \leq \mathcal{R}(1) + \mathcal{R}'(1)(\mathfrak{q} - 1)$ for all $\mathfrak{q} \in [\overline{\mathfrak{q}}, \infty)$. Lemma 2 shows that $\mathcal{R}(\mathfrak{q}) \leq \mathcal{R}(\mathfrak{q}_S) + \mathcal{R}'(\mathfrak{q}_S)(\mathfrak{q} - \mathfrak{q}_S)$ for all $\mathfrak{q} \geq 0$. Note that the concavity of $\mathcal{R}(\mathfrak{q})$ in the relevant range implies $\mathcal{R}'(\mathfrak{q}_S) > \mathcal{R}'(1)$, which together with the second of the previous inequalities yields $\mathcal{R}(\mathfrak{q}) \leq \mathcal{R}(\mathfrak{q}_S) + \mathcal{R}'(1)(\mathfrak{q} - \mathfrak{q}_S)$ for all $\mathfrak{q} \in [0, \mathfrak{q}_S]$. Applying the first inequality at $\mathfrak{q} = \mathfrak{q}_S$ establishes that $\mathcal{R}(\mathfrak{q}_S) \leq \mathcal{R}(1) + \mathcal{R}'(1)(\mathfrak{q}_S - 1)$. By combining these results it follows that $\mathcal{R}(\mathfrak{q}) \leq \mathcal{R}(1) + \mathcal{R}'(1)(\mathfrak{q} - 1)$ for all $\mathfrak{q} \geq 0$. Translating this into a property of the original revenue function $\mathscr{R}(q; P_B, \mathcal{E})$ using [B.2] and [B.7] yields the following for all q:

$$\mathscr{R}(q; P_B, \mathcal{E}) \leq \mathscr{R}(q_1; P_B, \mathcal{E}) + \mathscr{R}'(q_1; P_B, \mathcal{E})(q - q_1). \tag{C.17}$$

When $\lambda \in [\overline{\lambda}(\epsilon, \eta), 1]$ the other case to consider is $1 < \mathfrak{q}_N$. Using an exactly analogous argument to that given above, it is deduced that $\mathcal{R}(\mathfrak{q}) \leq \mathcal{R}(1) + \mathcal{R}'(1)(\mathfrak{q} - 1)$ for all $\mathfrak{q} \geq 0$ in this case as well. Hence [C.17] holds in both cases. The convexity of the total cost function $\mathscr{C}(q; W)$ together with [C.17] proves that no pricing strategy can improve on that used in the one-price equilibrium.

Non-existence of equilibria with more than two prices

Take any two prices p_1 and p_2 offered by a firm at a positive fraction of shopping moments, and define $\rho_1 \equiv p_1/P_B$ and $\rho_2 \equiv p_2/P_B$ in accordance with [B.2]. Denote the quantities sold by q_1 and q_2 and define $\mathfrak{q}_1 \equiv (P_B^{\epsilon}/\mathcal{E})q_1$ and $\mathfrak{q}_2 \equiv (P_B^{\epsilon}/\mathcal{E})q_2$ also in accordance with [B.2]. Using the first-order conditions [13] together with [B.2] and [B.7], it follows that $\mathfrak{q}_1$ and $\mathfrak{q}_2$ must satisfy the system of equations [B.16] in place of $\mathfrak{q}_S$ and $\mathfrak{q}_N$. But as Lemma 2 demonstrates that the solution to this system of equations is unique, there is a maximum of two distinct prices in any firm's profit-maximizing strategy. This completes the proof.

D Proof of Proposition 1

(i) The first-order conditions are of course necessary. For sufficiency, note using the argument in the proof of Theorem 1 that the first-order conditions in [15] are equivalent to the equations in [C.1]. As Lemma 3 shows, the equations in [C.1] have a unique solution. Since an equilibrium is known to exist by Theorem 1, the first-order conditions must also be sufficient.

(ii) Lemma 3 shows that μ, χ, μ_S and μ_N are uniquely determined as functions of ϵ and η when the inequality [16] is satisfied, as is necessary for the two-price equilibrium to exist.

(iii) Lemma 3 implicitly determines the purchase multipliers v_S and v_N using the expressions for $\rho_S \equiv p_S/P_B$ and $\rho_N = p_N/P_B$ in [B.31] and the fact that $v_S = (p_S/P_B)^{-(\eta-\epsilon)}$ and $v_N = (p_N/P_B)^{-(\eta-\epsilon)}$ from [10]. Hence Lemma 3 shows that these variables depend only on ϵ, η and λ. In conjunction with equation [20], knowledge of ρ_S and ρ_N from [B.31] yields a linear equation for s after dividing both sides of [20] by P_B. This shows that it too only depends on ϵ, η and λ.

(iv) Substituting the bounds for λ from [A.6] into equation [C.5] proves the first two results. Differentiating [C.5] with respect to λ yields the third result. This completes the proof.

E Proof of Theorem 2

Log linearizations

The notational convention adopted here is that a variable without a time subscript denotes its flexible-price steady-state value as determined in section 3, and the corresponding sans serif letter denotes the log deviation of the variable from its steady-state value (except for the sales frequency s, where it denotes just the deviation from steady state, and the inflation rate, where it denotes the log deviation of the gross rate).

Consider first the demand function faced by firms. The levels of demand $q_{S,\ell,t}$ and $q_{N,\ell,t}$ at the sale and normal prices are obtained from [22], which have the following log-linearized forms:

$$\mathsf{q}_{S,\ell,t} = \left(\frac{(1-\lambda)v_S}{\lambda + (1-\lambda)v_S}\right)\mathsf{v}_{S,\ell,t} - \epsilon(\mathsf{p}_{S,\ell,t} - \mathsf{P}_t) + \mathsf{Y}_t, \quad \text{and} \tag{E.1a}$$

$$\mathsf{q}_{N,\ell,t} = \left(\frac{(1-\lambda)v_N}{\lambda + (1-\lambda)v_N}\right)\mathsf{v}_{N,\ell,t} - \epsilon(\mathsf{R}_{N,t-\ell} - \mathsf{P}_t) + \mathsf{Y}_t, \tag{E.1b}$$

where the expressions are given in terms of log deviations of the purchase multipliers $v_{S,\ell,t}$ and $v_{N,\ell,t}$ from [10]:

$$\mathsf{v}_{S,\ell,t} = -(\eta - \epsilon)\left(\mathsf{p}_{S,\ell,t} - \mathsf{P}_{B,t}\right), \quad \text{and} \quad \mathsf{v}_{N,\ell,t} = -(\eta - \epsilon)\left(\mathsf{R}_{N,t-\ell} - \mathsf{P}_{B,t}\right). \tag{E.2}$$

By substituting the purchase multipliers into the demand functions [E.1], the following expressions are found:

$$\mathsf{q}_{S,\ell,t} = -\left(\frac{\lambda\epsilon + (1-\lambda)\eta v_S}{\lambda + (1-\lambda)v_S}\right)\mathsf{p}_{S,\ell,t} + (\eta - \epsilon)\left(\frac{(1-\lambda)v_S}{\lambda + (1-\lambda)v_S}\right)\mathsf{P}_{B,t} + \epsilon\mathsf{P}_t + \mathsf{Y}_t, \quad \text{and} \tag{E.3a}$$

$$\mathsf{q}_{N,\ell,t} = -\left(\frac{\lambda\epsilon + (1-\lambda)\eta v_N}{\lambda + (1-\lambda)v_N}\right)\mathsf{R}_{N,t-\ell} + (\eta - \epsilon)\left(\frac{(1-\lambda)v_N}{\lambda + (1-\lambda)v_N}\right)\mathsf{P}_{B,t} + \epsilon\mathsf{P}_t + \mathsf{Y}_t. \tag{E.3b}$$

From equation [17], the log-linearized optimal markups at given sale and normal prices are:

$$\mu_{S,\ell,t} = -\mathfrak{c}_S\mathsf{v}_{S,\ell,t}, \quad \text{with } \mathfrak{c}_S \equiv \frac{\lambda(1-\lambda)(\eta-\epsilon)v_S}{(\lambda\epsilon + (1-\lambda)\eta v_S)\left(\lambda(\epsilon-1) + (1-\lambda)(\eta-1)v_S\right)}, \quad \text{and} \tag{E.4a}$$

$$\mu_{N,\ell,t} = -\mathfrak{c}_N\mathsf{v}_{N,\ell,t}, \quad \text{with } \mathfrak{c}_N \equiv \frac{\lambda(1-\lambda)(\eta-\epsilon)v_N}{(\lambda\epsilon + (1-\lambda)\eta v_N)\left(\lambda(\epsilon-1) + (1-\lambda)(\eta-1)v_N\right)}, \tag{E.4b}$$

which are given in terms of the purchase multipliers from [E.2]. Overall demand $Q_{\ell,t} = s_{\ell,t}q_{S,\ell,t} + (1 - s_{\ell,t})q_{N,\ell,t}$ is log-linearized as follows:

$$\mathsf{Q}_{\ell,t} = \left(\frac{\chi - 1}{s\chi + (1-s)}\right)\mathsf{s}_{\ell,t} + \left(\frac{s\chi}{s\chi + (1-s)}\right)\mathsf{q}_{S,\ell,t} + \left(\frac{1-s}{s\chi + (1-s)}\right)\mathsf{q}_{N,\ell,t}. \tag{E.5}$$

Define the following weighted averages of variables across the distribution of normal-price vintages. First, the average sale frequency:

$$\mathsf{s}_t \equiv (1-\phi_p)\sum_{\ell=0}^{\infty}\phi_p^\ell\mathsf{s}_{\ell,t}.$$

Now the average normal price, the average quantity sold, and the purchase multiplier associated with the normal price:

$$\mathsf{P}_{N,t} \equiv (1-\phi_p)\sum_{\ell=0}^{\infty}\phi_p^\ell\mathsf{R}_{N,t-\ell}, \quad \mathsf{q}_{N,t} \equiv (1-\phi_p)\sum_{\ell=0}^{\infty}\phi_p^\ell\mathsf{q}_{N,\ell,t}, \quad \text{and} \quad \mathsf{v}_{N,t} \equiv (1-\phi_p)\sum_{\ell=0}^{\infty}\phi_p^\ell\mathsf{v}_{N,\ell,t}. \tag{E.6}$$

Finally, the average sale price and associated average quantity and purchase multiplier:

$$\mathsf{P}_{S,t} \equiv (1-\phi_p)\sum_{\ell=0}^{\infty}\phi_p^\ell\mathsf{p}_{S,\ell,t}, \quad \mathsf{q}_{S,t} \equiv (1-\phi_p)\sum_{\ell=0}^{\infty}\phi_p^\ell\mathsf{q}_{S,\ell,t}, \quad \text{and} \quad \mathsf{v}_{S,t} \equiv (1-\phi_p)\sum_{\ell=0}^{\infty}\phi_p^\ell\mathsf{v}_{S,\ell,t}. \tag{E.7}$$

The bargain hunters' price index $P_{B,t}$ as given in [29] is log-linearized as follows:

$$\mathsf{P}_{B,t} = \vartheta_B \mathsf{P}_{S,t} + (1-\vartheta_B)\mathsf{P}_{N,t} - \varphi_B \mathsf{s}_t, \quad \text{where} \qquad [\text{E.8}]$$
$$\vartheta_B \equiv \left(\frac{s}{s+(1-s)\mu^{\eta-1}}\right), \quad \text{and} \quad \varphi_B \equiv \frac{1}{\eta-1}\left(\frac{1-\mu^{\eta-1}}{s+(1-s)\mu^{\eta-1}}\right),$$

using the averages defined above. The coefficients satisfy $0 \leq \vartheta_B \leq 1$ and $\varphi_B \geq 0$. By analogy with the expression for $P_{B,t}$ in [29], define a price index $P_{L,t}$ corresponding to the average purchase price for a hypothetical loyal customer:

$$P_{L,t} = \left((1-\phi_p)\sum_{\ell=0}^{\infty}\phi_p^{\ell}\left\{s_{\ell,t}p_{S,\ell,t}^{1-\epsilon} + (1-s_{\ell,t})R_{N,t-\ell}^{1-\epsilon}\right\}\right)^{\frac{1}{1-\epsilon}}. \qquad [\text{E.9}]$$

This has the following log linearization:

$$\mathsf{P}_{L,t} = \vartheta_L \mathsf{P}_{S,t} + (1-\vartheta_L)\mathsf{P}_{N,t} - \varphi_L \mathsf{s}_t, \quad \text{where} \qquad [\text{E.10}]$$
$$\vartheta_L \equiv \left(\frac{s}{s+(1-s)\mu^{\epsilon-1}}\right), \quad \text{and} \quad \varphi_L \equiv \frac{1}{\epsilon-1}\left(\frac{1-\mu^{\epsilon-1}}{s+(1-s)\mu^{\epsilon-1}}\right),$$

where the coefficients satisfy $0 \leq \vartheta_L \leq 1$ and $\varphi_L \geq 0$.

Note that [28], [29] and [E.9] imply that the price level P_t can be expressed in terms of $P_{L,t}$ and $P_{B,t}$:

$$P_t = \left(\lambda P_{L,t}^{1-\epsilon} + (1-\lambda)P_{B,t}^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}},$$

which can be log linearized to yield:

$$\mathsf{P}_t = (1-\varpi)\mathsf{P}_{L,t} + \varpi \mathsf{P}_{B,t}, \quad \text{where} \quad \varpi = \frac{(1-\lambda)}{(1-\lambda)+\lambda\hbar^{\epsilon-1}}, \quad \text{and} \quad \hbar = \frac{\left(s+(1-s)\mu^{\epsilon-1}\right)^{\frac{1}{\epsilon-1}}}{\left(s+(1-s)\mu^{\eta-1}\right)^{\frac{1}{\eta-1}}}, \qquad [\text{E.11}]$$

with $\hbar$ being a bargain hunter's cost of consumption relative to a loyal customer, that is $\hbar \equiv P_B/P_L$, and ϖ denoting the weight on the bargain hunters' price index in the overall aggregate price level ($0 \leq \varpi \leq 1$). It is convenient to express the price level P_t in terms of the averages $\mathsf{P}_{S,t}$, $\mathsf{P}_{N,t}$ and s_t:

$$\mathsf{P}_t = \vartheta_P \mathsf{P}_{S,t} + (1-\vartheta_P)\mathsf{P}_{N,t} - \varphi_P \mathsf{s}_t, \quad \text{where} \quad \vartheta_P = (1-\varpi)\vartheta_L + \varpi\vartheta_B, \quad \text{and} \quad \varphi_P = (1-\varpi)\varphi_L + \varpi\varphi_B. \qquad [\text{E.12}]$$

Note that $0 \leq \vartheta_P \leq 1$ and $\varphi_P \geq 0$ follow from the properties of the coefficients ϑ_B, ϑ_L, φ_L, φ_B and ϖ.

The log linearization of the production function [8] is

$$\mathsf{Q}_{\ell,t} = \alpha \mathsf{H}_{\ell,t}, \quad \text{where} \quad \alpha \equiv \frac{\mathcal{F}^{-1}(Q)\mathcal{F}'(\mathcal{F}^{-1}(Q))}{\mathcal{F}(\mathcal{F}^{-1}(Q))}. \qquad [\text{E.13}]$$

The nominal marginal cost function corresponding to [9] has the following log-linear form:

$$\mathsf{X}_{\ell,t} = \gamma \mathsf{Q}_{\ell,t} + \mathsf{W}_t, \quad \text{where} \quad \gamma \equiv \frac{Q\mathscr{C}''(Q;W)}{\mathscr{C}'(Q;W)} = \left(-\frac{\mathcal{F}^{-1}(Q)\mathcal{F}''(\mathcal{F}^{-1}(Q))}{\mathcal{F}'(\mathcal{F}^{-1}(Q))}\right)\left(\frac{Q}{\mathcal{F}^{-1}(Q)\mathcal{F}'(\mathcal{F}^{-1}(Q))}\right). \qquad [\text{E.14}]$$

(i) The log-linearized first-order condition for the sales frequency (the first equation in [27]) is

$$(\chi-1)\mathsf{X}_{\ell,t} = \mu_S \chi \mathsf{p}_{S,\ell,t} - \mu_N \mathsf{R}_{N,t-\ell} + (\mu_S - 1)\chi(\mathsf{q}_{S,\ell,t} - \mathsf{q}_{N,\ell,t}), \qquad [\text{E.15}]$$

where the fact that $\chi = (\mu_N - 1)/(\mu_S - 1)$ is used to simplify the expression. By using equation [E.3]:

$$(\chi - 1)\mathsf{X}_{\ell,t} = \left(\mu_S - (\mu_S - 1)\left(\frac{\lambda\epsilon + (1-\lambda)\eta v_S}{\lambda + (1-\lambda)v_S}\right)\right)\chi \mathsf{p}_{S,\ell,t} - \left(\mu_N - (\mu_N - 1)\left(\frac{\lambda\epsilon + (1-\lambda)\eta v_N}{\lambda + (1-\lambda)v_N}\right)\right)\mathsf{R}_{N,t-\ell} + (\eta - \epsilon)\left(\frac{(1-\lambda)v_S}{\lambda + (1-\lambda)v_S} - \frac{(1-\lambda)v_N}{\lambda + (1-\lambda)v_N}\right)(\mu_S - 1)\chi \mathsf{P}_{B,t}.$$

Given the expressions for μ_S and μ_N in [17], the coefficients of both $\mathsf{p}_{S,\ell,t}$ and $\mathsf{R}_{N,\ell,t}$ in the above are zero. Since $\chi > 1$, this equation implies $\mathsf{X}_{\ell,t}$ is independent of $\mathsf{p}_{S,\ell,t}$ and $\mathsf{R}_{N,t-\ell}$. Using $\chi = (\mu_N - 1)/(\mu_S - 1)$ yields:

$$(\chi - 1)\mathsf{X}_{\ell,t} = (\chi - 1)\mathsf{P}_{B,t} - \left(1 - (\eta - \epsilon)\left(\frac{(1-\lambda)v_S}{\lambda + (1-\lambda)v_S}\right)(\mu_S - 1)\right)\chi \mathsf{P}_{B,t} + \left(1 - (\eta - \epsilon)\left(\frac{(1-\lambda)v_N}{\lambda + (1-\lambda)v_N}\right)(\mu_N - 1)\right)\mathsf{P}_{B,t}. \quad \text{[E.16]}$$

After substituting the expressions for μ_S and μ_N from [18], the above equation reduces to

$$(\chi - 1)\mathsf{X}_{\ell,t} = (\chi - 1)\mathsf{P}_{B,t} + (\epsilon - 1)\left((\mu_S - 1)\chi - (\mu_N - 1)\right)\mathsf{P}_{B,t},$$

and noting that the coefficient on the final term is zero, it follows that $\mathsf{X}_{\ell,t} = \mathsf{P}_{B,t}$ for all ℓ. Hence, all firms have the same marginal cost, $\mathsf{X}_t = \mathsf{P}_{B,t}$, irrespective of their normal-price vintage.

The optimal $p_{S,\ell,t}$ is characterized by the second equation in [27]. In log-linear terms it is

$$\mathsf{p}_{S,\ell,t} = \upmu_{S,\ell,t} + \mathsf{X}_t.$$

By substituting the expressions for the log-linearized optimal sale markup from [E.4] and the sale purchase multiplier from [E.2], and using $\mathsf{X}_t = \mathsf{P}_{B,t}$:

$$(1 - (\eta - \epsilon)\mathfrak{c}_S)(\mathsf{p}_{S,\ell,t} - \mathsf{X}_t) = 0, \quad \text{[E.17]}$$

so $\mathsf{p}_{S,\ell,t} = \mathsf{X}_t$ if the coefficient in the above is different from zero. The expressions for $\mathfrak{c}_S$ from [E.4] and μ_S from [18] imply

$$\frac{(1 - (\eta - \epsilon)\mathfrak{c}_S)}{\mu_S} = \frac{(\lambda(\epsilon - 1) + (1-\lambda)(\eta - 1)v_S)(\lambda\epsilon + (1-\lambda)\eta v_S) - (\eta - \epsilon)^2\lambda(1-\lambda)v_S}{(\lambda\epsilon + (1-\lambda)\eta v_S)^2}.$$

Using [B.8] and noting that $v_S = \rho_S^{\epsilon - \eta}$ it follows that $1 - (\eta - \epsilon)\mathfrak{c}_S = \mu_S \mathcal{D}'(\rho_S)\mathcal{R}''(\mathcal{D}(\rho_S))$, where the functions $\mathcal{D}(\rho)$ and $\mathcal{R}(\mathfrak{q})$ are defined in [B.1] and [B.4]. The coefficient in [E.17] is strictly positive because $\mathcal{D}'(\rho_S) < 0$ and Lemma 2 shows that $\mathcal{R}''(\mathcal{D}(\rho_S)) < 0$, and therefore $\mathsf{p}_{S,\ell,t} = \mathsf{X}_t$.

Since all firms face the same wage W_t, and as the argument above shows that all have the same nominal marginal cost X_t, the log linearization of nominal marginal cost in [E.14] shows that all must produce the same total quantity Q_t when $\gamma > 0$.

The log-linearization of the first-order condition [26] for the optimal reset price $R_{N,t}$ simplifies to

$$\sum_{\ell=0}^{\infty}(\beta\phi_p)^{\ell}\mathbb{E}_t\left[\mathsf{R}_{N,t} - \upmu_{N,\ell,t+\ell} - \mathsf{X}_{t+\ell}\right] = 0, \quad \text{[E.18]}$$

where $\upmu_{N,\ell,t}$ is the log-deviation of the optimal markup $\mu_{N,\ell,t} \equiv \upmu(R_{N,t-\ell}; P_{B,t})$. The optimal markup function is log-linearized in [E.4] and is given in terms of the corresponding purchase multiplier, itself log-linearized in [E.2]. Putting those results together, it follows that $\upmu_{N,\ell,t+\ell} = (\eta - \epsilon)\mathfrak{c}_N(\mathsf{R}_{N,t} - \mathsf{P}_{B,t+\ell})$. So by using $\mathsf{X}_t = \mathsf{P}_{B,t}$ and substituting these results into [E.18]:

$$(1-(\eta-\epsilon)\mathfrak{c}_N)\sum_{\ell=0}^{\infty}(\beta\phi_p)^{\ell}\mathbb{E}_t\left[R_{N,t}-\mathsf{X}_{t+\ell}\right]=0.$$

An exactly analogous argument to the proof of $1-(\eta-\epsilon)\mathfrak{c}_S > 0$ above shows that $1-(\eta-\epsilon)\mathfrak{c}_N > 0$ also holds. Hence:

$$\mathsf{R}_{N,t} = (1-\beta\phi_p)\sum_{\ell=0}^{\infty}(\beta\phi_p)^{\ell}\mathbb{E}_t\mathsf{X}_{t+\ell}. \tag{E.19}$$

(ii) By using $\mathsf{P}_{S,t} = \mathsf{X}_t$ and substituting this into [E.12] it is demonstrated that

$$\varphi_P\mathsf{s}_t = \vartheta_P(\mathsf{X}_t-\mathsf{P}_t)+(1-\vartheta_P)(\mathsf{P}_{N,t}-\mathsf{P}_t). \tag{E.20}$$

Likewise, by using $\mathsf{P}_{B,t} = \mathsf{X}_t$ and performing similar substitutions in the expression for $\mathsf{P}_{B,t}$ from [E.8]:

$$\varphi_B\mathsf{s}_t = (1-\vartheta_B)(\mathsf{P}_{N,t}-\mathsf{X}_t). \tag{E.21}$$

Equation [E.20] can be written as

$$\varphi_P\mathsf{s}_t = \vartheta_P(\mathsf{X}_t-\mathsf{P}_t)+(1-\vartheta_P)\left((\mathsf{P}_{N,t}-\mathsf{X}_t)+(\mathsf{X}_t-\mathsf{P}_t)\right),$$

and s_t is eliminated using [E.21]. After some rearrangement this leads to

$$\mathsf{X}_t-\mathsf{P}_{N,t} = \frac{1}{1-\psi}\mathsf{x}_t, \tag{E.22}$$

where $\mathsf{x}_t = \mathsf{X}_t - \mathsf{P}_t$ is real marginal cost and ψ is defined as follows:

$$\psi = \frac{(1-\vartheta_B)\varphi_P+\vartheta_P\varphi_B}{\varphi_B}. \tag{E.23}$$

Note that the recursive form of the expression for $\mathsf{P}_{N,t}$ in [E.6] is

$$\mathsf{P}_{N,t} = \phi_p\mathsf{P}_{N,t-1}+(1-\phi_p)\mathsf{R}_{N,t}, \tag{E.24}$$

and the recursive form of the equation [E.19] for $\mathsf{R}_{N,t}$ is:

$$\mathsf{R}_{N,t} = \beta\phi_p\mathbb{E}_t\mathsf{R}_{t+1}+(1-\beta\phi_p)\mathsf{X}_t. \tag{E.25}$$

Then multiplying both sides of the above by $(1-\phi_p)$ and substituting in the recursive equation for $\mathsf{P}_{N,t}$ yields

$$\mathsf{P}_{N,t}-\phi_p\mathsf{P}_{N,t-1} = \beta\phi_p\mathbb{E}_t\left[\mathsf{P}_{N,t+1}-\phi_p\mathsf{P}_{N,t}\right]+(1-\phi_p)(1-\beta\phi_p)\mathsf{X}_t,$$

which can be written in terms of normal-price inflation $\pi_{N,t} \equiv \mathsf{P}_{N,t}-\mathsf{P}_{N,t-1}$:

$$\pi_{N,t} = \beta\mathbb{E}_t\pi_{N,t+1}+\kappa(\mathsf{X}_t-\mathsf{P}_{N,t}), \tag{E.26}$$

and where $\kappa = (1-\phi_p)(1-\beta\phi_p)/\phi_p$ is as defined in the statement of the theorem.

Taking the first difference of [E.21] yields

$$\Delta\mathsf{s}_t = -\frac{(1-\vartheta_B)}{\varphi_B}\left(\Delta\mathsf{X}_t-\pi_{N,t}\right). \tag{E.27}$$

Now use [E.12] and make the substitution $\mathsf{P}_{S,t} = \mathsf{X}_t$ as before, and then take first differences and rearrange:

$$\pi_t = \pi_{N,t}+\vartheta_P\left(\Delta\mathsf{X}_t-\pi_{N,t}\right)-\varphi_P\Delta\mathsf{s}_t.$$

By eliminating $\mathbf{\Delta}\mathsf{s}_t$ from this equation using [E.27]:

$$\pi_t = \pi_{N,t} + \psi\left(\mathbf{\Delta}\mathsf{X}_t - \pi_{N,t}\right).$$

Substituting the first difference of equation [E.22] into the above yields

$$\pi_{N,t} = \pi_t - \frac{\psi}{1-\psi}\mathbf{\Delta}\mathsf{x}_t.$$

Combining this equation with [E.22] and [E.26] implies

$$\left(\pi_t - \frac{\psi}{1-\psi}\mathbf{\Delta}\mathsf{x}_t\right) = \beta\mathbb{E}_t\left[\pi_{t+1} - \frac{\psi}{1-\psi}\mathbf{\Delta}\mathsf{x}_{t+1}\right] + \frac{\kappa}{1-\psi}\mathsf{x}_t,$$

which is rearranged to yield the result [32]. Recursive forward substitution of equation [32] leads to

$$\pi_t = \frac{1}{1-\psi}\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\left[\kappa\mathsf{x}_{t+\ell} + \psi\left(\mathbf{\Delta}\mathsf{x}_{t+\ell} - \beta\mathbf{\Delta}\mathsf{x}_{t+1+\ell}\right)\right].$$

Notice that all $\mathbf{\Delta}\mathsf{x}_{t+\ell}$ terms apart from $\mathbf{\Delta}\mathsf{x}_t$ cancel out because each occurs twice with opposite signs. Hence equation [33] is obtained.

(iii) Equation [E.23] implies that an expression for $1-\psi$ is

$$1-\psi = \frac{(1-\vartheta_P)\varphi_B - (1-\vartheta_B)\varphi_P}{\varphi_B}. \tag{E.28}$$

It follows from [E.12] that $(1-\vartheta_P) = (1-\varpi)(1-\vartheta_L) + \varpi(1-\vartheta_B)$. Together with the expression for φ_P from the same equation, [E.28] implies

$$1-\psi = \frac{\left((1-\varpi)(1-\vartheta_L) + \varpi(1-\vartheta_B)\right)\varphi_B - (1-\vartheta_B)\left((1-\varpi)\varphi_L + \varpi\varphi_B\right)}{\varphi_B},$$

and by rearranging this expression:

$$1-\psi = (1-\varpi)\varphi_L\left(\frac{1-\vartheta_L}{\varphi_L} - \frac{1-\vartheta_B}{\varphi_B}\right). \tag{E.29}$$

Define the function

$$\Phi(\zeta;\mu) \equiv \frac{\mu^{-\zeta}-1}{\zeta} \tag{E.30}$$

in terms of the markup ratio μ. An alternative expression for this function is $\Phi(\zeta;\mu) = (e^{(-\log\mu)\zeta}-1)/\zeta$, which shows that it has derivative

$$\Phi'(\zeta;\mu) = \frac{\left((-\log\mu)\zeta - 1\right)e^{(-\log\mu)\zeta} + 1}{\zeta^2}.$$

Now define another function

$$\mathcal{J}(z) \equiv 1 + (z-1)e^z,$$

and note that $\mathcal{J}'(z) = ze^z$. Since $\mathcal{J}(0) = 0$, and $\mathcal{J}'(z) > 0$ for all $z > 0$, it follows that $\mathcal{J}(z) > 0$ for all $z > 0$. Then note

$$\Phi'(\zeta;\mu) = \frac{\mathcal{J}\left((-\log\mu)\zeta\right)}{\zeta^2},$$

which proves that $\Phi(\zeta;\mu)$ is strictly increasing in ζ when $\zeta > 0$ since $0 < \mu < 1$.

The expressions for ϑ_L and φ_L given in [E.10] are now used to demonstrate that:

$$\frac{1-\vartheta_L}{\varphi_L} = (1-s)\left(\frac{\epsilon-1}{(\mu^{-1})^{\epsilon-1}-1}\right) = \frac{1-s}{\Phi(\epsilon-1;\mu)}. \qquad \text{[E.31]}$$

Similarly, the expressions for ϑ_B and φ_B from [E.8] yield

$$\frac{1-\vartheta_B}{\varphi_B} = (1-s)\left(\frac{\eta-1}{(\mu^{-1})^{\eta-1}-1}\right) = \frac{1-s}{\Phi(\eta-1;\mu)}. \qquad \text{[E.32]}$$

These formulæ are then substituted into [E.29] to obtain:

$$1-\psi = (1-\varpi)(1-s)\varphi_L\left(\frac{1}{\Phi(\epsilon-1;\mu)} - \frac{1}{\Phi(\eta-1;\mu)}\right).$$

The expression for ψ in [E.23] together with the properties of ϑ_B, ϑ_P, φ_B and φ_P derived earlier demonstrates that $\psi \geq 0$. The inequality $\psi \leq 1$ follows from $\Phi(\zeta;\mu)$ being an increasing function of ζ together with $\eta > \epsilon$ and the properties of ϖ and φ_L. Thus, it is established that $0 \leq \psi \leq 1$.

Now use [E.31] to obtain the following:

$$1-\psi = (1-\varpi)(1-\vartheta_L)\left(1-\Theta(\epsilon,\eta;\mu)\right), \quad \text{where } \Theta(\epsilon,\eta;\mu) \equiv \frac{\Phi(\epsilon-1;\mu)}{\Phi(\eta-1;\mu)}. \qquad \text{[E.33]}$$

Note that the expression for P_B in [20] can be substituted into $v(p_S;P_B)$ from [7] to obtain:

$$v_S = \frac{1}{(s+(1-s)\mu^{\eta-1})^{\frac{\eta-\epsilon}{\eta-1}}},$$

and which by combining this with the expression for $\hbar$ from [E.11] yields

$$\hbar^{\epsilon-1} = \frac{1}{v_S}\left(\frac{s+(1-s)\mu^{\epsilon-1}}{s+(1-s)\mu^{\eta-1}}\right).$$

Thus, the weight $1-\varpi$ given in [E.11] is

$$1-\varpi = \frac{\lambda(s+(1-s)\mu^{\epsilon-1})}{\lambda(s+(1-s)\mu^{\epsilon-1}) + (1-\lambda)v_S(s+(1-s)\mu^{\eta-1})}.$$

Substituting this into [E.33] and using the formula for ϑ_L from [E.10] implies

$$1-\psi = \frac{\lambda(1-s)\mu^{\epsilon-1}}{\lambda(s+(1-s)\mu^{\epsilon-1}) + (1-\lambda)v_S(s+(1-s)\mu^{\eta-1})}\left(1-\Theta(\epsilon,\eta;\mu)\right).$$

Since the purchase multipliers are given by $v_N = \rho_N^{-(\eta-\epsilon)}$ and $v_S = \rho_S^{-(\eta-\epsilon)}$, the expressions for ρ_S and ρ_N from Lemma 3 imply that

$$(1-\lambda)v_N = \lambda z, \quad \text{and} \quad (1-\lambda)v_S = \mu^{\epsilon-\eta}\lambda z, \qquad \text{[E.34]}$$

where $z = \mathfrak{z}(\mu;\epsilon,\eta)$ is the value of the function in [A.3]. Substituting v_S into the expression for $1-\psi$ above yields

$$1-\psi = (1-s)(1-\Theta(\epsilon,\eta;\mu))\frac{\mu^{\epsilon-1}}{(s+(1-s)\mu^{\epsilon-1}) + \mu^{\epsilon-\eta}z(s+(1-s)\mu^{\eta-1})}.$$

After further rearrangement this implies

$$1-\psi = \frac{(1-\Theta(\epsilon,\eta;\mu))(1-s)}{(1+z) + \left((\mu^{1-\epsilon}-1) + (\mu^{1-\eta}-1)z\right)s}. \qquad \text{[E.35]}$$

For parameters consistent with a two-price equilibrium, Lemma 3 shows that $z = \mathfrak{z}(\mu; \epsilon, \eta)$ must be a positive real number. The definition of $\Phi(\zeta; \mu)$ in [E.30] implies that it is non-negative when $0 < \mu < 1$ and $\zeta > 0$. Since $\eta > \epsilon > 1$ and as $\Phi(\zeta; \mu)$ is increasing in ζ, the definition of $\Theta(\epsilon, \eta; \mu)$ in [E.33] ensures that $0 \leq \Theta(\epsilon, \eta; \mu) \leq 1$. Hence, because all terms in the expression above for $1 - \psi$ are positive, the derivative with respect to s (holding ϵ and η constant, and hence μ and z constant by Lemma 3) is negative. Proposition 1 shows that λ and s are negatively related (holding ϵ and η constant), so ψ is strictly decreasing in λ.

By using [A.4], it follows that $\mu\chi = \mu^{1-\epsilon}(1 + \mu^{\epsilon-\eta}z)/(1 + z)$, and hence

$$s\mu\chi + (1 - s) = \frac{1}{1 + z}\left((1 + z) + \left((\mu^{1-\epsilon} - 1) + (\mu^{1-\eta} - 1)z\right) s\right).$$

This expression is substituted into [E.35] to yield

$$1 - \psi = \frac{(1 - \Theta(\epsilon, \eta; \mu))(1 - s)}{(1 + z)(s\mu\chi + (1 - s))}. \tag{E.36}$$

Note that $\psi = 1$ requires the right-hand side of this expression to be zero. There are four terms to consider. First, $s = 1$ is the only way the expression can be zero as a result of the $1 - s$ term. Now consider the terms in the denominator. Since $\mu = p_S/p_N$ and $\chi = q_S/q_N$, the second term in the denominator is linked to the GDP share transacted at the normal price:

$$\frac{1}{s\mu\chi + (1 - s)} = \frac{1}{1 - s}\left(\frac{(1 - s)p_N q_N}{s p_S q_S + (1 - s)p_N q_N}\right).$$

So when $s < 1$, $(s\mu\chi + (1 - s)) \to \infty$ only if $(1 - s)p_N q_N/(s p_S q_S + (1 - s)p_N q_N) \to 0$, that is, the GDP share traded at the sticky normal price tends to zero. The other term in the denominator is $1 + z$, where $z = \mathfrak{z}(\mu; \epsilon, \eta)$, which is the smallest root of the quadratic [B.33]. As the proof of Lemma 3 demonstrates, this quadratic must always have two positive real roots in the relevant parameter range. The product of these roots is obtained from the coefficients of the quadratic in [B.33]:

$$\left(\frac{\epsilon(\epsilon - 1)}{\eta(\eta - 1)}\right)\mu^{\eta-\epsilon},$$

which is always less than one, hence $1 + z$ is finite, so the only way the denominator of [E.36] can approach infinity is through the normal-price GDP share approaching zero.

The final possibility to consider is $\Theta(\epsilon, \eta; \mu) = 1$. The function $\Theta(\epsilon, \eta; \mu)$ from [E.33] can be written as:

$$\Theta(\epsilon, \eta; \mu) = \left(\frac{\eta - 1}{\epsilon - 1}\right)\left(\frac{e^{(-\log\mu)(\epsilon-1)} - 1}{e^{(-\log\mu)(\eta-1)} - 1}\right),$$

and by L'Hôpital's rule:

$$\lim_{\mu \to 1} \Theta(\epsilon, \eta; \mu) = 1,$$

for any elasticities ϵ and η such that $1 < \epsilon < \eta$, so $\mu = 1$ is also a possible way that $\psi = 1$ could occur. Now take any other parameters ϵ and η such that $0 \leq \mu < 1$. The non-monotonicity condition [16] is necessary for an equilibrium with $\mu < 1$ to exist. Note that [16] implies that ϵ can never approach η in the region of parameters consistent with $\mu < 1$. Since $\Phi(\zeta; \mu)$ is known to be strictly increasing in ζ for any $0 < \mu < 1$, and that η is bounded away from ϵ, it follows that $\Phi(\epsilon - 1; \mu) < \Phi(\eta - 1; \mu)$ and thus $\Theta(\epsilon, \eta; \mu) < 1$ for any $\mu < 1$. This argument establishes that $\mu = 1$ is the only other possible way that $\psi = 1$ can occur, and so completes the proof.

The arguments developed in the proof above lead to the following set of results characterizing the fluctuations in other variables of interest.

Lemma 4 *The Phillips curve in [32] is a relationship between aggregate inflation π_t and real marginal cost x_t. Underlying this relationship are the following:*

(i) The average sale discount $P_{N,t} - P_{S,t}$ is determined by real marginal cost x_t. There is a negative relationship between $P_{N,t} - P_{S,t}$ and x_t, and the magnitude of the response of the average sale discount to real marginal cost is decreasing in λ.

(ii) The average quantity ratio $q_{S,t} - q_{N,t}$ is determined by real marginal cost x_t. There is a positive relationship between $q_{S,t} - q_{N,t}$ and x_t, and the magnitude of the response of the average quantity ratio to real marginal cost is decreasing in λ.

(iii) The average sales frequency s_t is determined by real marginal cost x_t. There is a negative relationship between s_t and x_t, and the magnitude of the response of the average sales frequency to real marginal cost is decreasing in λ.

(iv) A firm with a normal price above the average has a sale discount above the average and a sales frequency above the average.

(v) Relative price distortions $Q_t - Y_t$ are negatively related to real marginal cost x_t.

Proof (i) Let $\mu_t = P_{S,t} - P_{N,t}$. Using the result $P_{S,t} = X_t$ from Theorem 2 and [E.22], it follows that

$$\mu_t = \frac{1}{1-\psi} x_t. \qquad \text{[E.37]}$$

The coefficient on x_t is known to be positive because of the inequality for ψ derived in Theorem 2. Its magnitude is decreasing in λ because ψ is negatively related to λ, as shown in Theorem 2.

(ii) Let $\chi_t = q_{S,t} - q_{N,t}$. The log-linearized demand functions and purchase multipliers in [E.1] and [E.2] imply

$$\chi_t = -\zeta_N (P_{N,t} - P_{S,t}),$$

with ζ_N being the steady-state price elasticity at the normal price, and where $P_{S,t} = P_{B,t}$ has been used. Substitution of the result in [E.37] yields

$$\chi_t = \frac{\zeta_N}{1-\psi} x_t.$$

Using the inequality for ψ from Theorem 2 and $\zeta_N > 0$, it follows that the coefficient of x_t in the above is positive. By combining the expression for ζ_N from [14] and equation [E.34]:

$$\zeta_N = \frac{\epsilon + z\eta}{1+z}. \qquad \text{[E.38]}$$

Since $z = \mathfrak{z}(\mu; \epsilon, \eta)$, it follows from Lemma 3 that ζ_N is independent of λ. Hence, since Theorem 2 shows that ψ is decreasing in λ, the coefficient of x_t in the equation for χ_t is also decreasing in λ.

(iii) For the average sales frequency s_t, use equation [E.21] together with $X_t = P_{S,t}$ and the expression for μ_t in [E.37] to obtain:

$$s_t = -\left(\frac{1-\vartheta_B}{\varphi_B}\right)\left(\frac{1}{1-\psi}\right) x_t. \qquad \text{[E.39]}$$

It has been shown that $0 \leq \vartheta_B \leq 1$, $\varphi_B \geq 0$, and $0 \leq \psi \leq 1$, so it follows that the coefficient of x_t above is negative. By substituting the expressions for $(1-\vartheta_B)/\varphi_B$ from [E.32] and $1-\psi$ from [E.35] into the above:

$$\left(\frac{1-\vartheta_B}{\varphi_B}\right)\left(\frac{1}{1-\psi}\right) = \frac{(1+z) + \left((\mu^{1-\epsilon} - 1) + (\mu^{1-\eta} - 1)z\right)s}{\Phi(\eta - 1; \mu)\left(1 - \Theta(\epsilon, \eta; \mu)\right)}.$$

Since all the terms in the denominator and μ and z in the numerator are independent of λ, it follows that the magnitude of this coefficient is decreasing in λ because s is decreasing in λ.

(iv) Since Theorem 2 implies that $\mathsf{p}_{S,\ell,t} = \mathsf{P}_{S,t}$ for all ℓ, it follows that:

$$(\mathsf{R}_{N,t-\ell} - \mathsf{p}_{S,\ell,t}) - (\mathsf{P}_{N,t} - \mathsf{P}_{S,t}) = (\mathsf{R}_{N,t-\ell} - \mathsf{P}_{N,t}),$$

where $\mathsf{R}_{N,t-\ell} - \mathsf{P}_{N,t}$ clearly has a positive coefficient. A further consequence of $\mathsf{p}_{S,\ell,t} = \mathsf{P}_{S,t}$ is that $\mathsf{q}_{S,\ell,t} = \mathsf{q}_{S,t}$ for all ℓ. The demand function in [E.3] implies $\mathsf{q}_{N,\ell,t} - \mathsf{q}_{N,t} = -\zeta_N(\mathsf{R}_{N,t-\ell} - \mathsf{P}_{N,t})$. Together with equation [E.5] and the result $\mathsf{Q}_{\ell,t} = \mathsf{Q}_t$ from Theorem 2:

$$\mathsf{s}_{\ell,t} - \mathsf{s}_t = \frac{(1-s)\zeta_N}{\chi - 1}(\mathsf{R}_{N,t-\ell} - \mathsf{P}_{N,t}),$$

with the coefficient on $\mathsf{R}_{N,t-\ell} - \mathsf{P}_{N,t}$ in the above being positive.

(v) Let $\Delta_t = \mathsf{Y}_t - \mathsf{Q}_t$. From the expression for the log-linearized demand function and purchase multipliers in [E.1] and [E.2], the following individual demand functions are obtained:

$$\mathsf{q}_{S,t} = -\epsilon \mathsf{x}_t + \mathsf{Y}_t, \quad \mathsf{q}_{N,t} = -\epsilon \mathsf{x}_t + \mathsf{Y}_t - \zeta_N(\mathsf{P}_{N,t} - \mathsf{P}_{S,t}),$$

where the results $\mathsf{P}_{S,t} = \mathsf{P}_{B,t} = \mathsf{X}_t$ from Theorem 2 have been used. By substituting these into the expression for total quantity from [E.5]:

$$\mathsf{Q}_t = \mathsf{Y}_t - \epsilon \mathsf{x}_t - \zeta_N \left(\frac{(1-s)}{s\chi + (1-s)}\right)(\mathsf{P}_{N,t} - \mathsf{P}_{S,t}) + \left(\frac{\chi - 1}{s\chi + (1-s)}\right)\mathsf{s}_t.$$

Substituting [E.37] and [E.39] in the above expression yields

$$\Delta_t \equiv \mathsf{Y}_t - \mathsf{Q}_t = \left(\epsilon + \frac{1}{(s\chi + (1-s))(1-\psi)}\left((\chi - 1)\left(\frac{1-\vartheta_B}{\varphi_B}\right) - (1-s)\zeta_N\right)\right)\mathsf{x}_t.$$

This is written as $\Delta_t = \delta \mathsf{x}_t$, with the coefficient δ of real marginal cost x_t defined by:

$$\delta = \frac{s\chi\mu + (1-s)}{s\chi + (1-s)}\left(\epsilon \frac{s\chi + (1-s)}{s\chi\mu + (1-s)} + \wp\right), \tag{E.40}$$

and where the term $\wp$ is:

$$\wp = \frac{1}{(1-\psi)(s\mu\chi + (1-s))}\left((\chi - 1)\left(\frac{1-\vartheta_B}{\varphi_B}\right) - (1-s)\zeta_N\right).$$

By substituting the expression for $1 - \psi$ from [E.36] and rearranging:

$$\wp = \frac{1+z}{1 - \Theta(\epsilon, \eta; \mu)}\left((\chi - 1)\left(\frac{1-\vartheta_B}{\varphi_B(1-s)}\right) - \zeta_N\right).$$

Equation [E.32] then implies

$$\wp = \frac{1+z}{1 - \Theta(\epsilon, \eta; \mu)}\left(\frac{\chi - 1}{\Phi(\eta - 1; \mu)} - \zeta_N\right). \tag{E.41}$$

Noting that equation [A.4] can be used to express $\chi - 1$ as follows:

$$\chi - 1 = \frac{(\mu^{-\epsilon} - 1) + z(\mu^{-\eta} - 1)}{1+z},$$

and substituting this together with the formula for ζ_N in [E.38] into the expression for $\wp$ from [E.41]:

$$\wp = \frac{(\mu^{-\epsilon} - 1) + z(\mu^{-\eta} - 1) - (\epsilon + \eta z)\Phi(\eta - 1; \mu)}{(1 - \Theta(\epsilon, \eta; \mu))\,\Phi(\eta - 1; \mu)}.$$

By using the definitions of the functions $\Phi(\zeta;\mu)$ and $\Theta(\epsilon,\eta;\mu)$ from [E.30] and [E.33]:

$$\wp = \frac{\epsilon\left(\Phi(\epsilon;\mu) - \Phi(\eta-1;\mu)\right) + z\eta\left(\Phi(\eta;\mu) - \Phi(\eta-1;\mu)\right)}{\Phi(\eta-1;\mu) - \Phi(\epsilon-1;\mu)}.$$

The expression for δ from [E.40] can thus be written as:

$$\delta = \frac{s\chi\mu + (1-s)}{s\chi + (1-s)}\left(\epsilon\frac{s\chi + (1-s)}{s\chi\mu + (1-s)} + \frac{\epsilon\left(\Phi(\epsilon;\mu) - \Phi(\eta-1;\mu)\right) + z\eta\left(\Phi(\eta;\mu) - \Phi(\eta-1;\mu)\right)}{\Phi(\eta-1;\mu) - \Phi(\epsilon-1;\mu)}\right).$$

The final expression for δ is obtained by adding and subtracting ϵ inside the brackets:

$$\delta = \frac{s\chi\epsilon(1-\mu)}{s\chi + (1-s)} + \frac{s\chi\mu + (1-s)}{s\chi + (1-s)}\left(\frac{\epsilon\left(\Phi(\epsilon;\mu) - \Phi(\epsilon-1;\mu)\right) + z\eta\left(\Phi(\eta;\mu) - \Phi(\eta-1;\mu)\right)}{\Phi(\eta-1;\mu) - \Phi(\epsilon-1;\mu)}\right).$$

Since the function $\Phi(\zeta;\mu)$ from [E.30] is known to be strictly increasing in ζ, it follows that δ is positive. This completes the proof. ■

F DSGE model derivations

Wage-setting behaviour

When each firm chooses its use of the continuum of labour inputs to minimize the cost of obtaining a unit of H from equation [30], the minimized cost is given by the wage index

$$W \equiv \left(\int W(\imath)^{1-\varsigma} d\imath\right)^{\frac{1}{1-\varsigma}}, \qquad \text{[F.1]}$$

and the cost-minimizing labour demand functions are

$$H(\imath) = \left(\frac{W(\imath)}{W}\right)^{-\varsigma} H. \qquad \text{[F.2]}$$

As households are selected to update their wages at random, as they enjoy the same consumption, and as they face the same demand function for their labour services, all households setting a new wage at time t choose the same wage. This common wage is referred to as the reset wage, and is denoted by $R_{W,t}$. It is chosen to maximize expected utility over the lifetime of the wage subject to the labour demand function [F.2]. As shown by Erceg, Henderson and Levin (2000), the first-order condition for this maximization problem is

$$\sum_{\ell=0}^{\infty}(\beta\phi_w)^{\ell}\mathbb{E}_t\left[\frac{W_{t+\ell}^{\varsigma}H_{t+\ell}\upsilon_c(Y_{t+\ell})}{\upsilon_c(Y_t)}\left\{\frac{R_{W,t}}{P_{t+\ell}} - \frac{\varsigma}{\varsigma-1}\frac{\nu_h\left(R_{W,t}^{-\varsigma}W_{t+\ell}^{\varsigma}H_{t+\ell}\right)}{\upsilon_c(Y_{t+\ell})}\right\}\right] = 0. \qquad \text{[F.3]}$$

The wage index W_t in [F.1] then evolves according to

$$W_t = \left((1-\phi_w)\sum_{\ell=0}^{\infty}\phi_w^{\ell}R_{W,t-\ell}^{1-\varsigma}\right)^{\frac{1}{1-\varsigma}}. \qquad \text{[F.4]}$$

The presence of market power in wage setting means that the equation [24] determining steady-state output Y is replaced by

$$x = \frac{\varsigma}{\varsigma-1}\frac{\nu_h\left(\mathcal{F}^{-1}(Y/\Delta)\right)}{\upsilon_c(Y)\mathcal{F}'\left(\mathcal{F}^{-1}(Y/\Delta)\right)}.$$

All other equations determining the steady state are unaffected.

Log linearizations

The DSGE model is log linearized around the flexible-price equilibrium characterized in section 3. The notational convention is that a variable without a time subscript denotes its flexible-price steady-state value, and the corresponding sans serif letter denotes the log deviation of the variable from its steady-state value (except for the inflation rate and the nominal interest rate, where it denotes the log deviation of the corresponding gross rates).

The log linearization of the intertemporal IS equation in [34] is

$$\mathsf{Y}_t = \mathbb{E}_t\mathsf{Y}_{t+1} - \theta_c\left(\mathsf{i}_t - \mathbb{E}_t\pi_{t+1}\right), \quad \text{where } \theta_c \equiv -\left(\frac{Y\upsilon_{cc}(Y)}{\upsilon_c(Y)}\right)^{-1}. \qquad \text{[F.5]}$$

The intertemporal elasticity of substitution is θ_c. Money demand is implied by the binding cash-in-advance constraint in [34]. It is log linearized as follows:

$$\mathsf{M}_t - \mathsf{P}_t = \mathsf{Y}_t. \qquad \text{[F.6]}$$

The money supply rule [35] has the following log-linear form:

$$\boldsymbol{\Delta}\mathsf{M}_t = \mathfrak{p}\boldsymbol{\Delta}\mathsf{M}_{t-1} + (1-\mathfrak{p})\mathsf{e}_t. \qquad \text{[F.7]}$$

The log-linearized version of equation [F.3] for the utility-maximizing reset wage is

$$\mathsf{R}_{W,t} = (1-\beta\phi_w)\sum_{\ell=0}^{\infty}(\beta\phi_w)^{\ell}\mathbb{E}_t\left[\left(\frac{1}{1+\varsigma\theta_h^{-1}}\right)(\mathsf{P}_{t+\ell}+\mathsf{w}^*_{t+\ell}) + \left(\frac{\varsigma\theta_h^{-1}}{1+\varsigma\theta_h^{-1}}\right)\mathsf{W}_{t+\ell}\right], \qquad \text{[F.8]}$$

with w^*_t being the desired real wage in the absence of constraints on wage adjustment:

$$\mathsf{w}^*_t = \theta_h^{-1}\mathsf{H}_t + \theta_c^{-1}\mathsf{Y}, \quad \text{where } \theta_h \equiv \left(\frac{\mathcal{F}^{-1}(Y/\Delta)\nu_{hh}(\mathcal{F}^{-1}(Y/\Delta))}{\nu_h(\mathcal{F}^{-1}(Y/\Delta))}\right)^{-1}. \qquad \text{[F.9]}$$

The Frisch elasticity of labour supply is θ_h. Equation [F.8] has the following recursive form:

$$\mathsf{R}_{W,t} = \beta\phi_w\mathbb{E}_t\mathsf{R}_{W,t+1} + (1-\beta\phi_w)\left(\left(\frac{1}{1+\varsigma\theta_h^{-1}}\right)(\mathsf{P}_t+\mathsf{w}^*_t) + \left(\frac{\varsigma\theta_h^{-1}}{1+\varsigma\theta_h^{-1}}\right)\mathsf{W}_t\right). \qquad \text{[F.10]}$$

The log-linearized wage index [F.4] is

$$\mathsf{W}_t = \sum_{\ell=0}^{\infty}(1-\phi_w)\phi_w^{\ell}\mathsf{R}_{W,t-\ell},$$

which also has a recursive form:

$$\mathsf{W}_t = \phi_w\mathsf{W}_{t-1} + (1-\phi_w)\mathsf{R}_{W,t}. \qquad \text{[F.11]}$$

Combining the reset wage equation [F.10] with the wage index equation [F.11] yields an expression for wage inflation $\pi_{W,t} \equiv \mathsf{W}_t - \mathsf{W}_{t-1}$:

$$\pi_{W,t} = \beta\mathbb{E}_t\pi_{W,t+1} + \frac{(1-\phi_w)(1-\beta\phi_w)}{\phi_w}\frac{1}{1+\varsigma\theta_h^{-1}}\left(\mathsf{w}^*_t - \mathsf{w}_t\right), \qquad \text{[F.12]}$$

where w^*_t is defined in [F.9].

By averaging over normal-price vintages, equations [E.13] and [E.14] imply:

$$\mathsf{Q}_t = \alpha\mathsf{H}_t, \quad \text{and } \mathsf{x}_t = \mathsf{w}_t + \gamma\mathsf{Q}_t. \qquad \text{[F.13]}$$

Substituting $\mathsf{Y}_t = \mathsf{Q}_t + \delta\mathsf{x}_t$ from Lemma 4 into the above yields [A.9b]. Using equation [F.13] to eliminate

H_t from [F.9] implies:

$$w_t^* = \frac{\theta_h^{-1}}{\alpha} Q_t + \theta_c^{-1} Y_t.$$

Then by using $Q_t = Y_t - \delta x_t$ to eliminate Q_t and substituting in the expression for x_t from [A.9b] leads to the following expression for $w_t^* - w_t$:

$$w_t^* - w_t = \left(\theta_c^{-1} + \frac{1}{1+\gamma\delta}\frac{\theta_h^{-1}}{\alpha}\right) Y_t - \left(1 + \frac{\delta}{1+\gamma\delta}\frac{\theta_h^{-1}}{\alpha}\right) w_t.$$

Replacing $w_t^* - w_t$ in [F.12] with the expression above yields [A.9c].

G Two-sector model

DSGE model

The steady state of the two-sector model from section 5 is derived exactly as for the one-sector model by taking the sale sector as representative of the whole economy. This steady state is characterized in section 3.4 and can be computed as described in Appendix A.

The system of equations of the two-sector DSGE model with sales is

$$\bar{\pi}_t = \beta \mathbb{E}_t \bar{\pi}_{t+1} + \frac{1}{1-\bar{\psi}} \left(\kappa x_t + \bar{\psi}\left(\Delta x_t - \beta \mathbb{E}_t \Delta x_{t+1}\right)\right) + \left(\frac{1-\sigma}{1-\bar{\psi}}\left(\kappa \rho_t + \Delta\rho_t - \beta \mathbb{E}_t \Delta\rho_{t+1}\right)\right); \qquad \text{[G.1a]}$$

$$\Delta\rho_t = \beta \mathbb{E}_t \Delta\rho_{t+1} + \frac{\kappa}{1+\xi\gamma}\left(\frac{\gamma((1-\psi)\delta + \psi\epsilon - \xi)}{1-\bar{\psi}} x_t - \frac{\gamma(1-\psi)(\epsilon - (1-\sigma)\delta) + (1-\bar{\psi}) + (1-\sigma)\xi\gamma}{1-\bar{\psi}} \rho_t\right); \qquad \text{[G.1b]}$$

$$Y_t = \bar{Y}_t + \epsilon\left(\frac{1-\sigma}{1-\bar{\psi}}\right)\left((1-\psi)\rho_t - \psi x_t\right); \qquad \text{[G.1c]}$$

$$Y_t = Q_t + \delta\left(\frac{1-\psi}{1-\bar{\psi}}\right)\left(x_t + (1-\sigma)\rho_t\right); \qquad \text{[G.1d]}$$

$$x_t = w_t + \gamma Q_t; \qquad \text{[G.1e]}$$

$$\pi_{W,t} = \beta \mathbb{E}_t \pi_{W,t+1} + \frac{(1-\phi_w)(1-\beta\phi_w)}{\phi_w}\frac{1}{1+\varsigma\theta_h^{-1}}(w_t^* - w_t); \qquad \text{[G.1f]}$$

$$w_t^* = \left(\theta_c^{-1} + \frac{\theta_h^{-1}}{\alpha}\frac{\varDelta}{\sigma + (1-\sigma)\varDelta}\right)\bar{Y}_t + \frac{\theta_h^{-1}}{\alpha}\frac{\sigma}{\sigma + (1-\sigma)\varDelta}\left(Q_t - \varDelta Y_t\right); \qquad \text{[G.1g]}$$

$$\Delta w_t = \pi_{W,t} - \bar{\pi}_t; \qquad \text{[G.1h]}$$

$$\bar{Y}_t = \mathbb{E}_t \bar{Y}_{t+1} - \theta_c\left(i_t - \mathbb{E}_t \bar{\pi}_{t+1}\right); \qquad \text{[G.1i]}$$

$$\Delta\bar{Y}_t = \Delta M_t - \bar{\pi}_t; \qquad \text{[G.1j]}$$

$$\Delta M_t = \mathfrak{p}\Delta M_{t-1} + (1-\mathfrak{p})e_t. \qquad \text{[G.1k]}$$

A bar above a variable denotes the log-deviation averaged across both sale and non-sale sectors, using the appropriate weights (σ and $1-\sigma$), and this convention is also employed for the Phillips curve coefficient ψ, with $\bar{\psi}$ denoting the average Phillips curve coefficient $\sigma\psi$. All variables without a bar refer either to economy-wide aggregates, or sale-sector variables as used in earlier sections, as appropriate. The coefficients $\varDelta$, ψ, δ, ξ and κ are calculated using the same formulæ as those for the one-sector economy given in appendix A taking the sale sector as representative of the whole economy.

Derivation of the two-sector model

In the following, the notational conventions in addition to those already described are that large script letters denote non-sale sector variables and small script letters denote the corresponding log deviations of the non-sale sector variables.

The aggregate price level is now

$$\bar{P}_t = \left(\sigma P_t^{1-\epsilon} + (1-\sigma)\mathcal{P}_t^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}},$$

which has the log linear form:

$$\bar{\mathsf{P}}_t = \sigma \mathsf{P}_t + (1-\sigma)\mathscr{P}_t. \qquad \text{[G.2]}$$

The log-linearized price level $\mathscr{P}_t$ in the non-sale sector is a weighted average of past reset prices $\mathscr{R}_t$ in that sector:

$$\mathscr{P}_t = \phi_p \mathscr{P}_{t-1} + (1-\phi_p)\mathscr{R}_t. \qquad \text{[G.3]}$$

The log-linearized first-order condition for the non-sale sector reset price is standard:

$$\mathscr{R}_t = \beta\phi_p \mathbb{E}_t \mathscr{R}_{t+1} + (1-\beta\phi_p)\left(\frac{1}{1+\xi\gamma}\mathscr{X}_t + \frac{\xi\gamma}{1+\xi\gamma}\mathscr{P}_t\right), \qquad \text{[G.4]}$$

where ξ is the constant price elasticity in that sector and γ is the elasticity of marginal cost with respect to output at the firm level.

Optimization by households implies the following overall relative demand between the sale and non-sale sectors:

$$\frac{\mathcal{Y}_t}{Y_t} = \left(\frac{\mathcal{P}_t}{P_t}\right)^{-\epsilon},$$

which has the log-linear form:

$$\mathscr{Y}_t - \mathsf{Y}_t = -\epsilon(\mathscr{P}_t - \mathsf{P}_t). \qquad \text{[G.5]}$$

Define $\rho_t \equiv \mathscr{P}_t - \mathsf{P}_{N,t}$ to be the average relative price between the non-sale sector and the normal prices in the sale sector. Substituting the sale-sector price level equation into the aggregate price level leads to

$$\bar{\mathsf{P}}_t = (1-\bar{\vartheta}_P)\mathsf{P}_{N,t} + \bar{\vartheta}_P \mathsf{P}_{S,t} - \bar{\varphi}_P \mathsf{s}_t + (1-\sigma)\rho_t, \qquad \text{[G.6]}$$

where $\bar{\vartheta}_P = \sigma\vartheta_P$ and $\bar{\varphi}_P = \sigma\varphi_P$ are defined (by analogy with the aggregate Phillips curve coefficient $\bar{\psi}$).

Real marginal cost x_t for the sale sector is defined in the usual way. By using equation [G.6]:

$$\mathsf{x}_t = (1-\bar{\vartheta}_P)(\mathsf{P}_{S,t} - \mathsf{P}_{N,t}) + \bar{\varphi}_P \mathsf{s}_t - (1-\sigma)\rho_t,$$

where $\mathsf{X}_t = \mathsf{P}_{S,t}$ has been substituted. Then by using [E.21] to eliminate s_t and rearranging:

$$\mathsf{x}_t = \left(\frac{(1-\bar{\vartheta}_P)\varphi_B - (1-\vartheta_B)\bar{\varphi}_P}{\varphi_B}\right)(\mathsf{P}_{S,t} - \mathsf{P}_{N,t}) - (1-\sigma)\rho_t.$$

Noting that the coefficient in parentheses is $1-\sigma\psi$, which is also equal to $1-\bar{\psi}$ using the definition of $\bar{\psi}$, the equation above can be solved for $\mathsf{P}_{S,t} - \mathsf{P}_{N,t}$:

$$\mathsf{P}_{S,t} - \mathsf{P}_{N,t} = \frac{1}{1-\bar{\psi}}\left(\mathsf{x}_t + (1-\sigma)\rho_t\right). \qquad \text{[G.7]}$$

Using equation [E.21] again, the sales frequency s_t is given by

$$\mathsf{s}_t = -\left(\frac{1-\vartheta_B}{\varphi_B}\right)\left(\frac{1}{1-\bar{\psi}}\right)\left(\mathsf{x}_t + (1-\sigma)\rho_t\right). \qquad \text{[G.8]}$$

Taking equation [E.12] and substituting the expressions for $\mathsf{P}_{S,t} - \mathsf{P}_{N,t}$ and s_t derived above:

$$\mathsf{P}_t - \mathsf{P}_{N,t} = \frac{\psi}{1-\bar{\psi}}\left(\mathsf{x}_t + (1-\sigma)\rho_t\right),$$

which uses the formula for ψ derived in Theorem 2. Note that $\mathsf{X}_t - \mathsf{P}_t = (\mathsf{P}_{S,t} - \mathsf{P}_{N,t}) + (\mathsf{P}_{N,t} - \mathsf{P}_t)$, so

$$\mathsf{X}_t - \mathsf{P}_t = \frac{1-\psi}{1-\bar{\psi}}\left(\mathsf{x}_t + (1-\sigma)\rho_t\right). \qquad \text{[G.9]}$$

Similarly, note that $\mathsf{P}_t - \bar{\mathsf{P}}_t = (\mathsf{P}_t - \mathsf{X}_t) + \mathsf{x}_t$. Then substituting the expression for $\mathsf{X}_t - \mathsf{P}_t$ and simplifying yields:

$$\mathsf{P}_t - \bar{\mathsf{P}}_t = \frac{1-\sigma}{1-\bar{\psi}}\left(\psi\mathsf{x}_t - (1-\psi)\rho_t\right). \qquad \text{[G.10]}$$

An analogous log-linearization of the cost function in the non-sale sector leads to

$$\mathcal{X}_t = \gamma\mathcal{Q}_t + \mathsf{W}_t,$$

where the assumption about the non-sale sector production function guarantees it has the same elasticity of marginal cost with respect to output as in the sale sector. Note that $\mathcal{Q}_t = \mathcal{Y}_t$ in the non-sale sector since all output in that sector is sold at the same price in the steady state. The derivation of the link between Y_t and Q_t in the sale sector continues to hold subject to P_t being the price level for the sale sector alone:

$$\mathsf{Y}_t = \mathsf{Q}_t + \delta(\mathsf{X}_t - \mathsf{P}_t).$$

Hence the marginal cost differential between the two sectors is

$$\mathcal{X}_t - \mathsf{X}_t = \gamma\left((\mathcal{Y}_t - \mathsf{Y}_t) + \delta(\mathsf{X}_t - \mathsf{P}_t)\right). \qquad \text{[G.11]}$$

Using the demand function [G.5] and the aggregate price index [G.2], relative demand is given by

$$\mathcal{Y}_t - \mathsf{Y}_t = \frac{\epsilon}{1-\sigma}(\mathsf{P}_t - \bar{\mathsf{P}}_t). \qquad \text{[G.12]}$$

By substituting this into [G.11] and using [G.9] and [G.10], the marginal cost differential is

$$\mathcal{X}_t - \mathsf{X}_t = \frac{\gamma}{1-\bar{\psi}}\left((\epsilon\psi + \delta(1-\psi))\mathsf{x}_t + (1-\psi)(\delta(1-\sigma)-\epsilon)\rho_t\right).$$

Since price-setting behaviour in the non-sale sector is entirely standard, the usual derivation of the New Keynesian Phillips curve from [G.3] and [G.4] yields

$$\boldsymbol{\Delta}\mathcal{P}_t = \beta\mathbb{E}_t\boldsymbol{\Delta}\mathcal{P}_{t+1} + \frac{\kappa}{1+\xi\gamma}(\mathcal{X}_t - \mathcal{P}_t).$$

Together with [E.26], the differential ρ_t between $\mathcal{P}_t$ and $\mathsf{P}_{N,t}$ is determined by the equation:

$$\boldsymbol{\Delta}\rho_t = \beta\mathbb{E}_t\boldsymbol{\Delta}\rho_{t+1} + \frac{\kappa}{1+\xi\gamma}\left((\mathcal{X}_t - \mathsf{X}_t) - \xi\gamma(\mathsf{P}_{S,t} - \mathsf{P}_{N,t}) - \rho_t\right),$$

which is derived by using $\mathsf{X}_t = \mathsf{P}_{S,t}$. Substituting [G.7] and [G.11] into the above leads to [G.1b] after some rearrangement.

To obtain equation [G.1c], note that log-linearized aggregate output is $\bar{\mathsf{Y}}_t = \sigma\mathsf{Y}_t + (1-\sigma)\mathcal{Y}_t$, which is equivalent to $\mathsf{Y}_t - \bar{\mathsf{Y}}_t = -(1-\sigma)(\mathcal{Y}_t - \mathsf{Y}_t)$. Using [G.12] and substituting the expression for $\mathsf{P}_t - \bar{\mathsf{P}}_t$ from [G.10] yields the result.

Equation [G.1d] follows from substituting [G.9] into $\mathsf{Y}_t = \mathsf{Q}_t + \delta(\mathsf{X}_t - \mathsf{P}_t)$, which is taken from Lemma 4.

By writing equation [G.6] as $\bar{\mathsf{P}}_t = \mathsf{P}_{N,t} + \bar{\vartheta}_P(\mathsf{P}_{S,t} - \mathsf{P}_{N,t}) - \bar{\varphi}_P\mathsf{s}_t + (1-\sigma)\rho_t$, substituting in [G.7] and [G.8], and then taking first differences:

$$\bar{\pi}_t = \pi_{N,t} + \frac{\bar{\psi}}{1-\bar{\psi}}\boldsymbol{\Delta}\mathsf{x}_t + \left(\frac{1-\sigma}{1-\bar{\psi}}\right)\boldsymbol{\Delta}\rho_t. \qquad \text{[G.13]}$$

Then combine equation [E.26] with [G.7] to obtain:

$$\pi_{N,t} = \beta\mathbb{E}_t\pi_{N,t+1} + \frac{\kappa}{1-\bar{\psi}}\left(\mathsf{x}_t + (1-\sigma)\rho_t\right).$$

Using equation [G.13] to write down an expression for $\bar{\pi}_t - \beta\mathbb{E}_t\bar{\pi}_{t+1}$ and substituting for $\pi_{N,t} - \beta\mathbb{E}_t\pi_{N,t+1}$ from above yields the Phillips curve [G.1a].

Note that the choice of ξ (which equalizes the average markups in the two sectors) and the production function $\mathfrak{F}(\mathcal{H})$ in the non-sale sector imply that $Y = \mathcal{Y}$, and hence $Q/\mathcal{Q} = \Delta$. Since the production functions in the two sectors are related by $\mathfrak{F}(\mathcal{H}) = \Delta\mathcal{F}(\Delta^{-1}\mathcal{H})$, it follows that $H/\mathcal{H} = \Delta$. This means that the total labour usage equation $\bar{H}_t = \sigma H_t + (1-\sigma)\mathcal{H}_t$ is log linearized as follows:

$$\bar{\mathsf{H}}_t = \left(\frac{\sigma}{\sigma + (1-\sigma)\Delta}\right)\mathsf{H}_t + \left(\frac{(1-\sigma)\Delta}{\sigma + (1-\sigma)\Delta}\right)\mathscr{H}_t.$$

The log-linearized production functions are the same in the two sectors, so $\mathsf{Q}_t = \alpha\mathsf{H}_t$ and $\mathscr{Q}_t = \alpha\mathscr{H}_t$. By substituting these into the above equation:

$$\bar{\mathsf{H}}_t = \frac{1}{\alpha}\left(\left(\frac{\sigma}{\sigma + (1-\sigma)\Delta}\right)\mathsf{Q}_t + \left(\frac{(1-\sigma)\Delta}{\sigma + (1-\sigma)\Delta}\right)\mathscr{Q}_t\right).$$

By using $\mathscr{Y}_t = \mathscr{Q}_t$ and noting that $\mathscr{Y}_t = (\bar{\mathsf{Y}}_t - \sigma\mathsf{Y}_t)/(1-\sigma)$:

$$\bar{\mathsf{H}}_t = \frac{1}{\alpha}\frac{1}{\sigma + (1-\sigma)\Delta}\left(\Delta\bar{\mathsf{Y}}_t + \sigma(\mathsf{Q}_t - \Delta\mathsf{Y}_t)\right).$$

Substituting this expression into [F.9] and rearranging yields [G.1g].

H Proof of Proposition 2

(i) Note that Proposition 1 implies μ is only a function of ϵ and η. This is also true of $z = \mathfrak{z}(\mu;\epsilon,\eta)$, as can be seen from equation [A.3]. The value of s is then determined by λ (recall that Proposition 1 shows for every $s \in (0,1)$ there is a value of λ generating this s).

Hence, the equilibrium value of ψ can be obtained as a function of s, ϵ and η. This is denoted by $\Psi(s;\epsilon,\eta)$. From [E.35], the function is:

$$\Psi(s;\epsilon,\eta) = 1 - \frac{(1-\Theta(\epsilon,\eta;\mu))(1-s)}{(1+z) + \left((\mu^{1-\epsilon}-1) + (\mu^{1-\eta}-1)z\right)s}.$$

It has already been shown in Theorem 2 that $\Psi(s;\epsilon,\eta)$ is non-negative. By taking the first derivative with respect to s (holding ϵ and η constant, and hence varying only λ implicitly):

$$\Psi'(s;\epsilon,\eta) = \frac{1-\Theta(\epsilon,\eta;\mu)}{(1+z) + \left((\mu^{1-\epsilon}-1) + (\mu^{1-\eta}-1)z\right)s}\left(1 + \frac{\left((\mu^{1-\epsilon}-1) + (\mu^{1-\eta}-1)z\right)s}{(1+z) + \left((\mu^{1-\epsilon}-1) + (\mu^{1-\eta}-1)z\right)s}\right),$$

which is always strictly positive using the same logic from the proof of Theorem 2. Finally, taking the second derivative yields

$$\Psi''(s;\epsilon,\eta) = -2\frac{(1-\Theta(\epsilon,\eta;\mu))\left((\mu^{1-\epsilon}-1) + (\mu^{1-\eta}-1)z\right)}{(1+z) + \left((\mu^{1-\epsilon}-1) + (\mu^{1-\eta}-1)z\right)s}\left(1 + \frac{\left((\mu^{1-\epsilon}-1) + (\mu^{1-\eta}-1)z\right)s}{(1+z) + \left((\mu^{1-\epsilon}-1) + (\mu^{1-\eta}-1)z\right)s}\right),$$

which is always strictly negative. This establishes that the function $\Psi(s;\epsilon,\eta)$ is non-negative-valued, strictly increasing and strictly concave.

(ii) The two-sector model's Phillips curve in the general case is given in equation [G.1a] following the derivation in appendix G. Note that when $\gamma = 0$, the only stable solution of [G.1b] is $\rho_t = 0$. By substituting this result into [G.1a], it is clear that the resulting equation reduces to the Phillips curve with sales in [32] with coefficient $\bar{\psi}$ in place of ψ. Finally, note that $\gamma = 0$ implies that x_t is real marginal cost for both sectors, and hence for the aggregate economy. This completes the proof.