Legislative Bargaining under Weighted Voting: Corrigendum

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Abstract

This note corrects a mistake in the proofs of Propositions 3 and 4 of Snyder, Ting and Ansolabehere (2005). It also corrects a mistake in the statement of Proposition 4, and characterizes the distribution of payoffs more fully than done in Proposition 4.

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1 Set-up and Preliminary Results

We begin by reminding the reader of key definitions and equations in Snyder, Ting and Ansolabehere (2005) (henceforth, STA). The baseline model is a distributive bargaining game in the framework of Baron-Ferejohn (1989) (closed-rule and infinite-horizon). Types are numbers $t \in \{1, 2, ...T\}$ where weights are positive integers and $w_t < w_{t+1}$ for all $t < T$. Call $\mathbf{N}^r$ the set of legislators in the r-replicated game. Call a coalition $\mathbf{C}^r$ winning if $\sum_t n_t(\mathbf{C}^r)w_t \geq r\underline{w}$. Call $\mathbf{C}^r_t$ the least-cost winning coalition when t is a proposer, not including herself. Also,

$$v^r_t = p^r_t(1 - \underline{v}^r_t) + (1 - p^r_t)q^r_t v^r_t \tag{1}$$

$$\theta^r_t \equiv \frac{v^r_t r w}{w_t} \tag{2}$$

$$\theta^r \equiv \min_t \{\theta^r_t\} \tag{3}$$

$$\mathbf{T}^r_L = \{t \in \mathbf{T} | \theta^r_t = \theta^r\} \tag{4}$$

The following hold:

Comment 1. *In a stationary equilibrium, the continuation values for players of the same type are equal.*

Lemma 2. *In a stationary equilibrium, there exists a finite $\overline{r}_C$ such that for any $r \geq \overline{r}_C$, $\sum_{t \in \mathbf{T}^r_L} w_t n_t \geq \underline{w}$*

Lemma 3. *In a stationary equilibrium, if $r \geq \bar{r}_C$, as defined in Lemma 2, then $q_t^r < 2Tw_T/(rn_t)$ for all $t \notin \mathbf{T}_L^r$*

2 Corrections of Propositions 3 and 4

We first prove a few remarks, which will be useful in establishing the equivalent of Propositions 3 and 4 in STA.

Remark 1. *$\exists\{r_i\}$ an infinite subsequence s.t.*

a) $\mathbf{T}_L^r = \overline{\mathbf{T}}_L \; \forall r \in \{r_i\}$

b) $\lim_{i\to\infty} \theta^{r_i}$ *exists. Let* $\lim_{i\to\infty} \theta^{r_i} \equiv \theta^{\{r_i\}}$

c) $lim_{i\to\infty} \underline{v}_t^{r_i} = \frac{\underline{w}}{w}\theta^{\{r_i\}}$

d) *If $\theta^r < 1 \; \forall r \in \{r_i\}$ and $p_t^r = \frac{1}{rn} \; \forall t$, then $\exists r^* | \forall r \geq r^* \; (r \in \{r_i\})$, $t' \in \mathbf{T}_L^r, t'' \notin \mathbf{T}_L^r \Rightarrow w_{t'} > w_{t''}$.*

Proof. a) is obvious since T is finite. b) is obvious since $\forall r$, $\theta^r \in (0, 1]$. For c), note that $\forall r \geq \bar{r}_C$,

$$\underline{v}_t^r \in \left[\theta^r \frac{r\underline{w} - w_T}{rw}, \theta^r \frac{r\underline{w} + w_T - 1 - w_1}{rw}\right] \quad (5)$$

To find the lower and upper bounds, recall that, by lemma 1, it is possible for any proposer to form a winning coalition by only allying herself exclusively with legislators of the cheapest price. To find the lower bound, note that the best scenario for the proposer is to have maximal voting weight (w_T), pay all coalition members the minimum price (θ^r) and form a coalition which achieves exactly the minimum-winning voting weight ($r\underline{w}$). To find the upper bound, note that the worst scenario is for the proposer to have minimal voting weight (w_1), pay all coalition members the minimum price (θ^r) and "over-pay" by weight $w_T - 1$ (this would happen if all coalition partners have weight w_T, so that taking out any coalition partner no longer ensures a win).

For d), note that

$$
\begin{aligned}
& t' \in \mathbf{T}_L^r, t'' \notin \mathbf{T}_L^r \\
\Rightarrow \quad & \theta_{t'}^r < \theta_{t''}^r \\
\Rightarrow \quad & \frac{1-\underline{v}_{t'}^r}{1-\underline{v}_{t''}^r} < \left[\frac{1-(1-p_{t'}^r)q_{t'}^r}{1-(1-p_{t''}^r)q_{t''}^r}\right]\frac{w_{t'}}{w_{t''}} \leq \left[\frac{1}{1-(1-p_{t''}^r)2Tw_T/(rn_{t''})}\right]\frac{w_{t'}}{w_{t''}} \\
\Rightarrow \quad & 1 < \frac{w_{t'}}{w_{t''}}
\end{aligned}
$$

where the second implication follows from $p_{t'}^r = p_{t''}^r = \frac{1}{rn}$ and lemma 1 and the last implication follows from taking r to infinity and comment 1. $\square$

This remark states that we can find an infinite subsequence in which the set of cheap types does not change and the limit of the price of the cheapest types exists. Also, if some types are more expensive than others, then we can divide them cleanly, such that cheap types have high voting weights and expensive types have low voting weights. With these, we can show:

Proposition 3. *Suppose $p_t^r = 1/rn$ for all t. Also, suppose $n > (w - \underline{w})/w_1$. There exists a finite $\overline{r}_2$ such that if $r > \overline{r}_2$, then in any stationary equilibrium, $v_t^r = w_t/(rw)$ (i.e. $\theta_t = 1$) for all t.*

Proof. Let $n > (w - \underline{w})/w_1$. Assume that there is an infinite subsequence $\{r_i\}$ such that $\forall r \in \{r_i\}$, $\theta^r < 1$.

First, let us show that $\theta^{\{r_i\}} = 1$. By the adding up constraint and Remark 1d), $\exists r^*, t_0 | \forall r \geq r^*$ $(r \in \{r_i\})$

$$\begin{aligned}
1 &= \sum_{t \geq t_0} r n_t v_t^r + \sum_{t < t_0} r n_t v_t^r \\
1 &= \lim_{i \to \infty} \left(\sum_{t \geq t_0} \frac{n_t w_t \theta_t^{r_i}}{w} + \sum_{t < t_0} r_i n_t v_t^{r_i} \right) \\
1 &= \theta^{\{r_i\}} \sum_{t \geq t_0} \frac{n_t w_t}{w} + \sum_{t < t_0} \frac{n_t}{n} \left[1 - \theta^{\{r_i\}} \frac{\underline{w}}{w} \right] \\
\sum_{t \geq t_0} \frac{n_t}{n} &= \theta^{\{r_i\}} \left[\sum_{t \geq t_0} \frac{n_t w_t}{w} - \left(1 - \sum_{t \geq t_0} \frac{n_t}{n} \right) \frac{\underline{w}}{w} \right] \qquad (6)
\end{aligned}$$

Now write

$$\psi(t_0) = \sum_{t \geq t_0} \frac{n_t w_t}{w} - \left(1 - \sum_{t \geq t_0} \frac{n_t}{n} \right) \frac{\underline{w}}{w} - \sum_{t \geq t_0} \frac{n_t}{n}$$

Then $\theta^{\{r_i\}} < 1$ only if $\psi(t_0) > 0$, by lemma 1. Now note that $\psi(1) = 0$ and

$$\begin{aligned}
\psi(t_0 + 1) - \psi(t_0) &= -\frac{n_{t_0} w_{t_0}}{w} + \left(1 - \frac{\underline{w}}{w} \right) \frac{n_{t_0}}{n} \\
&= -\frac{n_{t_0} w_{t_0}}{nw} \left[n - \frac{w - \underline{w}}{w_{t_0}} \right] < 0
\end{aligned}$$

since $n - \frac{w-\underline{w}}{w_{t_0}} \geq n - \frac{w-\underline{w}}{w_1} > 0$. A contradiction. Therefore, $\theta^{\{r_i\}} = 1$.

This shows that the limit of the price of cheapest legislators must be 1. Now let us show that the price of cheapest legislators must be 1 for finite values of r, which would then produce a contradiction. By remark 1d), $\forall r \in \{r_i\}, \theta^r < 1 \Rightarrow 1 \notin \mathbf{T}_L^r$. Therefore, using (1), (2) and lemma 1,

$$\begin{aligned} 1 - \underline{v}_1^r &> \frac{nw_1}{w}[1 - (1 - p_1^r)q_1^r] \\ &\Rightarrow 1 - (\theta^{\{r_i\}})\frac{\underline{w}}{w} \geq \frac{nw_1}{w} \\ &\Rightarrow n \leq \frac{w - \underline{w}}{w_1} \end{aligned}$$

A contradiction. □

This proof therefore shows that Proposition 3 holds as stated in STA. Unfortunately, Proposition 4 as stated in STA is incorrect. Instead, we can show:

Proposition 4′. *Suppose $p_t^r = 1/rn$ for all t. Also, suppose $n < (w - \underline{w})/w_1$. There exists a finite $\overline{r}_3$ such that if $r > \overline{r}_3$, then in any stationary equilibrium, there is a type t_0 and a number $\theta^r = \min_t\{\theta_t^r\} < 1$ such that $v_t^r = \theta^r w_t/(rw)$ for all $t > t_0$ and $v_t^r > \theta^r w_t/(rw)$ for all $t \leq t_0$.*

Proof. First, let us show that if $n < \frac{w-\underline{w}}{w_{t'}}$, then $\exists r^{**} | \forall r \geq r^{**}, \theta_{t'}^r > 1$. By contradiction, let there be an infinite subsequence $\{r_i\}$ such that $\forall r \in \{r_i\}$,

$\theta^r_{t'} \leq 1$ and let $\lim_{i\to\infty} \theta^{r_i} \equiv \theta^{\{r_i\}}$. $\theta^r_{t'} \leq 1$ implies

$$\begin{aligned} 1 - \underline{v}^r_{t'} &\leq \frac{nw_{t'}}{w}[1-(1-p^r_{t'})q^r_{t'}] \leq \frac{nw_{t'}}{w} \\ &\Longrightarrow 1-\theta^{\{r_i\}}\left[\frac{\underline{w}}{w}\right] \leq \frac{nw_{t'}}{w} \\ &\Longrightarrow \frac{w-\underline{w}}{w_{t'}} \leq n \end{aligned}$$

A contradiction. Therefore, if $n < \frac{w-\underline{w}}{w_1}$, then $\exists \overline{r}_3 | \forall r \geq \overline{r}_3, \theta^r_1 > 1$ and, using remark 1d), there is a t_0 such that $t \in \mathbf{T}^r_L$ if and only if $t > t_0$. □

This is a weaker statement than Proposition 4 in STA, which reads as follows:

Proposition 4. *Suppose $p^r_t = 1/rn$ for all t. Also, suppose $n \leq (w-\underline{w})/w_1$. There exists a finite $\overline{r}_3$ such that if $r > \overline{r}_3$, then in any stationary equilibrium, there is a unique type $t_0 = \max\{t | n \leq (w-\underline{w})/w_t\}$ and a number $\theta^r = \min\{\theta^r_t\} < 1$ such that $v_t = \theta^r w_t/(rw)$ for all $t > t_0$ and $v_t > w_t/(rw)$ for all $t < t_0$.*

This proposition is wrong because it is false to claim a) $n > (w-\underline{w})/w_t \Rightarrow t \in \mathbf{T}^r_L$ and b) $t \notin \mathbf{T}^r_L \Rightarrow \theta^r_t > 1$. The following is a counter-example.

Let the electorate be described as follows: $(w_1, w_2, w_3) = (1, 2, 7)$, $(n_1, n_2, n_3) = (10, 5, 9)$, with $\underline{w} = \frac{w+1}{2} = 42$. (Note, this game is not homogeneous, but the weights given are minimal integer weights.) First, it is easy to check that

$n \leq (w - \underline{w})/w_1$ so that the condition of Proposition $4'$ is respected. Write $\lim_{i\to\infty} \theta_t^{r_i} \equiv \theta_t^{\{r_i\}}$. Then it is straightforward to compute that

$$(\theta_1^{\{r_i\}}, \theta_2^{\{r_i\}}, \theta_3^{\{r_i\}}) = (1.976, 0.988, 0.847)$$

where we have rounded up at the third decimal. Therefore, we have a case where $n > (w - \underline{w})/w_2$ and yet $2 \notin \mathbf{T}_L^r$ and $\theta_2^{\{r_i\}} < 1$.

We can in fact be more explict on the vector of values $\{\theta_t^{\{r_i\}}\}$. In the process, we show that we can in fact gather more information on the limit prices $\theta_t^{\{r_i\}}$ than stated in Proposition 4 of STA.

Corollary 1. *Suppose* $p_t^r = 1/rn$ *for all* t. *Also, suppose* $n \leq (w - \underline{w})/w_1$. *Then* $\forall t \in \mathbf{T}_L^r$,

$$\theta_t^{\{r_i\}} = \theta^{\{r_i\}} = \frac{\sum\limits_{t \geq t_0} \frac{n_t}{n}}{\sum\limits_{t \geq t_0} \frac{n_t w_t}{w} - \left(1 - \sum\limits_{t \geq t_0} \frac{n_t}{n}\right) \frac{\underline{w}}{w}} \tag{7}$$

and $\forall t \notin \mathbf{T}_L^r$

$$\theta_t^{\{r_i\}} = \frac{w - \theta^{\{r_i\}} \underline{w}}{n w_t} \tag{8}$$

Proof. (7) follows directly from (6). For $t \notin \mathbf{T}_L^r$, using $p_t^r = \frac{1}{rn}$ and (1), we get

$$r v_t^r = \frac{1 - \underline{v}_t^r}{n\left[1 - (1 - p_t^r) q_t^r\right]}$$

so that

$$\lim_{i\to\infty} r_i v_t^{r_i} = \frac{1 - \theta^{\{r_i\}} \frac{\underline{w}}{w}}{n}$$

and, using (2), we obtain (8). □

Therefore, we see that for any type who is not in the set of cheapest types, the limit price $\theta_t^{\{r_i\}}$ is inversely proportional to the type's voting weight (or, in other words, their value $r_i v_t^{r_i}$ converges to the same number). This is an additional result which was not included in Proposition 4. The intuition is as follows: given lemma 2, a type $t \notin \mathbf{T}_L^{r_i}$ is picked in a winning coalition only because of "rounding" problems. In other words, a proposer could buy votes only from the cheapest types, but she may still decide to pick a partner of type $t \notin \mathbf{T}_L^{r_i}$ because she would need to buy "too many" votes if she only picked from types $t \in \mathbf{T}_L^{r_i}$. As r_i goes to infinity, this rounding issue vanishes. Therefore, the probability that any type $t \notin \mathbf{T}_L^{r_i}$ is picked in a winning coalition converges to 0 (which follows from lemma 3). Also, as r_i goes to infinity, any difference in the price paid by any proposer vanishes. In other words, proposers of different types have different voting weights, and therefore differ in the voting weights they need to purchase to form a winning coalition. But this difference vanishes as r_i goes to infinity, since the proposer's own weight becomes relatively small, compared to the size of the coalition that must be gathered (in other words, $\underline{v}_t^{r_i}$ converges to the same value for any t, using 5).

References

[1] Baron, David P., and John A. Ferejohn. 1989. "Bargaining in Legislatures." *American Political Science Review*, 83(4): 1181-1206.

[2] Snyder, James M., Jr., Michael M. Ting, and Stephen Ansolabehere. 2005. "Legislative Bargaining under Weighted Voting." *American Economic Review*, 95(4): 981-1004.