

Online Appendix for 'Liquidity Trap and Excessive Leverage'

By ANTON KORINEK AND ALP SIMSEK

APPENDIX A: OMITTED PROOFS FOR THE BASELINE MODEL

This appendix presents the proofs of the results for the baseline model and its variants analyzed in Sections II, III, IV, V, and VI.

Appendix B develops the extension with flexible MPC differences between borrowers and lenders, described and used in Sections III.C, IV.A, and IV.B.

Appendix C analyzes various extensions, omitted from the main text, which illustrate the robustness of our main results to alternative specifications.

Proving the constrained efficiency of the myopic monetary policy. Suppose the monetary authority does not have commitment power and follows the monetary policy in (10) for each date $t + 1 \geq 2$. We claim that it is also constrained efficient to follow this policy at time $t \geq 1$, in the sense that the monetary authority cannot improve all households' welfare by deviating from it. This establishes the constrained efficiency of the policy at each $t \geq 1$. Proposition 6 in Section V establishes further the constrained efficiency of the policy at date 0 (once macroprudential policies are in place).

To prove the claim, first consider dates $t \geq 2$. Consider r_{t+1} is sufficiently close to r_{t+1}^* so that the borrowers are constrained (the other case is similar). Then, the allocations for date $t + 1$ onwards are the same as the frictionless benchmark in (11), which do not depend on r_{t+1} . In addition, the allocations at date t are subject to the feasibility constraints,

$$(A1) \quad c_t^b + c_t^l = 2e_t \leq 2e^*.$$

Here, the inequality follows since e^* maximizes the net income per household given the technology [cf. Eq. (7)]. Note also that setting $r_{t+1} = r_{t+1}^*$ obtains the upper bound in (A1). Thus, deviating from this policy cannot improve the welfare of one group of households without hurting the others.

Next consider date $t = 1$ for the case $r_2^* < 0$ (the case $r_2^* \geq 0$ is identical to the above analysis). In this case, the monetary authority faces the additional constraint, $r_2 \geq 0$. The same steps as above then imply a tighter feasibility constraint,

$$c_1^b + c_1^l = 2e_1 \leq e_1 - (d_1 - \phi) + \bar{c}_1^l < 2e^*.$$

Moreover, setting $r_2 = 0$ obtains the upper bound (in the weak inequality). Thus, the policy in (10) is also constrained efficient at this date, proving the claim. ■

Proof of Lemma 1. (i) If $r_{t+1} > 0$, then, by assumption, the monetary policy is unconstrained and implements $e_t = e^*$. (ii) If $r_{t+1} = 0$, then we have $e_t = \frac{c_t^b + c_t^l}{2} = n_t - v(n_t)$

by the resource constraints and symmetry. This further implies $e_t \leq e^*$ since e^* is the maximum value of net income per household, $n_t - v(n_t)$ [cf. Eq. (7)]. ■

Completing the characterization of the equilibrium at date 1 in Section II. When $d_1 > \bar{d}_1$, the equilibrium features a liquidity trap as described in the main text. Consider the case, $d_1 \leq \bar{d}_1$. We also assume $d_1 \geq \phi$, which will ensure that the borrowers are constrained at date 1 (recall that Section II restricts attention to these cases). For $d_1 \in [\phi, \bar{d}_1]$, we conjecture an equilibrium with an efficient level of net income, $e_1 = e^*$, along with a nonnegative interest rate, $r_2 \geq 0$. In such an equilibrium, lenders' Euler equation can be written as,

$$(A2) \quad f(d_1, r_2) \equiv u' \left(e^* + d_1 - \frac{\phi}{1+r_2} \right) - \beta^l (1+r_2) u' (e^* + \phi (1-\beta^l)) = 0.$$

The left hand side defines the function, $f(d_1, r_2)$, which is strictly decreasing in both d_1 and r_2 . Note also that $f(\bar{d}_1, 0)$ from the definition of \bar{d}_1 , and that $\lim_{r_2 \rightarrow \infty} f(d_1, r_2) < 0$ for any $d_1 \in [\phi, \bar{d}_1]$. Thus, there is a unique level of the interest rate, $r_2(d_1) \geq 0$, that ensures $f(d_1, r_2(d_1)) = 0$. Note also that $r_2(d_1)$ is strictly decreasing in d_1 , and the adjustment term, $d_1 - \frac{\phi}{1+r_2(d_1)}$, is strictly increasing in d_1 .

Next, we check that borrowers choose the constrained debt level, $d_2 = \phi$, given this interest rate. This is the case as long as,

$$u' \left(e^* - \left(d_1 - \frac{\phi}{1+r_2(d_1)} \right) \right) \geq \beta^b (1+r_2(d_1)) u' (e^* - \phi (1-\beta^l)).$$

This condition holds when $d_1 = \phi$, since this debt level ensures the economy is at the steady state starting at date 1. Increasing d_1 increases the current marginal utility on the left hand side (since $d_1 - \frac{\phi}{1+r_2(d_1)}$ is strictly increasing in d_1), while decreasing the future marginal utility on the right hand side. Hence, the condition also holds for $d_1 \in [\phi, \bar{d}_1]$, which completes the characterization of equilibrium at date 1. ■

Proof of Proposition 1. To prove this, suppose to the contrary that $d_1 < \bar{d}_1$, and thus, $e_0 = e_1 = e^*$. Let $\bar{r}_1(d_0)$ denote the solution to,

$$(A3) \quad \frac{1}{1+\bar{r}_1} = \frac{\beta^l u' (e^* + \bar{d}_1 - \phi)}{u' (e^* + d_0 - \bar{d}_1 / (1+\bar{r}_1))}.$$

Hence, $\bar{r}_1(d_0)$ is the interest rate at which lenders would hold assets \bar{d}_1 (more precisely, debt $-\bar{d}_1$) in equilibrium. Note that $\bar{r}_1(d_0)$ is a decreasing function of d_0 . This also implies that $d_0 - \bar{d}_1 / (1+\bar{r}_1(d_0))$ is increasing in d_0 .

The equilibrium debt level satisfies, $d_1 > \bar{d}_1$, contradicting our initial assumption, if

households' marginal rates of substitution calculated at the debt level \bar{d}_1 satisfy,

$$\left. \frac{\beta^l u'(c_1^l)}{u'(c_0^l)} \right|_{d_1=\bar{d}_1} > \left. \frac{\beta^b u'(c_1^b)}{u'(c_0^b)} \right|_{d_1=\bar{d}_1}.$$

This condition can be rewritten as,

$$(A4) \quad \frac{\beta^l u'(e^* + \bar{d}_1 - \phi)}{u'(e^* + d_0 - \bar{d}_1 / (1 + \bar{r}_1(d_0)))} > \frac{\beta^b u'(e^* - \bar{d}_1 + \phi)}{u'(e^* - d_0 + \bar{d}_1 / (1 + \bar{r}_1(d_0)))}.$$

Observe that the right hand side of this inequality is decreasing in β^b . Hence, for a given debt level d_0 , there is a threshold level of impatience $\bar{\beta}^b(d_0)$ such that the inequality holds for each $\beta^b \geq \bar{\beta}^b(d_0)$.

Similarly, since $d_0 - \bar{d}_1 / (1 + \bar{r}_1(d_0))$ is increasing in d_0 , the left hand side of (A4) is increasing in d_0 , while the right hand side is decreasing in d_0 . Hence, for a given level β^b , there is a threshold level $\bar{d}_0(\beta^b)$ such that the inequality holds for each $d_0 > \bar{d}_0(\beta^b)$. It follows that $d_1 > \bar{d}_1$ and $e_1 < e^*$ if the borrowers are sufficiently impatient or sufficiently indebted at date 0. ■

Characterizing the upper bound in Assumption (1). The upper bound \tilde{d}_1 is the level of debt that triggers a liquidity trap not only at date 1 but also at date 0. To characterize this level, consider the lenders' optimality condition (A3) corresponding to the interest rate $r_1 = 0$. Rewriting this expression, \tilde{d}_0 is found as the unique solution to,

$$u'(e^* + \tilde{d}_0 - \bar{d}_1) = \beta^l u'(e^* + \bar{d}_1 - \phi).$$

It can be checked that the competitive equilibrium features $r_1 > 0$ if and only if $d_0 < \tilde{d}_0$. It can also be checked that $\tilde{d}_0 > \bar{d}_0(\beta^b)$, where $\bar{d}_0(\beta^b)$ is the threshold level of debt that triggers a liquidity trap at date 1, which is characterized in the proof of Proposition 1. Hence, for each $\beta^b \leq \beta^l$, there is a non-empty set of initial debt levels $d_0 \in (\bar{d}_0(\beta^b), \tilde{d}_0]$ that triggers a recession at date 1 but not at date 0. Thus, Proposition 1 applies for a non-empty set of parameters. ■

Proof of Lemma 2. First consider the case $d_1 > \bar{d}_1$. Eq. (15) implies $\frac{de_1}{dd_1} = -1$. Eq. (17) then implies $\frac{\partial V^h}{\partial d_1} = -u'(u_1^h) < 0$.

Next consider the case $d_1 < \bar{d}_1$. Differentiating lenders' Euler equation (A2), we have:

$$\frac{dr_2}{dd_1} = \frac{u''(c_1^l)}{\beta^l u'(c_2^l) - u''(c_1^l) \phi / (1 + r_2)^2} < 0.$$

The change in borrowers' consumption, $\frac{dc_1^b}{dd_1}$, is then given by

$$(A5) \quad \eta = \frac{\phi}{(1+r_2)^2} \frac{-dr_2}{dd_1} = \frac{\phi}{(1+r_2)^2} \frac{-u''(c_1^l)}{\beta^l u'(c_2^l) - u''(c_1^l) \phi / (1+r_2)^2},$$

whereas the change in lenders' consumption satisfies $\frac{dc_1^l}{dd_1} = -\eta$. It can also be checked that $\eta \in (0, 1)$, completing the proof. ■

Proof of Proposition 2. Let $\nabla_{sub} V^h(d_1^h, d_1)$ denote the set of subgradients of function $V^h(\cdot)$ with respect to its second variable (aggregate debt level). If $d_1 \neq \bar{d}_1$, the function V^h is differentiable in its second variable. In this case, there is a unique subgradient characterized by Lemma 2. If $d_1 = \bar{d}_1$, then the function V^h has a kink at \bar{d}_1 due to the kink of the function $e_1(d_1)$ (see Eqs. (17) and (15)). In this case, there are multiple subgradients characterized by,

$$(A6) \quad \begin{aligned} \nabla_{sub} V^b(d_1^b, \bar{d}_1) &= [-u'(c_1^b), \eta u'(c_1^b)], \\ \nabla_{sub} V^l(d_1^l, \bar{d}_1) &= [-u'(c_1^l), -\eta u'(c_1^l)]. \end{aligned}$$

In particular, for each h , the subgradients lie in the interval between the right and the left derivatives of the function V^h characterized in Lemma 2.

Next consider the optimality conditions for problem (19), which can be written as:

$$(A7) \quad \frac{\beta^l [u'(c_1^l) + \delta^l]}{u'(c_0^l)} = \frac{\beta^b [u'(c_1^b) - \delta^b]}{u'(c_0^b)},$$

where $\delta^h \in \nabla_{sub} V^h(d_1^h, d_1)$ denotes the subgradient evaluated at $d_1 = d_1^b = -d_1^l$.²⁴ Any allocation that satisfies these conditions, along with the intratemporal condition $v'(n_0^h) = 1$ for each h , corresponds to a solution to problem (19) for appropriate Pareto weights (that satisfy $\frac{\gamma^b}{\gamma^l} = \frac{u'(c_0^b)}{u'(c_0^l)}$). Hence, it suffices to characterize the allocations that satisfy condition (A7).

First consider the case $d_1 < \bar{d}_1$. Using Lemma 2, condition (A7) becomes identical to the Euler equations (16), proving the first part. Next consider the case $d_1 > \bar{d}_1$. Using Lemma 2, condition (A7) is violated since the left hand side is zero while the right hand side is strictly positive. Hence, there is no constrained efficient allocation with $d_1 > \bar{d}_1$. Finally consider the case $d_1 = \bar{d}_1$. Using (A6), we have,

$$\frac{\beta^l [u'(c_1^l) + \delta^l]}{u'(c_0^l)} \in \left[0, \frac{(1-\eta) \beta^l u'(c_1^l)}{u'(c_0^l)} \right]$$

²⁴Note that we consider generalized first order conditions that apply also at points at which the objective function might have a kink.

and,

$$\frac{\beta^b [u'(c_1^b) - \delta^b]}{u'(c_0^b)} \in \left[\frac{(1 - \eta) \beta^b u'(c_1^b)}{u'(c_0^b)}, \frac{2\beta^b u'(c_1^b)}{u'(c_0^b)} \right].$$

Combining these expressions with condition (A7), we obtain $\frac{\beta^l u'(c_1^l)}{u'(c_0^l)} \geq \frac{\beta^b u'(c_1^b)}{u'(c_0^b)}$. Conversely, for any allocation that satisfies this inequality, there exists subgradients δ^l and δ^b such that condition (A7) holds. It follows that the optimal allocations with $d_1 \geq \bar{d}_1$ are characterized by $d_1 = \bar{d}_1$ and the inequality (20), completing the proof. ■

Proof of Corollary 1. Note from the discussion in Section III that lowering the debt level to \bar{d}_1 generates an ex-post Pareto improvement relative to the competitive equilibrium allocation in Proposition 1. It follows that setting the debt level $d_1 = \bar{d}_1$ while keeping the ex-ante allocations the same, $(c_0^h, n_0^h)_h = (c_0^{h,eq}, n_0^{h,eq})_h$, yields an ex-ante Pareto improvement. Specifically, this allocation strictly improves borrowers' welfare while leaving lenders indifferent. ■

Proof of Corollary 2. Consider constrained efficient allocations $((c_0^h, n_0^h)_h, d_1)$ with $d_1 \geq \bar{d}_1$ (the other case is straightforward) and $u'(c_0^l) \geq \beta^l u'(c_1^l)$. Note, from our analysis in Section II, that these allocations feature $e_1 = e^*$ and:

$$(A8) \quad c_1^l = \bar{c}_1^l = e^* + (\bar{d}_1 - \phi) \text{ and } c_1^b = e^* - (\bar{d}_1 - \phi).$$

We thus have a collection $(c_0^l, c_0^b, c_1^l, c_1^b)$ that satisfies the inequality (20) and Eq. (A8), along with the resource constraints at date 0.

We first prove that this allocation can be implemented with the debt limit $d_1^h \leq \bar{d}_1$, and an appropriate transfer T_0^b . The debt limit does not bind for lenders, so that the interest rate is given by $\frac{1}{1+r_1} = \frac{\beta^l u'(c_1^l)}{u'(c_0^l)}$. Given this interest rate, let T_0^b denote the unique solution to,

$$c_0^l = e^* + (d_0 - T_0^b) - \frac{\bar{d}_1}{1 + r_1}.$$

Note that T_0^b is the transfer level that ensures lenders' date 0 budget constraint holds as equality. With this transfer, lenders optimally choose $d_1^l = -\bar{d}_1$. Given the inequality in (20), borrowers are constrained and they also optimally choose $d_1^b = \bar{d}_1$. It follows that the debt limit along with the transfer T_0^b implements the constrained efficient allocation. Note also that the implementation does not violate the lower bound constraint (6) at date 0 because of Assumption (1).(ii).

We next show that the same allocation can also be implemented with the tax policy, τ_0^b . In this case, both borrowers and lenders are unconstrained and their Euler equations imply

$$\frac{1}{1 + r_1} = \frac{\beta^l u'(c_1^l)}{u'(c_0^l)} = \frac{\beta^b u'(c_1^b)}{u'(c_0^b)} \frac{1}{1 - \tau_0^b},$$

which also pins down the pre-tax interest rate r_1 . Moreover, lenders' budget constraint at date 0 can be written as:

$$c_0^l = e^* + (d_0 - T_0^b) - \frac{\bar{d}_1}{1 + r_1} + \frac{\bar{d}_1}{1 + r_1} \frac{\tau_0^b}{2},$$

where the last term captures the lump-sum rebates from the tax policy, and borrowers have a similar budget constraint. Let T_0^b denote the unique solution to lenders' budget constraint. With this transfer, and in view of Eq. (21), households optimally choose $d_1^b = -d_1^l = \bar{d}_1$. Thus, the tax policy, along with an appropriate transfer, also implements the constrained efficient allocation. Note that the implementation does not violate the lower bound constraint at date 0 because the net-of-tax interest rates for the two groups of households satisfy $\frac{1+r_1}{1-\tau_0^b} \geq 1 + r_1 \geq 1$ in view of Assumption (1).(ii). ■

Proof of Proposition 3. Under either condition (i), (ii), or (iii), we claim that there exists an equilibrium in which $d_{1,L} \geq \bar{d}_1$ and a recession is triggered in state L of date 1. By using the resource constraints and substituting $m_L = d_1 - d_{1,L}$, the optimality conditions (for the conjectured equilibrium) can be written as:

$$q_L = \frac{\beta^l \pi_L^l u'(\bar{c}_1^l)}{u'(e^* + d_0 - \frac{d_1}{1+r_1} + q_L(d_1 - d_{1,L}))} = \frac{\beta^b \pi_L^b u'(\bar{c}_1^l + 2(\phi - d_{1,L}))}{u'(e^* - (d_0 - \frac{d_1}{1+r_1}) - q_L(d_1 - d_{1,L}))}$$

$$\frac{q_L}{\frac{1}{1+r_1} - q_L} = \frac{\pi_L^l}{1 - \pi_L^l} \frac{u'(\bar{c}_1^l)}{u'(e^* + (1 - \beta) d_1)} = \frac{\pi_L^b}{1 - \pi_L^b} \frac{u'(\bar{c}_1^l + 2(\phi - d_{1,L}))}{u'(e^* - (1 - \beta) d_1)}.$$

Here, the second optimality condition is obtained by combining the two optimality conditions (27) and (28). It illustrates that households equate price to probability ratios between states L and H . The expression $q_H \equiv \frac{1}{1+r_1} - q_L$ is the shadow price for the H -state Arrow-Debreu security.

The previously displayed expressions represent 4 equations in 4 unknowns, d_1 , $d_{1,L}$, $\frac{1}{1+r_1}$, q_L . Given the regularity conditions, there is a unique solution. For the conjectured allocation to be an equilibrium, we also need the solution to satisfy $d_{1,L} \geq \bar{d}_1$. First consider conditions (i) or (ii), i.e., suppose $\pi_L^b = \pi_L^l$. In this case, a similar analysis as in the proof of Proposition 3 establishes that $d_{1,L} \geq \bar{d}_1$ is satisfied when $\beta^b \leq \bar{\beta}^b(d_0, \pi_L^b)$ or when $d_0 \geq \bar{d}_0(\beta^b, \pi_L^b)$ (for appropriate threshold functions $\bar{\beta}^b(\cdot)$ and $\bar{d}_0(\cdot)$). Next consider condition (iii). It can be checked that $d_{1,L}$ is decreasing in π_L^b , and that $\lim_{\pi_L^b \rightarrow 0} d_{1,L} > \bar{d}_1$ (since $\lim_{\pi_L^b \rightarrow 0} c_{1,L}^b = 0$). Thus, there exists a threshold function $\bar{\pi}_L^b(\beta^b, d_0)$ such that $d_{1,L} > \bar{d}_1$ whenever $\pi_L^b < \bar{\pi}_L^b(\beta^b, d_0)$, completing the proof. ■

Proof of Proposition 4. The proof proceeds similar to the proof of Proposition 2. The

optimality conditions for problem (29) imply $\frac{\beta^l \pi_H^l u'(c_{1,H}^l)}{u'(c_0^l)} = \frac{\beta^b \pi_H^b u'(c_{1,H}^b)}{u'(c_0^b)}$ and

$$(A9) \quad \frac{\beta^l \pi_L^l [u'(c_{1,L}^l) + \delta^l]}{u'(c_0^l)} = \frac{\beta^b \pi_L^b [u'(c_{1,L}^b) - \delta^b]}{u'(c_0^b)}.$$

Here, δ^l and δ^b denote respectively the subgradients of $V^h(d_{1,L}^h, d_{1,L}^l)$ with respect to the second variable, evaluated at $d_{1,L} = d_{1,L}^b = -d_{1,L}^l$. Conversely, it can be seen that any allocation that satisfies these equations corresponds to a solution to the planner's problem with appropriate Pareto weights. Hence, it suffices to characterize the allocations that satisfy condition (A9).

For the case $d_{1,L} < \bar{d}_1$, applying Lemma 2 to Eq. (A9) makes the condition equivalent to Eq. (28). For the case $d_{1,L} > \bar{d}_1$, the term on the left hand side of Eq. (A9) is zero whereas the right-hand side remains positive, implying that $d_{1,L} > \bar{d}_1$ is never optimal. Finally, for the case $d_{1,L} = \bar{d}_1$, condition (A9) implies the insurance inequality (30). Conversely, given the inequality in (30), there exists a subgradient such that the condition (A9) holds. This completes the characterization of the solution to problem (29). ■

Proof of Corollary 3. Follows from very similar steps as in the proof of Corollary 1. ■

Proof of Corollary 4. Follows from very similar steps as in the proof of Corollary 2. ■

We next establish the following lemma, which will be useful to prove Proposition 5.

LEMMA 3: *Consider a strictly increasing and strictly concave function $u(\cdot)$ such that $-u''(x)/u'(x)$ is a weakly decreasing function of x . Then,*

$$\frac{d}{dx} \left(\frac{u'(x+y)}{u'(x-y)} \right) \geq 0 \text{ and } \frac{d}{dy} \left(\frac{u'(x+y)}{u'(x-y)} \right) < 0,$$

for each $x, y \in \mathbb{R}_+$.

Proof of Lemma 3. Note that,

$$\frac{d}{dx} \left(\frac{u'(x+y)}{u'(x-y)} \right) = \frac{u'(x+y)}{u'(x-y)} \left(\frac{u''(x+y)}{u'(x+y)} - \frac{u''(x-y)}{u'(x-y)} \right) \geq 0,$$

where the inequality follows since $-u''(x)/u'(x)$ is weakly decreasing in x . We also have,

$$\frac{d}{dy} \left(\frac{u'(x+y)}{u'(x-y)} \right) = \frac{u''(x+y)u'(x-y) + u'(x+y)u''(x-y)}{u'(x-y)^2} < 0,$$

where the inequality follows since $u(\cdot)$ is strictly concave. ■

Proof of Proposition 5. Let $d_1(r_1)$ and $e_0(r_1)$ denote the solution to (35). It is also useful to define,

$$y(r_1) = d_0 - \frac{d_1(r_1)}{1+r_1},$$

which corresponds to lenders' consumption at date 0 in excess of their net income. Note that we have $y(r_1) > 0$ by assumption.

We first show that $e'_0(r_1) < 0$. Suppose, to reach a contradiction, $e'_0(r_1) \geq 0$. Lenders' Euler equation implies [cf. Eq. (35)],

$$(A10) \quad \frac{1}{1+r_1} u'(e_0(r_1) + y(r_1)) = \beta^l u'(\bar{c}_1^l).$$

Since $e'_0(r_1) \geq 0$, this expression implies $y'(r_1) < 0$. Using the definition of $y(\cdot)$, this further implies $d'_1(r_1) > 0$. Next consider borrowers' Euler equation [cf. Eq. (35)],

$$\frac{1}{1+r_1} = \frac{\beta^b u'(\bar{c}_1^b - 2(d_1(r_1) - \phi))}{u'(e_0(r_1) - y(r_1))}.$$

The left hand side is strictly decreasing in r_1 . However, since $e'_0(r_1) \geq 0$, $y'(r_1) < 0$ and $d'_1(r_1) > 0$, the right hand side is strictly increasing in r_1 . This yields a contradiction and proves $e'_0(r_1) < 0$.

We next establish that $d'_1(r_1) > 0$. Suppose, to reach a contradiction, $d'_1(r_1) \leq 0$. Using the definition of $y(\cdot)$, this further implies $y'(r_1) > 0$. Combining lenders' and borrowers' Euler equations, we also have [cf. Eq. (35)],

$$\frac{\beta^l u'(\bar{c}_1^l)}{\beta^b u'(\bar{c}_1^b - 2(d_1(r_1) - \phi))} = \frac{u'(e_0(r_1) + y(r_1))}{u'(e_0(r_1) - y(r_1))}.$$

The left hand side is weakly increasing in r_1 since $d'_1(r_1) \leq 0$. However, since $e'_0(r_1) < 0$ and $y'(r_1) > 0$, Lemma 3 implies the right hand side is strictly decreasing in r_1 . This yields a contradiction and shows $d'_1(r_1) > 0$, completing the proof. ■

Proof of Proposition 6. The planner faces the constrained planning problem (2) after replacing $u(c_0^h)$ with a more general utility function, $u^h(c_0^h)$. The same steps as in the proof of Proposition 2 show that the constrained efficient allocations feature the maximum level of net income, $e_0 = e^*$. Implementing this outcome requires setting $r_1 = r_1^*$, proving the result. ■

Proofs for the extension with fire sales. We next complete the characterization of the model with fire sales described in Section VI. To characterize the condition $p_2 \cdot \partial MRS / \partial p_1 < 1$ in more detail, note that

$$\frac{\partial MRS}{\partial p_1} = \frac{-(1-\beta)\phi u''(c_2^b) [(1-\phi)u'(c_1^b) - 2\phi\beta u'(c_2^b)] - (1-\phi)\phi u''(c_1^b) u'(c_2^b)}{[(1-\phi)u'(c_1^b) + \phi\beta u'(c_2^b)]^2}.$$

If we approximate $\beta \approx 1$, all but the last term disappear from the numerator. Furthermore, we approximate $u'(c_1^b) \approx \beta u'(c_2^b)$ which holds exactly in the neighborhood of where the constraint becomes binding. This simplifies the expression in the denominator. Taken together,

$$\frac{\partial MRS}{\partial p_1} \approx -\frac{\phi(1-\phi)u''(c_1^b)}{u'(c_1^b)} = \frac{\phi(1-\phi)}{\sigma c_1^b} < \frac{1}{p_2} \quad \text{or} \quad \phi(1-\phi) < \sigma \frac{c_1^b}{p_2},$$

where σ is the intertemporal elasticity of substitution. In short, the solution to equation (38) is unique and well-defined if the leverage parameter is sufficiently small compared to the consumption/asset price ratio. If the condition was violated, an infinitesimal increase at date 1 consumption would lead to a discrete upward jump in the asset price and relax the constraint by more than necessary to finance the marginal increase in consumption. This would violate the assumption that the equilibrium exhibits a binding borrowing constraint. Observe also that this is a common type of condition in models of financial amplification to guarantee uniqueness (see e.g. Lorenzoni, 2008; Jeanne and Korinek, 2010b). ■

APPENDIX B: EXTENSION WITH FLEXIBLE MPC DIFFERENCES

This appendix develops the extension with flexible MPC differences between borrowers and lenders. We first consider the case without uncertainty, completing the characterization in Section III.C. We then introduce uncertainty, with and without complete markets, and complete the characterizations in respectively Sections IV.A and IV.B.

1. Case without uncertainty

We analyze a slightly more general version of the model than described in Section III.C. In particular, suppose there are several groups of households denoted by $h = \{1, \dots, |H|\}$ each of which has mass ω^h and log utility. Each individual household experiences a shock at date 1 which turns her into one of two types, $\{h_{unc}, h_{con}\}$. As in the main text, type h_{unc} households are unconstrained and thus they have a low MPC equal to $1 - \beta$. Type h_{con} households mechanically target a constrained debt level, $d_2^{h_{con}} = \phi^h$. Consequently, these households have a high MPC equal to 1. Groups differ in terms of the fraction of the high MPC types they contain, $\alpha^h \in [0, 1]$, as well as the initial debt level, d_1^h . As before, the type shocks are uninsurable at date 0, so that households within the same group have the same level of debt at date 1, $d_1^{h_{unc}} = d_1^{h_{con}} = d_1^h$.

We assume that group 1 does not feature any constrained households, $\alpha^1 = 0$, and brings assets to date 1, $d_1^1 < 0$. This group captures fully unconstrained lenders as in the main text. Some of the remaining groups feature $d_1^h > 0$. These groups can be thought of as borrowers with heterogeneous MPCs, generalizing the borrowers in the main text. Some of the remaining groups might also feature $d_1^h \leq 0$. These groups can be thought of as lenders that have higher MPCs at date 1 than the baseline lenders (perhaps because they have relatively low assets and can become constrained with some probability). The

model in Section III.C is a special case with two groups of households: unconstrained lenders with mass $\omega^1 = \omega$ and a single group of borrowers with mass $\omega^2 = 1$ and constraints $\alpha^2 = \alpha$.

We next analyze the general model and obtain the results in Section III.C as a special case. As before, under appropriate conditions (that will be characterized below), date 1 features a liquidity trap equilibrium with $r_2 = 0$ and $e_1 \leq e^*$. In this equilibrium, type h_{con} households are forced into deleveraging. Thus, their outstanding debt for the next date is

$$(B1) \quad d_2^{h_{con}} = \phi^h.$$

In contrast, type h_{unc} households choose their consumption and outstanding debt according to the Euler equation,

$$u'(c_1^{h_{unc}}) = \beta u'(c_2^{h_{unc}}), \text{ with } c_1^{h_{unc}} = e_1 - d_1^h + d_2^{h_{unc}} \text{ and } c_2^{h_{unc}} = e^* - d_2^{h_{unc}} (1 - \beta).$$

After substituting log utility and rearranging terms, these households' debt choices satisfy

$$(B2) \quad e^* - d_2^{h_{unc}} = \beta (e_1 - d_1^h).$$

Finally, the relevant debt market clearing conditions can be written as:

$$(B3) \quad \sum_h \omega^h [\alpha^h d_1^h + (1 - \alpha^h) d_1^h] = 0,$$

$$(B4) \quad \text{and } \sum_h \omega^h [\alpha^h d_2^{h_{con}} + (1 - \alpha^h) d_2^{h_{unc}}] = 0.$$

The equilibrium is solved by combining Eqs. (B1)-(B4), which gives:

$$(B5) \quad (e^* - \beta e_1) \sum_h (1 - \alpha^h) \omega^h = \sum_h \alpha^h \omega^h (\beta d_1^h - \phi^h).$$

Loosely speaking, the right hand side of this expression provides a measure of aggregate deleveraging, that is, the reduction in debt level by all constrained (or high MPC) households. The left hand side captures the adjustment by all unconstrained (or low MPC) households. The economy experiences a liquidity trap as long as the right hand side is sufficiently large, that is,

$$(B6) \quad \sum_h \alpha^h \omega^h (\beta d_1^h - \phi^h) > e^* (1 - \beta) \sum_h (1 - \alpha^h) \omega^h.$$

Note that the liquidity trap is triggered when the constrained groups' initial debt level is large. When this is the case, Eq. (B5) pins down the equilibrium level of net income as a

function of each group's initial debt level:

$$(B7) \quad e_1(\{d_1^h\}_h) = \frac{e^*}{\beta} - \sum_h \frac{\omega^h \alpha^h}{\sum_h (1 - \alpha^h) \omega^h} \left(d_1^h - \frac{\phi^h}{\beta} \right).$$

Let $\bar{\omega} = \sum_h \omega^h$ denote the total mass of households and $\bar{\alpha} = \sum_h \frac{\omega^h}{\bar{\omega}} \alpha^h$ denote the average fraction of high MPC households. Then, using Eq. (B7), we obtain,

$$\frac{\bar{\omega}}{\omega^h} \frac{de_1}{dd_1^h} = -\frac{\alpha^h}{1 - \bar{\alpha}}.$$

To convert this expression into MPCs, note that $\alpha^h = \frac{MPC^h - MPC^1}{1 - MPC^1}$, where we recall that group 1 captures fully unconstrained lenders. Plugging in, we obtain:

$$(B8) \quad \frac{\bar{\omega}}{\omega^h} \frac{de_1}{dd_1^h} = -\frac{\alpha^h}{1 - \bar{\alpha}} = \frac{MPC^h - MPC^1}{1 - \overline{MPC}},$$

where $\overline{MPC} = \sum_h \frac{\omega^h}{\bar{\omega}} MPC^h$. The left hand side captures how raising group h households' total debt by 1 unit affects total demand. The expression is normalized by $\bar{\omega}/\omega^h$ since the group's total debt level is $\omega^h d_1^h$ and the total demand (or income) is $\bar{\omega} e_1$. Intuitively, increasing the debt of group h effectively transfers financial wealth from this group to unconstrained households. This in turn lowers aggregate demand, and more so when group h has high MPC relative to unconstrained households. The effect is amplified by the Keynesian income multiplier, captured by the denominator of the right-hand side.

To obtain Eq. (24), note that in the special case of Section III.C the equilibrium debt level satisfies $d_1 = d_1^2 = -d_1^1/\omega$. Thus, raising the equilibrium debt level amounts to raising the debt level of group 2 (borrowers) while reducing the debt level of group 1 (lenders). In particular, we have:

$$\bar{\omega} \frac{de_1}{dd_1} = \bar{\omega} \left(\frac{de_1}{dd_1^2} - \omega \frac{de_1}{dd_1^1} \right) = \bar{\omega} \frac{de_1}{dd_1^2} = \frac{MPC^2 - MPC^1}{1 - \overline{MPC}}.$$

This gives the expression in (24) after relabeling $b = 2$ and $l = 1$. Notice that the effect through lenders' debt (or financial wealth) drops out, because those households are unconstrained by assumption, which implies $\frac{de_1}{dd_1^1} = 0$ [cf. Eq. (B8)].

We next characterize the constrained optimal allocations for the range in which there is a liquidity trap. We use $V^h(d_1^h, e_1)$ to denote type h households' continuation utility conditional on their debt level d_1^h and the aggregate net income e_1 . With some abuse of notation, we also let $u'(c_1^h) = \alpha^h u'(c_1^{hcon}) + (1 - \alpha^h) u'(c_1^{hunc})$ denote group h households' expected marginal utility at date 1 before the realization of their types. The

constrained planning problem can be written as,

$$(B9) \quad \max_{(c_0^h, d_1^h)_h} \sum_h \gamma^h \omega^h [u(c_0^h) + \beta^h V^h(d_1^h, e_1)]$$

such that $\sum_h \omega^h d_1^h = 0$ for each h , and $\sum_h \omega^h c_0^h = \sum_h \omega^h e^*$.

Let A and B respectively denote the Lagrange multipliers for the constraints. The first order conditions can then be written as $\gamma^h u'(c_0^h) = B$ for each h , and

$$-\gamma^h \omega^h \beta^h u'(c_1^h) + \left[\sum_{\tilde{h}} \gamma^{\tilde{h}} \omega^{\tilde{h}} \beta^{\tilde{h}} u'(c_1^{\tilde{h}}) \right] \frac{de_1}{dd_1^h} = A \omega^h.$$

Combining the first order conditions, and using the fact that $\frac{de_1}{dd_1^1} = 0$ (since group 1 is unconstrained), we obtain

$$(B10) \quad \frac{\beta^1 u'(c_1^1)}{u'(c_0^1)} = \frac{\beta^h u'(c_1^h)}{u'(c_0^h)} - \left[\sum_{\tilde{h}} \frac{\omega^{\tilde{h}} \beta^{\tilde{h}} u'(c_1^{\tilde{h}})}{\bar{\omega} u'(c_0^{\tilde{h}})} \right] \left(\frac{\bar{\omega} de_1}{\omega^h dd_1^h} \right),$$

for each h . In particular, the planner penalizes the increase in group h household's total debt to the extent to which this will reduce total net demand, and therefore, the utility of the average household. Note that Eq. (25) in the main text follows as a special case.

We finally characterize the tax rates that implement the optimal allocations. Define the tax rate on group h , τ_0^h , as the solution to $\frac{\beta^1 u'(c_1^1)}{u'(c_0^1)} = \frac{\beta^h u'(c_1^h)}{u'(c_0^h)} \frac{1}{1-\tau_0^h}$ [cf. Eq. (21)]. Plugging these definitions into Eq. (B10), the optimal tax rates, $\{\tau_0^h\}$, solve the system,

$$\tau_0^h = -\frac{\bar{\omega} de_1}{\omega^h dd_1^h} (1 - \bar{\tau}) = \frac{MPC^h - MPC^1}{1 - MPC} (1 - \bar{\tau}) \text{ for each } h.$$

Here, $\bar{\tau} = \sum_{\tilde{h}} \frac{\omega^{\tilde{h}}}{\bar{\omega}} \tau_0^{\tilde{h}}$, denotes the average tax rate across all households, and the second equation uses Eq. (B8). The solution to the system is given by,

$$(B11) \quad \tau_0^h = \frac{MPC^h - MPC^1}{1 - MPC^1} \text{ for each } h.$$

Eq. (26) in the main text follows as a special case.

Note also that the implementation of the optimal policy with the tax rates in (B11) features two differences from the baseline implementation in Corollary 2. First, the planner uses non-anonymous policies in the sense that the tax rate τ_0^h is only applied to group h households (since different groups require different tax rates). Second, the rate τ_0^h is applied to all debt issuance by this group, d_0^h , as opposed to only positive debt issuance. A tax rate on a negative debt issuance $d_0^h < 0$ is in effect a subsidy for saving:

it raises the interest rate households receive to $\frac{1+r_1}{1-\tau_0^h}$. The planner can use these types of subsidies to raise the saving of lenders that have relatively high MPCs at date 1.

2. Case with uncertainty and complete markets

Next suppose that the economy at date 1 is in two possible states $s \in \{H, L\}$, as described in Section IV. Let $d_{1,H}^h, d_{1,L}^h$ denote group h households' debt level in state H and L , respectively. State H features the frictionless level of net income, $e_{1,H} = e^*$, along with consumption levels, $c_{1,H}^h = e^* + d_{1,H}^h(1 - \beta)$ for each h . State L is exactly the same as date 1 of the earlier model so that much of the analysis for the previous case continues to apply. Specifically, the economy experiences a demand-driven recession if the outstanding debt levels $\{d_{1,L}^h\}_h$ satisfy the inequality in (B6), and the net income $e_{1,L}$ is characterized by Eq. (B7).

The main difference in this case concerns the constrained efficient allocations and the optimal macroprudential policies at date 0 (for the range in which there is a liquidity trap in state L). If the market is complete in the sense that households can trade insurance contracts for state L , then the constrained efficient allocations solve the following analogue of problem (B9):

$$(B12) \quad \max_{(c_0^h, d_{1,L}^h, d_{1,H}^h)_h} \sum_h \gamma^h \omega^h \left[u(c_0^h) + \beta^h \sum_{s \in \{H, L\}} \pi_s^h V_s^h(d_{1,s}^h, e_1) \right]$$

such that $\sum_h \omega^h d_s^h = 0$ for each h and $s \in \{H, L\}$, and $\sum_h \omega^h c_0^h = \sum_h \omega^h e^*$.

The optimality conditions for c_0^h and $(d_{1,H}^h)_h$ imply full insurance for state H (except for the idiosyncratic shocks of households), that is, $\frac{\beta^1 \pi_H^1 u'(c_{1,H}^1)}{u'(c_0^1)} = \frac{\beta^h \pi_H^h u'(c_{1,H}^h)}{u'(c_0^h)}$ for each h . In contrast, the optimality conditions for c_0^h and $(d_{1,L}^h)_h$ imply the following analogue of Eq. (B10):

$$\frac{\beta^1 \pi_L^1 u'(c_{1,L}^1)}{u'(c_0^1)} = \frac{\beta^h \pi_L^h u'(c_{1,L}^h)}{u'(c_0^h)} - \left[\sum_{\tilde{h}} \frac{\omega^{\tilde{h}} \beta^{\tilde{h}} \pi_L^{\tilde{h}} u'(c_{1,L}^{\tilde{h}})}{\bar{\omega} u'(c_0^{\tilde{h}})} \right] \left(\frac{\bar{\omega}}{\omega^h} \frac{de_{1,L}}{dd_{1,L}^h} \right),$$

for each h .

We next define characterize the insurance subsidies that implement the optimal allocations. Define the insurance subsidy on group h , ζ_0^h , as the solution to $\frac{\beta^1 \pi_L^1 u'(c_1^1)}{u'(c_0^1)} = \frac{\beta^h \pi_L^h u'(c_1^h)}{u'(c_0^h)} \frac{1}{1-\zeta_0^h}$ [cf. Eq. (31)]. Plugging these definitions into the previously displayed equation implies that the insurance subsidies, $\{\zeta_0^h\}_h$, solve the system,

$$\zeta_0^h = - \left(\frac{\bar{\omega}}{\omega^h} \frac{de_{1,L}}{dd_{1,L}^h} \right) (1 - \bar{\zeta}_0) = \frac{MPC^h - MPC^1}{1 - MPC} (1 - \bar{\zeta}_0) \text{ for each } h.$$

Here, $\bar{\zeta}_0 = \sum_{\tilde{h}} \frac{\omega^{\tilde{h}}}{\bar{\omega}} \zeta_0^{\tilde{h}}$ denotes the average subsidy across all households, and the second equality follows from Eq. (B8). The solution to the system is given by

$$(B13) \quad \zeta_0^h = \frac{MPC^h - MPC^1}{1 - MPC^1} \text{ for each } h.$$

This also implies Eq. (32) in the main text and completes the characterization with complete markets.

3. Case with uncertainty and incomplete markets

Next consider the case in which the market is incomplete in the sense that state L is uninsurable. In this case, households are constrained to choose $m_L^h = 0$ (or equivalently, $d_{1,L}^h = d_{1,H}^h$). The constrained efficient allocations solve problem (B12) with the additional restriction that $d_1^h \equiv d_{1,H}^h = d_{1,L}^h$ for each h . The first order conditions imply the following analogue of Eq. (B10):

$$\frac{\beta^1 E^1 [u'(c_1^1)]}{u'(c_0^1)} = \frac{\beta^h E^h [u'(c_{1,H}^h)]}{u'(c_0^h)} - \left(\frac{\bar{\omega}}{\omega^h} \frac{de_{1,L}}{dd_1^h} \right) \sum_{\tilde{h}} \pi_L^{\tilde{h}} \frac{\omega^{\tilde{h}} \beta^{\tilde{h}} u'(c_{1,L}^{\tilde{h}})}{u'(c_0^{\tilde{h}})},$$

for each h . This also implies Eq. (33) in the main text.

In this case, we cannot provide an exact formula for the optimal tax rate due to the market incompleteness. For a back-of-the-envelope calculation, consider the simplifications in the main text, $\pi_L \equiv \pi_L^h$ and $u'(c_{1,L}^h) \simeq u'(c_{1,H}^h)$ for each h . The latter assumption also implies $\frac{\beta^h u'(c_{1,L}^h)}{u'(c_0^h)} \simeq \frac{\beta^h E^h [u'(c_{1,H}^h)]}{u'(c_0^h)}$ for each h at the no-tax allocation. Then, the optimal tax rates, $\{\tau_0^h\}$, satisfy the system [cf. Eq. (B11)],

$$(B14) \quad \tau_0^h = -\pi_L \left(\frac{\bar{\omega}}{\omega^h} \frac{de_{1,L}}{dd_1^h} \right) (1 - \bar{\tau}) = \pi_L \cdot \frac{MPC^h - MPC^1}{1 - MPC^1} (1 - \bar{\tau}) \text{ for each } h.$$

Unlike Eqs. (B11) and (B13), the tax formula in this case cannot be simplified further. To make progress, consider the solution to the optimal tax system as a function of the probability of deleveraging, $\{\tau_0^h(\pi_L)\}_h$. It can be seen that $\lim_{\pi_L \rightarrow 0} \bar{\tau}(\pi_L) = 0$, that is, the average tax rate approximates zero as the probability of deleveraging approaches zero. It follows that, when π_L is close to zero, Eq. (B14) can be further approximated as $\tau_0^h \simeq \pi_L \cdot \frac{MPC^h - MPC^1}{1 - MPC^1}$ for each h . Labeling 1 as l and setting $h = b$, we obtain Eq. (34) in the main text, completing the analysis with flexible MPCs.

APPENDIX C: OMITTED EXTENSIONS

This appendix presents several extensions of our baseline model, in order to illustrate that our main results are robust to these alternative specifications. Section C.1 develops

a version of the model in which nominal prices are partially flexible. Section C.2 considers the case in which the nominal price stickiness is in the labor market as opposed to the goods market. Section C.3 analyzes the case without subsidies to correct for the monopoly distortions. Section C.4 considers the case in which households have separable preferences between consumption and labor, as opposed to the GHH preferences analyzed in the main text.

1. *Partially sticky prices*

Our baseline model features an extreme form of nominal price stickiness. We next develop a version of the model in which prices are partially flexible. We show that, as long as the monetary policy follows an inflation targeting rule, this model yields the same real allocations as the baseline model, up to a first-order approximation. We also allow the inflation target to be greater than zero and show that, while a higher inflation target reduces the incidence of liquidity traps, it does not change our qualitative results.

For simplicity, we work with a stylized version of the New Keynesian model. We denote the gross nominal inflation rate at time t with $\Pi_t = \frac{P_t}{P_{t-1}}$. A fraction s of firms have sticky prices in the sense that they do not reoptimize their price level every period. However, these firms passively index their price changes to the long run inflation rate in the economy. Specifically, they set

$$P_t^{sticky} = P_{t-1}\Pi \text{ for each } t,$$

where $\Pi \geq 1$ denotes the (gross) inflation target that will be described below and P_{-1} is given. Given their predetermined price level, these firms solve problem (9) as in the baseline model. The remaining fraction $1 - s$ of firms have fully flexible prices, in the sense that they reoptimize their price level every period. These firms solve problem (8), which yields,

$$(C1) \quad P_t^{flex} = \frac{\varepsilon}{\varepsilon - 1} P_t w_t (1 - \tau(n_t)) = P_t w_t,$$

where we have used the assumption that $\tau(n_t) = 1/\varepsilon$ over the relevant range. The baseline model in Section I can be thought of as the special case with $s = 1$ and $\Pi = 1$.

Given the Dixit-Stiglitz technology in (4), the nominal price for the final consumption good satisfies $P_t^{1-\varepsilon} = s \left(P_t^{sticky}\right)^{1-\varepsilon} + (1 - s) \left(P_t^{flex}\right)^{1-\varepsilon}$. Log-linearizing this equation around an equilibrium in which all firms set the same price level $P_t^{sticky} \simeq P_t^{flex}$, and combining with the earlier pricing equations, we obtain:

$$\log \Pi_t \simeq \frac{1 - s}{s} \log w_t + \log \Pi.$$

Log-linearizing the wage level around the frictionless benchmark, we further obtain a

variant of the New Keynesian Phillips Curve,²⁵

$$(C2) \quad \log \Pi_t \simeq \frac{1-s}{s} \frac{v''(n^*)}{v'(n^*)n^*} \log \left(\frac{y_t}{y^*} \right) + \log \Pi.$$

Note that inflation depends on the current output gap $\log(y_t/y^*)$, in view of the flexible price firms, as well as the long run inflation target, in view of the sticky price firms with inflation indexation. The extent to which the current output gap influences inflation depends on the fraction of flexible firms, $1-s$, as well as the elasticity of marginal costs (or wages) with respect to the changes in employment.

Eq. (C2) summarizes the behavior of the supply side of the model up to log-linearization. On the demand side, we replace (10) with an inflation targeting monetary policy, specifically, we assume the monetary authority follows the rule:

$$(C3) \quad 1 + i_{t+1} = \begin{cases} \max \left(0, (1 + r_{t+1}^*) \Pi \left(\frac{\Pi_t}{\Pi} \right)^\psi \right) & \text{if } t \geq 1, \\ \max \left(0, (1 + r_{t+1}^*) \Pi_1 \left(\frac{\Pi_0}{\Pi} \right)^\psi \right) & \text{if } t = 0. \end{cases}$$

Here, r_{t+1}^* is the frictionless real interest rate defined as before, $\Pi \geq 0$ is the gross inflation target, and $\psi > 1$ is a coefficient that captures the responsiveness of monetary policy to inflation. This policy attempts to set the real interest rate equal to its frictionless level, r_{t+1}^* , while also keeping inflation at its target level.²⁶

We next turn to the characterization of equilibrium. First consider dates $t \geq 2$ at which $r_{t+1}^* = 1/\beta^l - 1 > 0$. At these dates, the equilibrium features the frictionless outcomes, $r_{t+1} = r_{t+1}^*$, $y_t = y^*$, $e_t = e^*$, along with inflation equal to its target level, $\Pi_t = \Pi$. This level of inflation corresponds to an equilibrium in view of Eq. (C3) and the zero output gap, $\log(y_t/y^*) = 0$. Intuitively, since the zero lower bound does not bind for any $\tilde{t} \geq 2$, the monetary policy in (C3) implements the frictionless outcomes while also stabilizing inflation.²⁷

Next consider date $t = 1$. The key observations is that, since the monetary policy will stabilize inflation starting date 2, households (rationally) expect the inflation to be equal to its target level, that is, $\Pi_2 = P_2/P_1 = \Pi$. Combining this with the nominal interest rate bound in (3), we obtain a bound on the real rate as in the baseline model,

$$(C4) \quad 1 + r_2 = \frac{1 + i_2}{\Pi} \geq \frac{1}{\Pi}.$$

²⁵Specifically, we first log-linearize Eqs. (4) and (5) around the frictionless benchmark (that features $y_t(v) = y^*$), which gives $\log(n_t/n^*) \simeq \log(y_t/y^*)$. We then log-linearize the optimal labor supply condition $w_t = v'(n_t)$, which gives $\log w_t \simeq \frac{v''(n^*)}{v'(n^*)n^*} \log(n_t/n^*)$. Plugging these expressions into the inflation equation yields Eq. (C2).

²⁶We make a distinction for date 0 because, as we will see, this will be the only date at which the expected inflation will deviate from the target (i.e., $\Pi_1 \neq \Pi$). The monetary policy at date 0 is adjusted to take this anticipated deviation into account.

²⁷We abstract away from the equilibria with self-fulfilling deflationary traps and inflationary panics (see Cochrane, 2011).

Note that, when the gross inflation target is equal to 1, the bound is the same as in the main text [cf. (6)]. Hence, in this case the equilibrium allocations are also exactly the same. For the more general case with $\Pi \geq 1$, the bound on the real interest rate is smaller than in the main text [cf. (C4)]. Thus, a higher inflation target Π reduces the incidence of a liquidity trap. That said, the qualitative analysis is the same as in Section II. Specifically, there is a threshold debt level $\bar{d}_1(\Pi)$, increasing in Π , such that the economy enters a liquidity trap as long as $d_1 > \bar{d}_1(\Pi)$.

One difference from the baseline model concerns the behavior of inflation at date 1. In particular, when $d_1 > \bar{d}_1(\Pi)$, the economy features a negative output gap, $\log(y_t/y^*) < 0$. Consequently, Eq. (C2) implies that the realized inflation at date 1 is below the target level, $\Pi_1 < \Pi$. Thus, the liquidity trap episode is also associated with some disinflation.

Finally, consider the equilibrium at date $t = 0$. The real interest rate at this date is also bounded by $1 + r_1 = \frac{1+i_1}{\Pi_1} \geq 1$. We modify the bound \bar{d}_0 in Assumption (1) appropriately so that this constraint does not bind in equilibrium. Under this assumption, the monetary policy in (C3) implements the frictionless outcome along with $\Pi_0 = P_0/P_{-1} = \Pi$. The equilibrium debt level is characterized by Eq. (16) as before, and features $d_1 > \bar{d}_1(\Pi)$ under conditions analogous to those in Proposition 1.

In sum, if monetary policy follows the rule in (C3) with $\Pi = 1$, then the model with partially sticky prices yields the same allocations as in the baseline model, up to a first-order approximation. Intuitively, although prices are somewhat flexible at the micro level, the aggregate prices between dates 1 and 2 continue to be sticky in view of the inflation targeting monetary policy. A higher inflation target Π alleviates the liquidity trap by relaxing the bound in (C4), but it does not change our qualitative results.

By the same logic, alternative monetary policies that increase inflation at date 2 above the target level Π can also alleviate the liquidity trap. Note, however, that these policies are not time consistent in our environment. Specifically, these policies create a dispersion of relative prices between sticky and flexible price firms, which lowers social welfare. If inflation is costly for this (or other) reasons, then a monetary authority without commitment power will find it optimal to follow the policy in (C3) together with $\Pi = 1$, which will generate the same real allocations as in the baseline model.

In our model, households at date 0 anticipate the path of future inflation and adjust the interest rate charged accordingly. However, if the deleveraging episode is unanticipated, or if it is stochastic and the financial market is incomplete (as in Section 5.2), then there is an additional force, debt deflation, that would aggravate the recession. When households have noncontingent nominal debt and inflation is lower than expected, the real burden of debt is inflated by the falling price level (see Eggertsson and Krugman 2012 for a formalization in the context of a liquidity trap). This has the power to significantly exacerbate the resulting recession and compound the aggregate demand externalities. As we discuss in more detail in Remark 2 of the main text, the US economy was fortunately spared any significant disinflation during the most recent macroeconomic slump so that the forces of debt deflation were weak.

2. Downward nominal wage rigidities

Our baseline model features nominal price stickiness in the goods market. We next illustrate that assuming instead downward nominal wage rigidity in the labor market (as in Eggertsson and Mehrotra 2014 or Schmitt-Grohé and Uribe 2012c) does not change our results, as long as monetary policy follows an inflation targeting rule.

Suppose each final good firm reoptimizes its price every period. Eq. (C1) then implies that the aggregate price level is given by,

$$(C5) \quad P_t = W_t \text{ for each } t,$$

where W_t denotes the nominal wage level (equivalently, the relative wage level is $w_t = 1$). Unlike in the earlier analysis, however, the nominal wage level is assumed to be sticky in the downward direction. In particular, wages cannot fall below a norm \bar{W}_t . Following Eggertsson and Mehrotra (2014), we also assume $\bar{W}_t = \gamma W_{t-1} + (1 - \gamma) P_t v'(n_t)$, where γ is a parameter that captures the degree of price rigidity and $P_{-1} = W_{-1}$ is given. If $\gamma = 1$, then the nominal wage level cannot fall below its level in the last period. For lower levels of γ , the wage level can fall to some extent (when the marginal cost of labor is low) but it is nonetheless constrained.

The key difference of this model concerns the behavior of households' labor supply. If the (nominal) marginal cost of labor is above the norm, $P_t v'(n_t) \geq \bar{W}_t$, then the constraint does not bind. In this case, the equilibrium wage level satisfies $W_t = P_t v'(n_t)$ and the labor supply is competitive. If instead the marginal cost is below the norm, $P_t v'(n_t) < \bar{W}_t$, then the wage level satisfies $W_t = \bar{W}_t$ and the labor supply is rationed (symmetrically across all households). Combining the two cases, the labor supply is given by,

$$(C6) \quad W_t = \max \{ \bar{W}_t, P_t v'(n_t) \}, \text{ where } \bar{W}_t = \gamma W_{t-1} + (1 - \gamma) P_t v'(n_t).$$

We also assume that monetary policy follows the inflation targeting rule in (C3). The rest of the equilibrium is unchanged.

The characterization of the equilibrium closely parallels the analysis in Section C.1. Specifically, starting date 2 onwards, the inflation targeting monetary policy implements the frictionless outcomes along with the target inflation level, $\Pi_t = \Pi$. Note that the labor market is in equilibrium because the nonnegative inflation target implies $P_t = W_t \geq P_{t-1} = W_{t-1}$ for each t , so that the wage constraint does not bind and $n_t = n^*$ (cf. Eqs. (C5) and (C6)).

At date 1, the expected inflation is equal to the target as before, $P_2/P_1 = \Pi$, which leads to the bound on the real rate in (C4). Given this bound, when the debt level exceeds a threshold $d_1 > \bar{d}_1(\Pi)$, the economy cannot replicate the frictionless outcome. In this case, the economy experiences a demand-driven recession as before. The recession puts downward pressure on the wage level (in view of the low marginal costs), but the wages can only fall so much. Specifically, the equilibrium at date 1 features [cf. Eqs. (C5) and

(C6)],

$$(C7) \quad P_1 = W_1 = \bar{W}_1 = \frac{\gamma P_0}{1 - (1 - \gamma) v'(n_1)}.$$

In this case, the wage constraint in (C6) binds and labor supply is rationed. The equilibrium level of employment satisfies $n_1 = y_1 < n^* = y^*$, and it is determined by aggregate demand at the constrained interest rate in (C4). Thus, the real equilibrium allocations at date 1 are as in Section C.1. Intuitively, the difference in this case is that the shortage of demand is countered by rationing in the labor market (as opposed to rationing in the goods market that translates into low wages and employment). The analysis of the equilibrium at date 0 also closely parallels that in Section C.1, illustrating that our results are robust to allowing for downward wage rigidity.

Note also that Eq. (C7) implies that the price level at date 1 satisfies, $P_1 < P_0$, because $v'(n_1) < v'(n^*) = 1$. Hence, as in Section C.1, the economy features disinflation (in fact, deflation) at date 1. However, the magnitude of the disinflation at date 1 is different, as it is governed by the degree of the nominal rigidity in the labor market as opposed to the goods market.

3. Role of monopoly subsidies

Our baseline model features linear subsidies for monopolists that depend on aggregate employment as follows,

$$(C8) \quad \tau(n_t) = \begin{cases} 1/\varepsilon, & \text{if } n_t \leq n^* \\ 0, & \text{if } n_t > n^* \end{cases}.$$

We first explain why we take away these subsidies for the range $n_t > n^*$. We then generalize our analysis to the case without monopoly subsidies $\tau = T_t = 0$.

To see why we assume (C8), consider the alternative assumption that $\tau = 1/\varepsilon$ regardless of the employment level. In this case, Proposition 1 features not one but two equilibria (for some parameters). Specifically, when $d_1 > \bar{d}_1$, there is a second equilibrium that has the same net allocations, e_1, c_1^l, c_1^b , but that differs in actual output and employment. The multiplicity emerges because, in view of the GHH preferences, the net income $e_1 = n_1 - v(n_1) < e^*$ can be obtained by two different levels of employment: one that features a recession $n_1^r < n^*$, and another one that features an excessive boom $n_1^b > n^*$. The two equilibria exhibit the same utility, $u(c_t^h)$, for all households and time periods. Thus, they are identical in terms of their welfare implications. However, the boom equilibrium is fragile in the sense that it is an artefact of positive monopoly subsidies. More specifically, once we take away the subsidies for $n_1 > n^*$ as in (C8), then the boom equilibrium disappears because monopolists' marginal cost exceeds their marginal product, that is, $w_1 = v'(n_1^b) > v'(n^*) = 1$. We assume (C8) since this enables us to focus on the recession equilibrium, which is less fragile (in the sense that it does not depend on the subsidies), while also providing a clean conceptual benchmark for welfare analysis.

We next consider the case without monopoly subsidies $\tau = T_t = 0$ and show that our analysis remains unchanged up to some relabeling. The main difference in this case concerns the frictionless employment and net income levels, which we respectively denote by n^{**} and e^{**} to emphasize their difference from the efficient levels, n^* and e^* . Specifically, the optimality condition for problem (8) implies $1 = p_t = \frac{\varepsilon}{\varepsilon-1} w_t = \frac{\varepsilon}{\varepsilon-1} v'(n^{**})$. This in turn implies $n^{**} < n^*$ and $e^* < e^{**}$ [cf. (7)]. We assume that the monetary policy continues to follow (10) after replacing e^* with e^{**} .²⁸ With this assumption, the analysis in Section II, including Proposition 1, remains unchanged after replacing e^* with e^{**} .

Moreover, the equilibrium at date 1 is actually unique in this case, not only in terms of net income but also in terms of actual output and employment. To see this, note that the equilibrium employment with $\tau = 0$ must satisfy $n_1 \leq n^*$, since otherwise monopolists would make negative profits [cf. problem (9)]. Next note that, given any feasible level of net income $e_1 = n_1 - v(n_1)$, there is a unique level of employment that also satisfies $n_1 \leq n^*$. When the debt level is below the threshold $d_1 \leq \bar{d}_1$, the unique equilibrium features $e_1 = e^{**}$ and $n_1 = n^{**}$. When the debt level is above the threshold $d_1 > \bar{d}_1$, the interest rate constraint binds and the unique equilibrium features a recession $e_1 < e^{**}$ and $n_1 < n^{**}$.

We next analyze the efficiency properties of the equilibrium. The analogue of the constrained planning problem (19) in this case is given by,

$$(C9) \quad \max_{((c_0^h)_h, d_1)} \sum_h \gamma^h (u(c_0^h) + \beta^h V^h(d_1^h, d_1))$$

such that $d_1 = d_1^b = -d_1^l$ and $\sum_h c_0^h = 2e^{**} = 2(n^{**} - v(n^{**}))$.

In particular, the planner is also subject to the monopoly distortions at date 0 (as well as future dates). Solving problem (C9), it follows that our main result, Proposition 2, is also unchanged after replacing e^* with e^{**} . The result generalizes because Eq. (15) continues to hold in this case, so that $de_1/dd_1 = -1$ and leverage continues to exert negative aggregate demand externalities.

4. Separable preferences

Our baseline model features households with GHH preferences over consumption and labor, $u(\tilde{c} - v(l))$. This section generalizes our main result to a setting with separable preferences, $u(c) - v(l)$. We also assume downward wage rigidities, as in Section C.2, which leads to a simpler analysis. However, the results also hold for price rigidities as in the baseline model with slightly different formulas (see Appendix A.5 in our NBER working paper version).

Consider the baseline environment in Section I, with the difference that households have separable preferences, $u(c_t^h) - v(n_t^h)$. Let $\tau_t^{lab,h} = 1 - \frac{u'(c_t^h)}{v'(n_t^h)}$ denote the households'

²⁸In particular, we assume that the monetary policy is not used to target the efficient level of output, because this would exert upwards pressure on inflation (see Eq. (C2) in Section C.1).

labor wedge, which captures the extent to which employment at date t is below (or above, if negative) its efficient level. Suppose the final good firms have flexible prices so that the nominal price equals the nominal wage level, $P_t = W_t$, as in Section C.2. However, nominal wages are rigid and cannot fall below a wage norm, $\bar{W}_t = W_{t-1}$, which we take to be last period's wage for simplicity. The labor supply of each household h satisfies,

$$(C10) \quad W_t = \max \left\{ \bar{W}_t, P_t \frac{v'(n_t^h)}{u'(c_t^h)} \right\}.$$

If the nominal price (and wage) level is sufficiently high, then $P_t \frac{v'(n_t^h)}{u'(c_t^h)} \geq \bar{W}_t$ holds so that the labor wedge is zero and the labor is supplied efficiently. Otherwise, the wage level satisfies $W_t = \bar{W}_t$ and then the aggregate labor supply is rationed and determined by aggregate demand, $\frac{n_t^b + n_t^l}{2} = \frac{c_t^b + c_t^l}{2}$. We also assume the symmetric rationing rule (that will be specified below) by which a unit decline in demand translates into a unit decline in the labor supply of each group.

First consider dates $t \geq 2$, at which the level of consumer debt is constant at the maximum permissible level $d_t = \phi$ and borrowers pay lenders a constant amount of interest $\left(1 - \frac{1}{1+r_{t+1}}\right) \phi = (1 - \beta^l) \phi$ at every date. In this case, the inflation targeting policy implements $\Pi_t = \Pi = 1$ along with a zero labor wedge, $\tau_t^{lab,h} = 0$. Households' labor supply is given by:

$$u'(n^{l*} + (1 - \beta^l) \phi) = v'(n^{l*}), \text{ and } u'(n^{b*} - (1 - \beta^l) \phi) = v'(n^{b*}).$$

The labor market is in equilibrium because $P_t = P_{t-1}$ also implies $W_t = W_{t-1} = \bar{W}_t$ (see Eq. (C10)).

Now consider date 1. As in the main text, there is a threshold, \bar{d}_1 , such that there is a liquidity trap only if $d_1 \geq \bar{d}_1$. First consider the case $d_1 = \bar{d}_1$, in which case $r_2 = 0$ but the outcomes are still efficient (in particular, the labor wage is zero). Consider lenders' Euler equation at zero interest rate, $u'(\bar{c}_1^l) = \beta^l u'(n^{l*} + (1 - \beta^l) \phi)$, which determines their consumption \bar{c}_1^l . Their intratemporal condition, $u'(\bar{c}_1^l) = v'(\bar{n}^l)$, determines a corresponding employment level, \bar{n}^l . This in turn pins down the threshold debt level \bar{d}_1 , from lenders' budget constraint, $\bar{c}_1^l = \bar{n}_1^l + \bar{d}_1 - \phi$. Borrowers' consumption is constrained, and their labor supply is the solution to their own intratemporal condition, $u'(\bar{n}_1^b - (\bar{d}_1 - \phi)) = v'(\bar{n}_1^b)$.

Next consider the equilibrium when $d_1 > \bar{d}_1$. In this case, the economy features a liquidity trap with $r_2 = 0$. The nominal wage (as well as the price) is equal to the norm, $W_1 = \bar{W}_1 = W_0$, and the labor supply is rationed. We assume labor supply is rationed symmetrically according to the rule:

$$(C11) \quad n_1^l = \bar{n}_1^l - \Delta \text{ and } n_1^b = \bar{n}_1^b - \Delta,$$

where Δ denotes the size of the decline in demand and output (relative to the $d_1 = \bar{d}_1$

benchmark) to be determined. To solve for Δ , note that lenders' consumption is equal to $c_1^l = \bar{c}_1^l$. Using lenders' budget constraint in equilibrium, $c_1^l = \bar{n}_1^l - \Delta + d_1 - \phi$, we solve for the endogenous decline in output as:

$$(C12) \quad \Delta = \bar{n}_1^l - \bar{c}_1^l + d_1 - \phi,$$

Eqs. (C11) and (C12) provide the analogue of Eq. (15) in the main text. Note that $\frac{dn_1^l}{dd_1} = -1$ and $\frac{dn_1^b}{dd_1} = -1$, which also implies $\frac{dy_1}{dd_1} = 0$. A unit increase in debt generates a unit decline in output, as well as each household's employment, similar to the main text.

The date 0 equilibrium is characterized by the Euler equations (16). Under conditions similar to those in Proposition 1, the equilibrium features $d_1 > \bar{d}_1$ and an anticipated recession.

We next analyze the efficiency properties of this equilibrium. First let $V^h(d_1^h, d_1)$ denote the utility of a household h conditional on entering date 1 with an individual level of debt d_1^h and an aggregate level of debt d_1 . For the range $d_1 > \bar{d}_1$, we have:

$$V^b(d_1^b, d_1) = u(n_1^b(d_1) - (d_1^b - \phi)) - v(n_1^b(d_1)) + \sum_{t=2}^{\infty} (\beta^b)^t (u(c^{b*}) - v(n^{b*})),$$

and $V^l(d_1^l, d_1) = u(n_1^l(d_1) + (d_1^l - \phi)) - v(n_1^l(d_1)) + \sum_{t=2}^{\infty} (\beta^l)^t (u(c^{l*}) - v(n^{l*})).$

The aggregate demand externalities are then given by:

$$(C13) \quad \frac{\partial V^h}{\partial d_1} = \frac{dn_1^h}{dd_1} (u'(c_1^h) - v'(n_1^h)) = -u'(c_1^h) \tau_1^{lab,h}.$$

Hence, the externalities depend on the effect of debt on household's income (which is 1) as well as the labor wedge, $\tau_1^{lab,h} = 1 - v'(n_1^h) / u'(c_1^h) \in (0, 1)$.

Next consider the ex-ante constrained planning problem, which is still given by (19). Combining the first order conditions with Eq. (C13) implies

$$(C14) \quad \frac{\beta^l u'(c_1^l)}{u'(c_0^l)} = \frac{\beta^b u'(c_1^b)}{u'(c_0^b)} - \left(\frac{\beta^b u'(c_1^b)}{u'(c_0^b)} \tau_1^{lab,b} + \frac{\beta^l u'(c_1^l)}{u'(c_0^l)} \tau_1^{lab,l} \right).$$

It follows that the analogues of Proposition 2 and its Corollaries 1 and 2 also hold for this case. Note, however, that the planner typically mitigates but does not fully avoid the recession. This is because $\tau_1^{lab,h} = 0$ when $d_1 = \bar{d}_1$, which implies that Eq. (C14) will also have interior solutions with $d_1 > \bar{d}_1$ and $\tau_1^{lab,h} > 0$. We can also characterize the optimal tax rate on debt issuance, τ_0^b , which satisfies $1 = \frac{1}{1-\tau_0^b} - \left(\frac{1}{1-\tau_0^b} \tau_1^{lab,b} + \tau_1^{lab,l} \right)$.

Solving this equation, we obtain,

$$(C15) \quad \tau_0^b = \frac{\tau_1^{lab,b} + \tau_1^{lab,l}}{1 + \tau_1^{lab,l}}.$$

As Eqs. (C13) – (C15) illustrate that, when preferences are separable, the size of the inefficiency as well as the optimal intervention also depends on the labor wedge. Intuitively, raising aggregate demand improves social welfare to the extent to which labor is underutilized, as captured by the labor wedge.²⁹

The labor wedge does not appear in the formulas for the baseline model with GHH preferences, $u(\tilde{c}_t^h - v(n_t))$, as these preferences generate an endogenous amplification mechanism. To see the amplification, recall that 1 unit of decline in debt in the baseline model increases the average net income by 1 unit, which means that it increases the average income by $1 + v(n_t)$ units [cf. Eq. (C12)]. Intuitively, keeping the interest rate constant, any increase in labor costs $v(n_t)$ creates a further increase in demand and output (because unconstrained households' intertemporal substitution depends on net consumption, $c_t^h = \tilde{c}_t^h - v(n_t^h)$). This amplification ensures that the disutility of labor, as well as the labor wedge, does not appear in the optimal subsidy or tax formulas in the main text.

²⁹The borrowers' and lenders' MPC differences do not show up in Eq. (C15) because we have analyzed the baseline version of the model in which borrowers' MPC is equal to 1. The optimal tax rate would also depend on the MPC differences, as in the main text, if we were to consider the version with flexible MPCs described in Section III.C.