Exclusive contracts and market dominance

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Online Appendix

1. Proofs for the baseline model

This Section provides the proofs of Propositions 1 and 2.

Proof of Proposition 1. Since the competitive fringe will always price at cost, to prove the proposition it suffices to show that the dominant firm's equilibrium pricing strategy is indeed optimal. To do so, we shall focus on direct mechanisms and find the optimal quantity $q_A(\theta)$. The next step is then to show that this coincides with the quantity schedules associated with the equilibrium price schedules reported in the statement of Proposition 1. It is then straightforward to conclude that those price schedules are indeed optimal.

To begin with, we report the equilibrium quantities associated with the price schedules described in Proposition 1. They are:

- when $c \le c^{\lim}$

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \le \breve{\theta}_A \\ q_A^{cr}(\theta) & \text{for } \theta \ge \breve{\theta}_A \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \le \breve{\theta}^e \\ q^e(\theta) & \text{for } \breve{\theta}^e \le \theta \le \breve{\theta}_A \\ q_B^{cr}(\theta) & \text{for } \theta \ge \breve{\theta}_A, \end{cases}$$

- when $c^{\lim} \le c \le c^m$

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \le \breve{\theta}^e \\ q^{\lim}(\theta) & \text{for } \breve{\theta}^e \le \theta \le \breve{\theta}_B \\ q_A^{cr}(\theta) & \text{for } \theta \ge \breve{\theta}_B \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \le \breve{\theta}_B \\ q_B^{cr}(\theta) & \text{for } \theta \ge \breve{\theta}_B, \end{cases}$$

- when $c \ge c^m$

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \le \breve{\theta}^m \\ q^m(\theta) & \text{for } \breve{\theta}^m \le \theta \le \theta^{\lim} \\ q^{\lim}(\theta) & \text{for } \theta^{\lim} \le \theta \le \breve{\theta}_B \\ q_A^{cr}(\theta) & \text{for } \theta \ge \breve{\theta}_B \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \le \breve{\theta}_B \\ q_B^{cr}(\theta) & \text{for } \theta \ge \breve{\theta}_B, \end{cases}$$

where the threshold $\breve{\theta}_A$ is implicitly defined as the largest θ such that $q_A^{cr}(\theta) = 0$, and thus is the marginal buyer for product A (just as $\breve{\theta}_B$ is the marginal buyer for product B) under common representation; $\breve{\theta}^m$ is the marginal buyer under monopoly; and $\breve{\theta}^e$ is defined by the condition $q^e(\theta) = 0$, and hence is the marginal buyer under competitive (and limit) pricing.

To show that these quantities are indeed optimal, consider the indirect payoff function $v(q_A, \theta)$, as defined by (4), when $P_B(q_B) = cq_B$. This is piecewise smooth, with two branches corresponding to the cases in which the quantity $\tilde{q}_B(q_A, \theta) = \arg\max_{q_B \geq 0} [u(q_A, q_B, \theta) - cq_B]$ is 0 or is strictly positive, and a kink between the two branches. It can be easily checked that $v(q_A, \theta)$ is globally concave in q_A. It also satisfies the single-crossing condition $v_{\theta q_A}(q_A, \theta) \geq 0$, since we have

$$v_\theta(q_A, \theta) = u_\theta(q_A, \tilde{q}_B(q_A, \theta), \theta)$$

and hence:

$$\begin{aligned} v_{\theta q_A} &= u_{\theta q_A} + \frac{d\tilde{q}_B(q_A, \theta)}{dq_A} u_{\theta q_B} \\ &= u_{\theta q_A} - \frac{u_{q_B q_A}}{u_{q_B q_B}} u_{\theta q_B} \geq 0, \end{aligned}$$

where the inequality follows by the fact that the goods are imperfect substitutes.

The single-crossing condition guarantees that the participation constraint binds only for the marginal buyer, whom we indicate here as $\tilde{\theta}$, and that firm A's optimisation program (A.1) can be written as

$$\begin{aligned} &\max_{q_A(\theta)} \int_{\tilde{\theta}}^{\theta_{\max}} [v(q_A(\theta), \theta) - U(\theta)] f(\theta) d\theta \\ \text{s.t.} \quad \frac{dU}{d\theta} &= v_\theta(q_A, \theta) \\ U(\tilde{\theta}) &= v(0, \tilde{\theta}) \end{aligned}$$

By a standard integration by parts, the problem reduces to finding the function $q_A(\theta)$ that pointwise maximises the indirect virtual surplus:

$$s(q_A, \theta) = v(q_A, \theta) - \frac{1 - F(\theta)}{f(\theta)} v_\theta(q_A, \theta).$$

Like the indirect payoff function, the indirect virtual surplus has two branches and a kink at $q_A = q^{\lim}(\theta)$. Generally speaking, for any θ the maximum can occur in either one of the two quadratic branches, or at the kink. By definition, $q^m(\theta) = \arg\max_{q_A} s(q_A, \theta)$ when the maximum lies on the first branch, and $q_A^{cr}(\theta) = \arg\max_{q_A} s(q_A, \theta)$ when it lies on the second. Furthermore, the kink is implicitly defined by the condition $u_{q_B}\left(q^{\lim}(\theta), 0, \theta\right) = c$ and hence occurs at $q^{\lim}(\theta)$.

Since $q_A(\theta)$ must pointwise maximise the virtual surplus, we can conclude that $q_A(\theta) = q_A^m(\theta)$ if the maximum is achieved on the upper branch, $q_A(\theta) = q_A^{cr}(\theta)$ if the maximum is achieved on the lower branch, and $q_A(\theta) = q^{\lim}(\theta)$ if the maximum is achieved at the kink. By assumption H1, $s(q_A, \theta)$ is globally concave in q_A. This implies that if $q^m(\theta) > q^{\lim}(\theta)$, then $s(q_A, \theta)$ is increasing at the kink and the maximum is achieved at $q^m(\theta)$. If instead $q^m(\theta) < q^{\lim}(\theta)$, then $s(q_A, \theta)$ is decreasing "to the right" of the kink, and one must further distinguish between two cases. If $q_A^{cr}(\theta) > q^{\lim}(\theta)$, then $s(q_A, \theta)$ is increasing "to the left" of the kink and so the maximum is achieved at the kink, $q^{\lim}(\theta)$. If instead $q_A^{cr}(\theta) < q^{\lim}(\theta)$, the maximum is achieved "to the left" of the kink and is $q_A^{cr}(\theta)$.

It remains to find out when each type of solution applies. By H2, the condition $q^m(\theta) > q^{\lim}(\theta)$ is equivalent to $\theta < \theta^{\lim}$. Since $q^m(\theta)$ is positive only for $\theta > \theta^m$, the monopoly solution is obtained if and only if the interval $\theta^m \leq \theta \leq \theta^{\lim}$ is not empty. This is true if only if $c > c^m$ (recall that c^m is defined as the lowest c such that $q^m(\theta) > q^{\lim}(\theta)$ for some θ). In this case, then, we have $q_A(\theta) = q^m(\theta)$ for $\theta^m \leq \theta \leq \theta^{\lim}$. Of course, the corresponding equilibrium quantity of good B must be nil.

Now suppose that $\theta > \theta^{\lim}$, so that $q^m(\theta) < q^{\lim}(\theta)$. In this case, the solution depends on whether $q_A^{cr}(\theta)$ is larger or smaller than $q_A^{\lim}(\theta)$. The limit pricing solution can emerge only if $q_A^{cr}(\theta) > q_A^{\lim}(\theta)$. By H2, the condition $q_A^{cr}(\theta) > q^{\lim}(\theta)$ reduces to $\theta < \breve{\theta}_B$. Since $q^{\lim}(\theta)$ is positive only for $\theta > \breve{\theta}^e$, the limit pricing solution is obtained if and only $\breve{\theta}^e < \breve{\theta}_B$. This condition is equivalent to $c \geq c^{\lim}$ (recall that $c^{\lim}$ is the lowest c such that $q_A^{cr}(\theta) > q^{\lim}(\theta)$ for some θ). When this condition holds, there exists an interval of states of demand in which the limit pricing solution applies. Again, the corresponding equilibrium quantity of good B must be nil.

Finally, consider the case in which $\theta \geq \breve{\theta}_B$, so that $q_A^{cr}(\theta) \leq q^{\lim}(\theta)$ and the maximum is achieved on the lower branch of the virtual surplus function. Here, we must distinguish between two sub-cases, depending on whether the solution is interior, or is a corner solution at $q_A(\theta) = 0$. Clearly, the solution is interior, and is $q_A^{cr}(\theta)$, when $\theta \geq \breve{\theta}_A$. In this case, the corresponding equilibrium quantity of good B is $q_B^{cr}(\theta) = \tilde{q}_B(q_A^{cr}(\theta), \theta)$. Now, notice that when $c < c^{\lim}$ we have $\breve{\theta}_B < \breve{\theta}_A$, whereas the inequality is reversed when $c \geq c^{\lim}$. This means that if $c \geq c^{\lim}$ and the maximum is achieved in the lower branch, it must necessarily be an interior solution. However, when $c < c^{\lim}$ we have $\breve{\theta}_B < \breve{\theta}_A$. In this case, for $\breve{\theta}^e \leq \theta \leq \breve{\theta}_A$, we have a corner solution for q_A, and the corresponding equilibrium quantity of good B is $q^e(\theta)$; for $\theta \geq \breve{\theta}_A$, the solution is again interior.

This completes the derivation of the optimal quantities in all possible cases. It is then easy to check that they coincide with the equilibrium quantities reported above, and that they are implemented by the equilibrium price schedules. Notice that since equilibrium quantities are everywhere continuos, the equilibrium price schedules must be continuous. (In fact, it can be verified that the equilibrium price schedules are also everywhere smooth.) Finally, notice that the constant terms that guarantee continuity are all negative, i.e. they are fixed subsidies.

To complete the proof, we finally show that $c^m > c^{\lim}$. For the purposes of this proof, let us define $\theta^m(c)$ and $\theta^{\lim}(c)$ as follows: $\theta^m(c)$ is such that $q^{\lim}(\theta^m) = q^m(\theta^m)$, and $\theta^{\lim}(c)$ such that $q^{\lim}(\theta^{\lim}) = q_A^{cr}(\theta^{\lim})$. Then, it is clear that c^m satisfies $q^{\lim}(\theta^m(c^m)) = q^m(\theta^m(c^m)) = 0$, and $c^{\lim}$ satisfies $q^{\lim}(\theta^{\lim}(c^{\lim})) = q_A^{cr}(\theta^{\lim}(c^{\lim})) = 0$.

We distinguish between two cases: $\theta^m(c) \leq \theta^{\lim}(c)$ and $\theta^m(c) > \theta^{\lim}(c)$. If

$\theta^m(c^m) \leq \theta^{\lim}(c^m)$ then

$$q^{\lim}(\theta^{\lim}(c^m)) = q_A^{cr}(\theta^{\lim}(c^m)) \geq q^{\lim}(\theta^m(c^m)) = q^m(\theta^m(c^m)) = 0,$$

where the inequality simply follows from the fact that all quantities are increasing in θ. Furthermore, since $q^{\lim}$ is decreasing in c and q_A^{cr} is increasing in c, it follows that $\theta^{\lim}$ is increasing in c. This immediately implies that $c^{\lim} < c^m$.

If instead $\theta^m(c^m) > \theta^{\lim}(c^m)$, then the equilibrium never entails monopoly pricing and hence the comparison between $c^{\lim}$ and c^m is irrelevant (strictly speaking, c^m is not even well defined). To see why this is so, notice that if $\theta^m > \theta^{\lim}$ monopoly pricing would only arise for intermediate types, with both lower and higher types buying both products under common representation. But this is impossible as it would entail multiple intersections between $q^m(\theta)$ and $q_A^{cr}(\theta)$, thus contradicting assumption H2. ■

Proof of Proposition 2. The strategy of proof is the same as for Proposition 1. Obviously, the competitive fringe will always price at cost, i.e. $P_B^E(q_B) = P_B^{NE}(q_B) = cq_B$. As for firm A, we shall focus on direct mechanisms and hence look for the optimal quantity $q_A(\theta)$, showing that it coincides with the equilibrium quantities implied by the equilibrium price schedules reported in the statement of the proposition. Such quantities are:

- when $c \leq \bar{c}$

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq \check{\theta}^e \\ q^e(\theta) & \text{for } \check{\theta}^e \leq \theta \leq \hat{\theta} \\ q_A^{cr}(\theta) & \text{for } \theta > \hat{\theta} \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \leq \hat{\theta} \\ q_B^{cr}(\theta) & \text{for } \theta > \hat{\theta}; \end{cases}$$

- when $\bar{c} \leq c \leq c^m$

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq \check{\theta}^e \\ q^e(\theta) & \text{for } \check{\theta}^e \leq \theta \leq \theta^+ \\ q^m(\theta) & \text{for } \theta^+ \leq \theta \leq \hat{\theta} \\ q^{cr}(\theta) & \text{for } \theta > \hat{\theta} \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \leq \hat{\theta} \\ q_B^{cr}(\theta) & \text{for } \theta > \hat{\theta}; \end{cases}$$

- when $c \geq c^m$

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq \theta^m \\ q^m(\theta) & \text{for } \theta^m \leq \theta \leq \hat{\theta} \\ q_A^{cr}(\theta) & \text{for } \theta > \hat{\theta} \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \leq \hat{\theta} \\ q_B^{cr}(\theta) & \text{for } \theta > \hat{\theta}, \end{cases}$$

where θ^+ is the solution to $q^e(\theta) = q^m(\theta)$ (this is unique by H3).

The separation property mentioned in the main text implies that the solution to the dominant firm's problem is formed by appropriately joining the solutions to the maximisation program (A.1) and (A.2). By assumption H4, the former applies to high-demand states ($\theta < \hat{\theta}$), the latter to low-demand ones ($\theta > \hat{\theta}$).

The solution to problem (A.1) has already been characterised in Proposition 1. Therefore, we start by focusing on problem (A.2). This is a standard monopolistic non-linear pricing problem with a utility function $u(q_A, 0, \theta)$, except that buyers now have a type-dependent reservation utility

$$U_A^R(\theta) = \max\left[u(0, q, \theta) - cq\right].$$

Thus, the problem becomes

$$\begin{aligned} &\max_{q_A(\theta)} \int_0^1 \left[u(q_A(\theta), 0, \theta) - U(\theta)\right] f(\theta) d\theta \\ \text{s.t. } \frac{dU}{d\theta} &= u_\theta(q_A(\theta), 0, \theta) \\ U(\theta) &\geq U_A^R(\theta). \end{aligned} \tag{OA.1}$$

Its solution is given in the following.

Lemma 1 *When $c \geq c^m$, the solution to problem (OA.1) is*

$$q_A(\theta) = \begin{cases} 0 & \text{for } 0 \leq \theta \leq \breve{\theta}^m \\ q^m(\theta) & \text{for } \theta \geq \breve{\theta}^m. \end{cases}$$

When instead $c \leq c^m$, the solution is

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq \breve{\theta}^e \\ q^e(\theta) & \text{for } \breve{\theta}^e \leq \theta \leq \theta^+ \\ q^m(\theta) & \text{for } \theta \geq \theta^+. \end{cases}$$

Proof. Consider first the unconstrained problem. Clearly, its solution is $q_A^m(\theta)$.

When $c \geq c^m$, we have $U^m(\theta) \geq U_A^R(\theta)$ for all θ, so the unconstrained solution applies. To show this, notice first of all that it follows from our definitions that

$$q^e(\theta) \leq q^{\lim}(\theta),$$

with equality only when both quantities vanish. Thus, $\breve{\theta}^e$ is the largest θ such that $q^e(\theta) = q^{\lim}(\theta) = 0$. The condition $c \geq c^m$ guarantees that $\breve{\theta}^m \geq \breve{\theta}^e$. By H3, this implies that $q_A^m(\theta) > q^e(\theta)$ for $\theta > \theta^m$. Since

$$U^m(\theta) = \int_{\breve{\theta}^m}^{\theta} u_\theta(q_A^m(s), 0, s) ds$$

whereas

$$U_A^R(\theta) = \int_{\breve{\theta}^m}^{\theta} u_\theta(q^e(s), 0, s) ds$$

it follows by the sorting condition $u_{\theta q_A} \geq 0$ that the participation constraint is always satisfied.

Now suppose that $c < c^m$, so that the type-dependent participation constraint must bind for a non-empty set of states of demand. To deal with this constraint, we use the results of Jullien (2000), and in particular Proposition 3. To apply that proposition, we must show that our problem satisfies the conditions of Weak Convexity, Potential Separation, Homogeneity, and Full Participation. Weak Convexity requires that $U^m(\theta)$ is more strongly convex than $U_A^R(\theta)$. This is implied by assumption H3. Following Jullien (2000), define the modified virtual surplus function

$$s^E(g, q_A, \theta) = u(q_A, 0, \theta) - \frac{g - F(\theta)}{f(\theta)} u_\theta(q_A, 0, \theta)$$

where the "weight" $g \in [0, 1]$ accounts for the possibility that the participation constraint may bind over any subset of the support of the distribution of θ. Pointwise maximisation of the modified virtual surplus function yields

$$\ell^E(g, \theta) = \arg\max_{q_A} s^E(g, q_A, \theta).$$

Potential Separation requires that $\ell^E(g, \theta)$ is non-decreasing in θ, which is here obviously true. Homogeneity is also obvious, as it requires that $U_A^R(\theta)$ can be implemented by a continuous and non decreasing quantity; in our case, this is $q^e(\theta)$ by construction. Finally, the condition of Full Participation requires that in equilibrium buyers obtain positive quantities whenever $\theta > \check{\theta}^e$, which is obvious given that the reservation utility of these types is strictly positive.

Proposition 3 in Jullien (2000) then implies that the solution to problem (OA.1) is

$$q_A(\theta) = \begin{cases} q^e(\theta) & \text{for } \check{\theta}^e \leq \theta \leq \theta^+ \\ q^m(\theta) & \text{for } \theta \geq \theta^+, \end{cases}$$

and obviously $q_A(\theta) = 0$ for $\theta \leq \check{\theta}^e$. ■

Next, we proceed to the characterisation of the optimal switching point, $\hat{\theta}$. To begin with, observe that condition H4 guarantees that the equilibrium rent function $U(\theta)$ is steeper under non-exclusivity than under exclusivity. This implies that the solution to the hybrid optimal control problem involves a unique switch from problem (A.2) (which applies to low-demand states) to problem (A.1) (which applies to high-demand states).

The next lemma says that the switch must be from exclusive dealing to a common representation equilibrium. In other words, at the switching point the solution to problem (A.2) is given by the common representation quantities $q_A^{cr}(\theta), q_B^{cr}(\theta) > 0$. This rules out the possibility that the switch occurs for types who obtain the monopoly or limit pricing quantity of product A.

Lemma 2 *When $\theta > \hat{\theta}$, both $q_A(\theta)$ and $q_B(\theta)$ are strictly positive.*

Proof. From condition (7), it is clear that when $v_\theta(q_A^{NE}(\theta),\theta) > u_\theta(q_A^E(\theta),0,\theta)$ (which is guaranteed by H4) it must be $P_A^{NE}(q_A^{cr}(\hat{\theta})) > P_A^E(q_A^E(\hat{\theta}))$, so the dominant firm extracts more rents, at the margin, from buyers who accept non-exclusive contracts than from those who accept exclusive ones. From this, it follows immediately that that $q_A^{NE}(\hat{\theta}) > 0$ (otherwise, $P_A^{NE}(q_A^{NE}(\hat{\theta}))$ must be nil). The proof that also $q_B^{NE}(\hat{\theta}) > 0$ is equally simple. If the solution entails $q_B(\theta) = 0$, it must be either $\max[q_A^m(\theta), q^e(\theta)]$ or $q_A^{\lim}(\theta)$. In the former case, the dominant firm would obtain the same rent from buyers who accept non-exclusive contracts as from those who accept the exclusive one; in the latter, it would actually obtain less. Since we have just shown that it must obtain more, these two cases are not possible. ■

While for $\theta > \hat{\theta}$ we always have the common representation quantities, for $\theta < \hat{\theta}$ we can have either the monopoly quantity $q_A^m(\theta)$ or the quantity $q^e(\theta)$. The former case arises when $c > \bar{c}$, the latter when $c \leq \bar{c}$, where the threshold $\bar{c}$ is implicitly defined as the solution to $\hat{\theta}(c) = \theta^+(c)$ and hence satisfies $\bar{c} < c^{\lim}$.

This completes the derivation of the equilibrium quantities in all possible cases. It is then easy to check that these equilibrium quantities are implemented by the price schedules reported in the statement of the proposition. ■

2. Uniform-quadratic model

In this Section, we provide the explicit solutions for the uniform-quadratic specification (2) and check that all our regularity assumptions are satisfied in this case.

Equilibrium quantities are:

$$q^m(\theta) = \frac{2\theta - 1}{1-\gamma},$$

$$q^{\lim}(\theta) = \frac{\theta - c}{\gamma},$$

$$q^e(\theta) = \frac{\theta - c}{1-\gamma}$$

$$q_A^{cr}(\theta) = 2\theta - 1 + c\frac{\gamma}{1-2\gamma},$$

and

$$q_B^{cr}(\theta) = \theta\frac{1-2\gamma}{1-\gamma} + \frac{\gamma}{1-\gamma} - c\frac{1-\gamma}{1-2\gamma}.$$

The critical thresholds are $c^m = \frac{1}{2}$, $c^{\lim} = \frac{1-2\gamma}{2-3\gamma}$, $\breve{\theta}^m = \frac{1}{2}$, $\theta^{\lim} = \frac{c(1-\gamma)-\gamma}{1-3\gamma}$, $\breve{\theta}^e = c$, $\breve{\theta}_A = \frac{1}{2} + c\frac{\gamma}{2(1-2\gamma)}$, $\breve{\theta}_B = c\frac{(1-\gamma)^2}{(1-2\gamma)^2} - \frac{\gamma}{1-2\gamma}$, and $\theta^+ = 1 - c$.

The price schedules are

$$P^m(q) = \frac{1}{2}q - \frac{1-\gamma}{4}q^2,$$

$$P^{\lim}(q) = cq - \left(\frac{1}{2} - \gamma\right) q^2,$$

and

$$P_A^{cr}(q_A) = \frac{1 - 2\gamma + c\gamma}{2(1-\gamma)} q_A - \frac{1-2\gamma}{4(1-\gamma)} q_A^2.$$

The explicit expressions for $\hat{\theta}$, $\bar{c}$ and $\Phi_A > 0$ are complicated and are reported in a Mathematica file which is available from the authors upon request.

We now verify that conditions H1-H4 are met. Consider condition H1 first. The indirect payoff function is :

$$v(q_A, \theta) = \begin{cases} \theta q_A - \dfrac{1-\gamma}{2} q_A^2 & \text{if } q_A \geq q^{\lim}(\theta) \\ A_0 + A_1 q_A + A_2 q_A^2 & \text{if } q_A \leq q^{\lim}(\theta), \end{cases}$$

where

$$A_0 = \frac{(\theta - c)^2}{2(1-\gamma)}, \quad A_1 = \frac{c\gamma + \theta(1-2\gamma)}{1-\gamma}, \quad \text{and } A_2 = -\frac{1-2\gamma}{2(1-\gamma)}.$$

On both branches, the coefficients of the quadratic terms are negative. Furthermore,

$$\begin{aligned} \left.\frac{\partial v(q_A, \theta)}{\partial q_A}\right|_{q_A < q_A^{\lim}(\theta)} &= \frac{c\gamma + \theta(1-2\gamma)}{1-\gamma} - \frac{1-2\gamma}{(1-\gamma)} q_A^{\lim}(\theta) \\ &\geq \left.\frac{\partial v(q_A, \theta)}{\partial q_A}\right|_{q_A > q_A^{\lim}(\theta)} = \theta - (1-\gamma)\, q_A^{\lim}(\theta), \end{aligned}$$

so the function v is globally concave in q_A. Since the additional term in the virtual surplus function, $(1-\theta)v_\theta(q_A, \theta)$, is linear in q_A, $s(q_A, \theta)$ is also globally concave in q_A.

Conditions H2 and H3 are obviously met, as the quantity schedules are linear and thus can intersect at most once. Finally, to verify H4 notice that $u_\theta(q_A^E(\theta), 0, \theta) = q^m(\theta)$ whereas $v_\theta(q_A^{NE}(\theta), \theta)$ is either $\max[q^m(\theta), q^{\lim}(\theta)]$ or

$$\frac{(\theta - c)}{(1-\gamma)} + \frac{(1-2\gamma)}{1-\gamma} q_A^{cr}(\theta) > q^m(\theta).$$

3. Proof of Proposition 3

We first characterise the non-linear pricing equilibrium (Lemma 3), then the equilibrium with exclusive contracts (Lemmas 4 and 5), and finally we prove the proposition by comparing the two.

Non-linear pricing. To find the non-linear pricing equilibrium, we adapt to the asymmetric case the solution procedure proposed by Martimort and Stole (2009) for the symmetric case (i.e. $c = 0$). This is a "guess and check" procedure that starts from the conjecture that the equilibrium price schedules are (piecewise) quadratic and then verifies it by identifying the coefficients of the price schedules. It is important to stress that this procedure makes a guess on the structure of the equilibrium, but does not restrict firms to quadratic price schedules.

The drawback of the guess and check procedure is that it cannot find equilibria in which the price schedules do not conform to the guess, if there are any. However, this is not a problem for the purposes of the proof. The reason is as follows. By the separation property, with exclusive contracts the common representation quantities must be the same as in the non-linear pricing equilibrium. If there were multiple non-linear pricing equilibria, for each of these there would exist a corresponding equilibrium with exclusive contracts. If one compares equilibria with and without exclusive contracts that share the same common representation quantities, the welfare comparison is exactly the same as the welfare comparison between the exclusive and non-exclusive equilibria that we explicitly identify.

The non-linear pricing equilibrium turns out to be similar to the competitive fringe model: depending on the size of its competitive advantage c and the intensity of demand θ, the dominant firm can engage in monopoly pricing, limit pricing, or it can accommodate its rival.

The monopoly price schedule is exactly the same as in the competitive fringe model. The limit pricing schedule is similar, except that now the unit cost c is replaced by the marginal price that firm B charges for the first unit it offers, $P_B'^{cr}(0)$. As for the price schedules under common representation, they are:

$$P_A^{cr}(q) = \alpha q + c\frac{\alpha\gamma}{1-2\gamma}q - \frac{\alpha}{2}q^2; \quad P_B^{cr}(q) = cq + \alpha\left[1 - \frac{c(1-\gamma)}{1-2\gamma}\right]q - \frac{\alpha}{2}q^2, \tag{OA.2}$$

so $P_B'^{cr}(0) = c + \alpha\left[1 - \frac{c(1-\gamma)}{1-2\gamma}\right]$.

Define $\tilde{c}$ as the lowest c such that there exists at least one state of demand θ for whom $q^m(\theta) > q^{\lim}(\theta)$. The exact structure of the equilibrium is as follows:

Lemma 3 *In the duopoly model, the following is a non-linear pricing equilibrium. Firm B offers the price schedule*

$$P_B(q) = P_B^{cr}(q)$$

and:

- *when $c \leq \tilde{c}$, firm A offers the price schedule*

$$P_A(q) = \begin{cases} P^{\lim}(q) & \text{for } 0 \leq q \leq q^{\lim}(\breve{\theta}_B) \\ P_A^{cr}(q) + \text{ constant} & \text{for } q \geq q^{\lim}(\breve{\theta}_B) \end{cases}$$

  where $\breve{\theta}_B$ is implicitly defined by the condition $q_B^{cr}(\breve{\theta}_B) = 0$ and the constant guarantees the continuity of the price schedule;

- *when $c \geq \tilde{c}$, firm A offers the price schedule*

$$P_A(q) = \begin{cases} P^m(q) & \text{for } 0 \leq q \leq q^m(\theta^{\lim}) \\ P^{\lim}(q) + \ \text{constant} & \text{for } q^m(\theta^{\lim}) \leq q \leq q^{\lim}(\breve{\theta}_B) \\ P_A^{cr}(q) + \ \text{constant} & \text{for } q \geq q^{\lim}(\breve{\theta}_B), \end{cases}$$

where $\theta^{\lim}$ is implicitly defined by the condition $q^m(\theta^{\lim}) = q^{\lim}(\theta^{\lim})$ and the constants guarantee the continuity of the price schedule.

Proof. Proceeding as in the proof of Propositions 1 and 2, we start by reporting the equilibrium quantities. These are

- when $c \leq \tilde{c}$,

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq P_B'^{cr}(0) \\ q^{\lim}(\theta) & \text{for } \breve{\theta}_A \leq \theta \leq \breve{\theta}_B \\ q_A^{cr}(\theta) & \text{for } \breve{\theta}_B \leq \theta \leq 1 \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \leq \breve{\theta}_B \\ q_B^{cr}(\theta) & \text{for } \breve{\theta}_B \leq \theta \leq 1; \end{cases}$$

- when $c > \tilde{c}$,

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq \frac{1}{2} \\ q^m(\theta) & \text{for } \frac{1}{2} < \theta \leq \theta^{\lim} \\ q^{\lim}(\theta) & \text{for } \theta^{\lim} < \theta \leq \breve{\theta}_B \\ q_A^{cr}(\theta) & \text{for } \theta > \breve{\theta}_B \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \leq \breve{\theta}_B \\ q_B^{cr}(\theta) & \text{for } \breve{\theta}_B \leq \theta \leq 1 \end{cases}$$

  where

$$P_B'^{cr}(0) = \alpha + c\left[1 - \frac{\alpha(1-\gamma)}{1-2\gamma}\right].$$

Like in the competitive fringe model, $\breve{\theta}_B$ is implicitly defined by the condition $q_B^{cr}(\breve{\theta}_B) = 0$ and θ^m by the condition $q^m(\theta^m) = q^{\lim}(\theta^m)$; now, however, the explicit expressions are different as $q_B^{cr}(\breve{\theta}_B)$ and $q^{\lim}(\theta^m)$ in the duopoly model differ from the competitive fringe model (Section 2 of this online appendix). They are:

$$\breve{\theta}_B = \alpha + c\frac{(1-\gamma)(1-\alpha)}{1-2\gamma}$$

and

$$\theta^m = \frac{(1-\gamma)}{1-3\gamma}P_B'^{cr}(0) - \frac{\gamma}{1-3\gamma}.$$

To prove the lemma, we must show that the equilibrium price schedules satisfy the best response property. Given its rival's price schedule, a firm is faced with an optimal non-linear pricing problem that can be solved by invoking the Revelation Principle and thus focusing on direct mechanisms. The strategy of the proof is to show that for each firm $i = A, B$ the optimal quantity $q_i(\theta)$, given $P_{-i}(q_{-i})$, coincides with the equilibrium quantity reported above. It is then straightforward to conclude that the price schedules that support these quantities must be equilibrium price schedules.

Given $P_{-i}(q_{-i})$, firm i faces a monopolistic screening problem where type θ has an indirect payoff function

$$v^i(q_i,\theta)=\max_{q_{-i}\geq 0}\left[u(q_i,q_{-i},\theta)-P_{-i}(q_{-i})\right],$$

which accounts for any benefit he can obtain by optimally trading with its rival. Since u is quadratic and $P_{-i}(q_{-i})$ is piecewise quadratic, v_i is also piecewise quadratic. It may have kinks, but we shall show that any such kink preserves concavity, so the indirect payoff function is globally concave.

Provided that the single-crossing condition holds, firm i's problem reduces to finding a function that pointwise maximises the "indirect virtual surplus"

$$s^i(q_i,\theta)=v^i(q_i,\theta)-c_iq_i-(1-\theta)v^i_\theta,$$

where c_i is zero for $i=A$ and c for $i=B$. It is then easy to verify ex post that the maximiser $q_i(\theta)$ satisfies the monotonicity condition.

Consider firm A's best response to the equilibrium price schedule of firm B, $P_B(q_B)$. The indirect payoff function is piecewise quadratic, with two branches corresponding to the case in which $\arg\max_{q_B\geq 0}\left[u(q_A,q_B,\theta)-P_B(q_B)\right]$ is 0 or is strictly positive, and a kink between the two branches:

$$v^A(q_A,\theta)=\begin{cases}\theta q_A-\dfrac{1-\gamma}{2}q_A^2 & \text{if } q_B=0 \text{ or, equivalently, } q_A\geq q^{\lim}(\theta)\\ A_0+A_1q_A+A_2q_A^2 & \text{if } q_B>0 \text{ or, equivalently, } q_A<q^{\lim}(\theta).\end{cases}$$

The coefficients A_0, A_1 and A_2 can be calculated as

$$\begin{aligned}A_0&=\frac{\left[(\theta-c)(1-2\gamma)-\alpha(1-c(1-\gamma)-2\gamma)\right]^2}{2(1-\gamma-\alpha)(1-2\gamma)^2},\\ A_1&=\gamma\frac{c(1-2\gamma)+\alpha(1-c(1-\gamma)-2\gamma)}{(1-\gamma-\alpha)(1-2\gamma)}+\theta\frac{1-2\gamma-\alpha}{1-\gamma-\alpha}\\ A_2&=-\frac{1-2\gamma+\alpha(1-\gamma)}{2(1-\gamma-\alpha)}<0.\end{aligned}$$

On both branches of the indirect payoff function, the coefficients of the quadratic terms are negative. In addition, it can be easily checked that

$$\left.\frac{\partial v^A(q_A,\theta)}{\partial q_A}\right|_{q_A\leq q_A^{\lim}(\theta)}\geq\left.\frac{\partial v^A(q_A,\theta)}{\partial q_A}\right|_{q_A>q_A^{\lim}(\theta)},$$

so the function $v^A(q_A,\theta)$ is globally concave in q_A. It can also be checked that the sorting condition $\frac{\partial^2 v^A}{\partial\theta\partial q_A}>0$ is satisfied as

$$\frac{\partial^2 v^A}{\partial\theta\partial q_A}=\begin{cases}1 & \text{if } q_A\geq q_A^{\lim}(\theta)\\ \dfrac{1-2\gamma-\alpha}{1-\gamma-\alpha}>0 & \text{if } q_A<q_A^{\lim}(\theta).\end{cases}$$

We can therefore obtain A's best response by pointwise maximising the virtual surplus function $s^A(q_A, \theta)$. Like the indirect payoff function, the virtual surplus function is piecewise quadratic with a kink. The maximum can occur in either one of the two quadratic branches, or at the kink. To be precise:

$$\arg\max_{q_A(\theta)}[\sigma^A(q_A,\theta)] = \begin{cases} \dfrac{2\theta-1}{1-\gamma} & \text{if } \gamma < \frac{1}{3} \text{ and } \frac{1}{2} \leq \theta \leq \theta^{\lim} \\ \dfrac{\theta - P_B'^{cr}(0)}{\gamma} & \begin{array}{l} \text{if } \gamma < \frac{1}{3} \text{ and } \theta^{\lim} \leq \theta \leq \breve{\theta}_B \\ \text{or if } \gamma \geq \frac{1}{3} \text{ and } P_B'^{cr}(0) \leq \theta \leq \breve{\theta}_B \end{array} \\ \dfrac{\theta-\alpha}{1-\alpha} + \dfrac{c\gamma}{1-2\gamma} & \text{if } \theta \geq \breve{\theta}_B. \end{cases}$$

But these are precisely the monopoly, limit-pricing and common representation quantities, respectively. Note also that the case in which $\gamma < \frac{1}{3}$ and $\frac{1}{2} \leq \theta \leq \theta^m$ cannot arise if $c < \tilde{c}$. In this case, the optimum is never achieved on the upper branch of the indirect payoff function; in other words, firm A's best response never involves setting the quantity at the monopoly level. It is therefore apparent that firm A's best response is to offer precisely the equilibrium quantities. This can be achieved by offering the equilibrium price schedules which verifies that firm A's equilibrium price schedule satisfies the best response property.

Consider now firm B. The procedure is the same as for firm A, but now we must distinguish between two cases, depending on whether A's price schedule comprises the lowest, monopoly, branch or not.

Consider first the case in which there is no monopoly branch in A's price schedule. The indirect payoff function of a buyer when trading with firm B then is

$$v^B(q_B,\theta) = \begin{cases} \theta q_B - \frac{1-\gamma}{2} q_B^2 & \text{if } q_B \geq q_B^{\lim}(\theta) \\ \hat{B}_0 + \hat{B}_1 q_B + \hat{B}_2 q_B^2 & \text{if } \check{q}_B(\theta) \leq q_B < q_B^{\lim}(\theta) \\ B_0 + B_1 q_B + B_2 q_B^2 & \text{if } 0 < q_B \leq \check{q}_B(\theta) \end{cases}$$

where

$$\begin{aligned} q_B^{\lim}(\theta) &= \frac{\theta-\alpha}{\gamma} - \frac{\alpha c}{1-2\gamma} \\ \check{q}_B(\theta) &= \frac{\theta - \alpha - c(1-\alpha)}{\gamma} + \frac{\alpha c}{1-2\gamma}. \end{aligned}$$

The first branch corresponds to firm B acting as a monopolist. Along the second branch, firm B competes against firm A's limit-pricing price schedule. Clearly, neither case can occur in equilibrium. Finally, the third branch corresponds to the case in which firm A accommodates the competitive fringe. The coefficients in $v^B(q_B, \theta)$ are

$$\hat{B}_0 = \frac{(\theta-c)^2}{2\gamma}; \quad \hat{B}_1 = c; \quad \hat{B}_2 = -\frac{1-2\gamma}{2}$$

and

$$B_0 = \frac{2\theta-1}{2(1-\gamma)}; \quad B_1 = \theta - \frac{\gamma}{1-\gamma}; \quad B_2 = -\frac{1-\gamma}{2}.$$

All branches are concave. Global concavity can be checked by comparing the left and right derivatives of $v^B(q_B, \theta)$ at the kinks, as we did for firm A. The sorting condition can also be checked as we did for firm A. We can therefore find B's best response by pointwise maximisation of the virtual surplus function.

It is easy to verify that there is never an interior maximum on the upper or intermediate branch of the virtual surplus function. This is equivalent to the intuitive result that (the less efficient) firm B is active only when firm A supplies the common representation quantity $q_A^{cr}(\theta)$. Pointwise maximisation of the relevant branch of virtual surplus function then leads to

$$\arg\max[\sigma^B(q_B, \theta)] = \frac{\theta - \alpha}{1 - \alpha} - c\frac{1 - \gamma}{1 - 2\gamma}.$$

This coincides with $q_B^{cr}(\theta)$, thereby confirming that the equilibrium price schedule $P_B(q_B)$ is indeed its best response to firm A's strategy.

The case where firm A's price schedule comprises also the monopoly branch is similar. The indirect payoff function $v^B(q_B, \theta)$, and hence the virtual surplus $s^B(q_B, \theta)$, now comprise four branches (all quadratic). The equation of the fourth branch, which corresponds to $0 < q_A < q^m(\theta)$, is

$$v^B(q_B, \theta) = \tilde{B}_0 + \tilde{B}_1 q_B + \tilde{B}_2 q_B^2$$

where

$$\tilde{B}_0 = \frac{(2\theta - 1)^2}{4(1 - \gamma)}; \quad \tilde{B}_1 = \frac{\theta + \gamma(1 - 3\gamma)}{1 - \gamma}; \quad \tilde{B}_2 = -\frac{1 - \gamma(2 + \gamma)}{2(1 - \gamma)}.$$

However, it turns out that the optimum still lies on the same branch as before and that it therefore entails a quantity equal to $q_B^{cr}(\theta)$. This observation completes the proof of the lemma. ■

Exclusive contracts. With exclusive contracts, in the duopoly model there may be scope for multiple equilibria. Generally speaking, multiple equilibria are endemic in models in which firms interact strategically by making contractual offers that are destined not to be accepted in equilibrium. For example, under duopoly there are always equilibria in which both firms charge exorbitant nonexclusive prices, forcing buyers into exclusive dealing agreements. However, such equilibria are ruled out since we focus on weekly undominated strategies.

When the competitive advantage is large (to be precise, the threshold is $\check{c} = \frac{2(1-2\gamma)}{5(1-\gamma)+\sqrt{1-2\gamma+9\gamma^2}}$ as we will show), this is in fact the only source of multiplicity. In other words, there is only one equilibrium in undominated strategies. When the competitive advantage is small, on the other hand, firms face non obvious coordination problems that enlarge the scope for multiplicity, as we shall see below.

We start the analysis from the case $c \geq \check{c}$, where for any given equilibrium common representation schedules, the equilibrium with exclusive contracts is unique. Such a unique equilibrium is similar to the equilibrium of the competitive fringe model in the large competitive advantage case. Like in that model, the non-exclusive tariffs coincide with the common representation tariffs that arise in the non-linear pricing equilibrium, except for constant terms. However, these constant terms, which we denote by Φ_A and Φ_B, are now determined as follows.

Let $\hat{\theta}$ denote the critical buyer $\hat{\theta}$ who is just indifferent between exclusive and non-exclusive contracts. For this buyer, the following condition must hold

$$u(q_A^E(\hat{\theta}), 0, \hat{\theta}) - P_A^E(q_A^E(\hat{\theta})) = u(q_A^{cr}(\hat{\theta}), q_B^{cr}(\hat{\theta}), \hat{\theta}) - P_A^{cr}(q_A^{cr}(\hat{\theta})) - P_B^{cr}(q_B^{cr}(\hat{\theta}) - \Phi_A - \Phi_B \tag{OA.3}$$

which is the counterpart of condition (6). Clearly, an increase in Φ_i will increase $\hat{\theta}$.

Intuitively, when choosing Φ_A and Φ_B, both firms are trading off market share and profitability. Consider, for instance, firm B. Since its exclusive contracts are not accepted (and in any case would not be profitable), it must try to induce more high-demand buyers, who value product variety more highly, to reject the exclusive contracts offered by firm A and buy both products. To get such buyers to purchase both products, firm B must lower its non-exclusive prices by adding a negative term (a lump-sum subsidy) to the tariff $P_B^{cr}(q)$. Firm A, by contrast, will add a fixed fee to the tariff $P_A^{cr}(q)$. The fixed fee is sufficiently large that the dominant firm earns more, at the margin, from buyers who choose common representation than from those who choose exclusive dealing (as in the competitive fringe model).

More formally, firm A's profit is

$$\int_{\tilde{\theta}}^{\hat{\theta}} P_A^E(q_A^E(\theta))d\theta + \int_{\hat{\theta}}^{1} \left[P_A^{cr}(q_A^{cr}(\theta)) + \Phi_A\right] d\theta,$$

and firm B's is

$$\int_{\hat{\theta}}^{1} \left[P_B^{cr}(q_B^{cr}(\theta)) - cq_B^{cr}(\theta) + \Phi_B\right] d\theta.$$

Since $\hat{\theta}$ is determined by (OA.3), the equilibrium conditions for Φ_A and Φ_B are:

$$\frac{P_A^{cr}(q_A^{cr}(\hat{\theta})) + \Phi_A - P_A^E(q_A^E(\hat{\theta}))}{q_A^{cr}(\hat{\theta}) + q_B^{cr}(\hat{\theta}) - q_A^E(\hat{\theta})} = 1 - \hat{\theta}, \tag{OA.4}$$

$$\frac{P_B^{cr}(q_B^{cr}(\hat{\theta})) + \Phi_B - cq_B^{cr}(\hat{\theta})}{q_A^{cr}(\hat{\theta}) + q_B^{cr}(\hat{\theta}) - q_A^E(\hat{\theta})} = 1 - \hat{\theta}. \tag{OA.5}$$

Conditions (OA.4) and (OA.5) are the duopoly counterpart of condition (7) in the competitive fringe model. The economic intuition is similar. It can be confirmed that in equilibrium $\Phi_A > 0$, $\Phi_B < 0$ and $\Phi_A + \Phi_B > 0$.

We are now ready to provide the characterisation of the equilibrium when exclusive contracts are permitted and the dominant firm's competitive advantage is large.

Lemma 4 *The following is an equilibrium in the duopoly model when firms can use exclusive contracts and the dominant firm's competitive advantage is large, i.e.* $c > \check{c}$.[1]

- *When* $\check{c} < c < c^m$ *the two firms offer the following exclusive price schedules*

$$\begin{array}{rl} P_B^E(q) = & cq \\ P_A^E(q) = & \left\{ \begin{array}{lll} cq & & \text{for } q \leq q^e(\theta^m) \\ P_A^m(q) + & \text{constant} & \text{for } q > q^e(\theta^m) \end{array} \right. \end{array}$$

  where θ^m *is such that* $q^e(\theta^m) = q_A^m(\theta^m)$ *and the constant guarantees the continuity of the price schedule, and the following non-exclusive price schedules*

$$\begin{array}{rll} P_A^{NE}(q) = & P_A^{cr}(q) + \Phi_A & \text{for } q \geq q_A^{cr}(\hat{\theta}) \\ P_B^{NE}(q) = & P_B^{cr}(q) + \Phi_B & \text{for } q \geq q_B^{cr}(\hat{\theta}) \end{array}$$

  where $\hat{\theta}$, Φ_A *and* Φ_B *are the solution to system (OA.3)-(OA.5).*

- *When* $c \geq c^m$ *the two firms offer the following price schedules*

$$P_A^E(q) = P^m(q)$$

  (firm B may not offer any exclusive contract at all), and

$$\begin{array}{rll} P_A^{NE}(q) = & P_A^{cr}(q) + \Phi_A & \text{for } q \geq q_A^{cr}(\hat{\theta}) \\ P_B^{NE}(q) = & P_B^{cr}(q) + \Phi_B & \text{for } q \geq q_B^{cr}(\hat{\theta}) \end{array}$$

  where $\hat{\theta}$, Φ_A *and* Φ_B *are defined as in the previous case.*

To avoid repetitions, it is convenient to prove this lemma after Lemma 5 below.

Lemma 5 applies to the case in which the dominant firm's competitive advantage is small ($c \leq \check{c}$). In this case, there is a multiplicity of equilibria that arises because the firms may or may not succeed in coordinating their strategies.

To understand the coordination problems that the firms face, consider the outcome of the competition for exclusives: firm B prices at cost, and firm A just undercuts it. Clearly, this is always a possible equilibrium. When the competitive advantage is small, however, both firms can obtain larger profits. This requires that the firms lower their non-exclusive prices in coordinated fashion, inducing some buyers to purchase both products. This move allows firms to extract the buyers' preference for variety.

Note that this coordination cannot take place when the competitive advantage is large. The reason for this is that the payoff function u by itself always

[1] As we have already noted, if there were different equilibrium price schedules under common representation, $P_i^{cr}(q)$, for each of them there would be corresponding equilibria with exclusive contracts with the same structure as that described in Lemma 4. The same remark applies also to Lemma 5 below.

entails a preference for variety, but the fact that $c > 0$ means that exclusion may be efficient. In particular, when demand is low efficiency requires that only good A must be produced. In other words, there is room for extracting the preference for variety only if the competitive advantage is not too large. The condition is precisely $c < \breve{c}$.

Assuming, then, that $c < \breve{c}$, note that if firms manage to coordinate their non-exclusive prices as described above, a new opportunity of coordination arises. Since certain exclusive contracts will no longer be accepted in equilibrium, firms have no longer an incentive to undercut one another's exclusive prices; therefore, they can also increase exclusive prices so as to reduce the intensity of competition.

Therefore, multiple equilibria arise when $c < \breve{c}$. Fortunately, however, the critical threshold c^* of Proposition 3 is greater than $\breve{c}$, as we shall confirm below. Thus, in order to prove Proposition 3 it is not necessary to provide a complete characterisation of the set of equilibria for $c < \breve{c}$. Rather, it suffices to focus on the "most cooperative" equilibrium, where prices and profits are largest (given that the firms are actually playing a non-cooperative game), and therefore welfare is lowest. Since even such most cooperative equilibrium results in a level of social welfare which is greater than in the non-linear pricing equilibrium, exclusive contracts are unambiguously pro-competitive when $c < \breve{c} < c^*$.

In the most cooperative equilibrium, the exclusive and non-exclusive price schedules must be determined simultaneously. The conditions that must be satisfied are the following. Let $U^E(\theta)$ be the (type-dependent) reservation utility that buyer θ could obtain by choosing his most preferred exclusive contract. To extract the buyer's preference for variety, the firms must introduce non-exclusive price schedules implicitly defined by the condition:

$$\max_{q_A, q_B} \left[u(q_A, q_B, \theta) - P_A^{NE}(q_A) - P_B^{NE}(q_B) \right] = U^E(\theta). \tag{OA.6}$$

(To avoid issue of equilibrium existence, we now assume that when buyers are indifferent in monetary terms, they prefer to purchase both goods.) These price schedules apply to low-demand buyers; high-demand buyers will actually obtain more than $U^E(\theta)$ simply thanks to the competition in non-exclusive contracts. Notice that equation (OA.6) does not pin down $P_A^{NE}(q_A)$ and $P_B^{NE}(q_B)$ uniquely. This reflects the fact that the preference for variety can be split between the two firms in different ways. All that matters is that the total payment requested by the firms does not exceed what the buyer is willing to pay in order to purchase both goods. Since we look for the equilibrium in which firms' profits are largest, we shall focus on the case in which the firms maximise the rents that they extract. This requires maximisation of the total surplus $u(q_A, q_B, \theta) - cq_B$, subject to the constraint that buyers must (exactly) obtain $U^E(\theta)$. Using the envelope theorem applied to the quadratic payoff specification (2), the constraint can be rewritten as

$$q_A(\theta) + q_B(\theta) = q^E(\theta), \tag{OA.7}$$

where $q^E(\theta)$ is the optimal quantity under exclusivity. Notice that $q^E(\theta)$ depends on what exclusive prices are sustainable in the most cooperative equilibrium and hence must be determined jointly with all other variables.

Generally speaking, the more efficient firm must produce more than the less efficient one. In particular, the problem of total-surplus maximisation may have a corner solution in which some low-demand types must buy good A only. In this case, exclusive contracts must be accepted in equilibrium by those types, and so competition in utility space implies that exclusive prices must fall to c. Therefore, for low-demand types $q_A(\theta)$ must coincide with $q^e(\theta)$, and $q_B(\theta)$ must vanish.

When instead the total-surplus maximisation problem has an interior solution, which is

$$q_A(\theta) = \frac{1}{2}q^E(\theta) + \frac{c}{2(1-2\gamma)}; \quad q_B(\theta) = \frac{1}{2}q^E(\theta) - \frac{c}{2(1-2\gamma)}, \tag{OA.8}$$

buyers purchase both products. The corresponding exclusive contracts are not actually accepted in equilibrium, and so there may be room for coordinating the exclusive prices too. The reason for this is that exclusive contracts affect the equilibrium outcome even if they are not accepted. The less aggressively firms bid for exclusivity, the lower the buyer's payoff under exclusive dealing, and hence the greater the payments firms can obtain for non-exclusive contracts. Thus, raising the exclusive prices is good for the firms' profits.

Let us denote by an upper bar the highest exclusive prices that can be sustained in a non-cooperative equilibrium. To find them, we can assume, with no loss of generality, that both firms offer the same exclusive price schedule $\bar{P}^E(q)$.[2] By construction, low-type buyers must be just indifferent between exclusive and non-exclusive contracts (equation (OA.6)). Thus, any arbitrarily small discount would trigger a switch to an exclusive contract. In equilibrium, no such deviation can be profitable. This implies the following no undercutting conditions:

$$\begin{aligned} P^E(q^E(\theta)) &\le P_A^{NE}(q_A^{cr}(\theta)); \\ P^E(q^E(\theta)) - cq^E(\theta) &\le P_B^{NE}(q_B^{cr}(\theta)) - cq_B^{cr}(\theta), \end{aligned} \tag{OA.9}$$

which in the most cooperative equilibrium must hold as equalities.

The most cooperative equilibrium is found by solving the system of equations (OA.6)-(OA.9). Specifically, denote by $\bar{q}^E(\theta)$ the optimal quantity associated with the exclusive prices $\bar{P}^E(q)$, and by $\bar{q}_i^{cr}(\theta)$ the values of $q_i(\theta)$ given by (OA.8) when $q^E(\theta) = \bar{q}^E(\theta)$. Rewrite (OA.6) as

$$u(\bar{q}_A^{cr}(\theta), \bar{q}_B^{cr}(\theta), \theta) - \bar{P}_A^{NE}\left(\bar{q}_A^{cr}(\theta)\right) - \bar{P}_B^{NE}\left(\bar{q}_B^{cr}(\theta)\right) = u(0, \bar{q}^E(\theta), \theta) - \bar{P}^E\left(\bar{q}^E(\theta)\right)$$

and use the no-undercutting conditions (OA.9) to get

$$\bar{P}^E\left(q^E(\theta)\right) = \left[u(\bar{q}_A^{cr}(\theta), \bar{q}_B^{cr}(\theta), \theta) - u(0, \bar{q}^E(\theta), \theta)\right] + c\left[\bar{q}^E(\theta) - \bar{q}_B^{cr}(\theta)\right].$$

[2] To prove that this does not entail any loss of generality, suppose to the contrary that one firm offered more attractive exclusive contracts than its rival. Since these contracts are not accepted in equilibrium, the firm could increase its exclusive prices without losing any profits on its exclusive contracts. In fact, the buyers' reservation utility would decrease, allowing both firms to increase their profits from non-exclusive contracts.

The term inside the first square brackets on right-hand side can be interpreted as the preference for variety, while the term inside the second square bracket is the cost saving. Using (OA.8), we finally get

$$\bar{P}^E(q) = \frac{c^2}{2(1-2\gamma)} + \frac{c}{2}q + \frac{1-2\gamma}{4}q^2,$$

and

$$\bar{P}^{cr}_A(q_A) = -cq + (1-2\gamma)q^2 + cq^e(\hat{\theta}); \quad \bar{P}^{cr}_B(q_B) = 2cq + (1-2\gamma)q^2,$$

where $\hat{\theta}$ is now the solution to $q^e(\hat{\theta}) = \bar{q}^{cr}_A(\hat{\theta})$ and the constant term in $\bar{P}^{cr}_A(q_A)$ guarantees smooth pasting from exclusive to non-exclusive contracts. The corresponding quantities are

$$\bar{q}^E(\theta) = \frac{2\theta - c}{3-4\gamma},$$

and

$$\bar{q}^{cr}_A(\theta) = \frac{2\theta - c}{2(3-4\gamma)} + \frac{c}{2(1-2\gamma)}; \quad \bar{q}^{cr}_B(\theta) = \frac{2\theta - c}{2(3-4\gamma)} - \frac{c}{2(1-2\gamma)}.$$

We are now ready to provide the characterisation of the most cooperative equilibrium.

Lemma 5 *Suppose that the dominant firm's competitive advantage is small: $c \leq \check{c}$. Then, in the duopoly model the most cooperative equilibrium with exclusive contracts is as follows. Both firms offer the exclusive price schedules*

$$P^E_A(q) = P^E_B(q) = \begin{cases} cq & \text{for } q \leq q^e(\hat{\theta}) \\ \bar{P}^E(q) & \text{for } q > q^e(\hat{\theta}) \end{cases}$$

with firm A slightly undercutting firm B, though. Furthermore:

$$P^{NE}_A(q) = \begin{cases} \bar{P}^{cr}_A(q) & \text{for } q \leq \bar{q}^{cr}_A(\bar{\theta}) \\ P^{cr}_A(q) + \text{constant} & \text{for } q \geq \bar{q}^{cr}_A(\bar{\theta}) \end{cases}$$
$$P^{NE}_B(q) = \begin{cases} \bar{P}^{cr}_B(q) & \text{for } q \leq \bar{q}^{cr}_B(\bar{\theta}) \\ P^{cr}_B(q) + \text{constant} & \text{for } q \geq \bar{q}^{cr}_B(\bar{\theta}) \end{cases}$$

where $\hat{\theta}$ is the solution to $q^e(\hat{\theta}) = \bar{q}^{cr}_A(\hat{\theta})$ and $\bar{\theta}$ the solution to $\bar{q}^{cr}_A(\bar{\theta}) = q^{cr}_A(\bar{\theta})$ (and to $\bar{q}^{cr}_B(\bar{\theta}) = q^{cr}_B(\bar{\theta})$), and the constants guarantee the continuity of the price schedules.

Proof. The equilibrium quantities are:

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq c \\ q^e(\theta) & \text{for } c \leq \theta \leq \hat{\theta} \\ \bar{q}^{cr}_A(\theta) & \text{for } \hat{\theta} \leq \theta \leq \bar{\theta} \\ q^{cr}_A(\theta) & \text{for } \bar{\theta} \leq \theta \leq 1 \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \leq \hat{\theta} \\ \bar{q}^{cr}_B(\theta) & \text{for } \hat{\theta} \leq \theta \leq \bar{\theta} \\ q^{cr}_B(\theta) & \text{for } \bar{\theta} \leq \theta \leq 1, \end{cases}$$

where $\hat{\theta}$ and $\bar{\theta}$, which are defined in the text of the lemma, are given by

$$\begin{aligned} \hat{\theta} &= \frac{c(2-3\gamma)}{1-2\gamma}, \\ \bar{\theta} &= \frac{c(1-2\gamma)+\alpha[3-c-2(2-c)\gamma]}{\alpha+2(1-2\gamma)}. \end{aligned}$$

The claim that this is the most cooperative equilibrium has been justified above. Here, we just verify that this is indeed an equilibrium of the game. The logic of the proof is the same as for Lemma 3. We must show that for each firm the equilibrium price schedules satisfy the best response property. When calculating the best response, we take $\{P_{-i}^E(q), P_{-i}^{NE}(q)\}$ as given. Hence, we can invoke the Revelation Principle and focus on direct mechanisms. Proceeding this way, we must show that for each firm $i = A, B$ the optimal quantities $q_i(\theta)$ coincide with the equilibrium quantities reported above. It is then straightforward to conclude that the price schedules $\left\{P_i^E(q), P_i^{NE}(q)\right\}$ that support these quantities must be equilibrium price schedules.

Given its rival's exclusive and non exclusive price schedules, a firm must solve a monopolistic screening problem in which the buyer has an indirect payoff function

$$v^i(q_i,\theta) = \max_{q_{-i}\geq 0}\left[u\left(q_i, q_{-i}, \theta\right) - P_{-i}^{NE}(q_{-i})\right],$$

and a reservation utility

$$U_i^R(\theta) = \max_{q_{-i}}\left[u(0, q_{-i}, \theta) - P_{-i}^E(q_{-i})\right].$$

Since firm i can impose exclusivity clauses, it must solve a "hybrid" optimal control problem in which the two control systems are

$$\begin{aligned} &\max_{q_i}\int\left[v^i(q_i,\theta) - U(\theta) - c_i q_i\right]d\theta \\ \text{s.t. } \frac{dU}{d\theta} &= v_\theta^i(q_i,\theta) \\ U(\theta) &\geq U_i^R(\theta) \end{aligned} \tag{OA.10}$$

if $q_{-i}(\theta) > 0$, and

$$\begin{aligned} &\max_{q_i}\int\left[u(q_i,0,\theta) - U(\theta) - c_i q_i\right]d\theta \\ \text{s.t. } \frac{dU}{d\theta} &= u_\theta(q_i,0,\theta) \\ U(\theta) &\geq U_i^R(\theta) \end{aligned} \tag{OA.11}$$

if $q_{-i}(\theta) = 0$. In both cases, $q_i(\theta)$ must be non-decreasing.

Problem (OA.11) is relevant only for the dominant firm. When it sets $q_B(\theta) = 0$, noting that problem (OA.11) coincides with problem (A.2) in the

Appendix, we can apply Lemma 1 and conclude that

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq c \\ q^e(\theta) & \text{for } c \leq \theta \leq 1-c \\ q_A^m(\theta) & \text{for } \theta \geq 1-c. \end{cases}$$

It is then easy to verify that $\hat{\theta}$ is now lower than $1-c$, so the only relevant part of the solution is $q^e(\theta)$.

Consider now problem (OA.10). Several properties of the solution to this problem must hold for both firms. By construction, the indirect payoff functions $v^i(q_i, \theta)$ are almost everywhere differentiable. At any point where the derivatives exist, by the envelope theorem we have

$$v_\theta^i(q_i, \theta) = q_i + \tilde{q}_{-i}(q_i, \theta),$$

where

$$\tilde{q}_{-i}(q_i, \theta) = \arg\max_{q_{-i}>0} \left[u\left(q_i, q_{-i}, \theta\right) - P_{-i}^{NE}(q_{-i})\right]$$

Generally speaking, the indirect payoff functions $v^i(q_i, \theta)$ have two branches, according to whether $\tilde{q}_{-i}(q_i, \theta) \leq \bar{q}_{-i}(\bar{\theta})$ or $\tilde{q}_{-i}(q_i, \theta) \geq \bar{q}_{-i}(\bar{\theta})$ respectively. When $\tilde{q}_{-i}(q_i, \theta) \leq \bar{q}_{-i}(\bar{\theta})$, we have $P_{-i}^{NE}(q_{-i}) = \bar{P}_{-i}^{cr}(q_{-i})$. When $\tilde{q}_{-i}(q_i, \theta) \geq \bar{q}_{-i}(\bar{\theta})$, we have $P_{-i}^{NE}(q_{-i}) = P_{-i}^{cr}(q_{-i})$ (plus a constant).

The indirect payoff functions $v^i(q_i, \theta)$ are continuous, a.e. differentiable, and satisfy $v_{\theta q_i}^i(q_i, \theta) > 0$. Continuity and a.e. differentiability follow directly from the definition of $v^i(q_i, \theta)$. To prove the sorting condition, observe that

$$v_{\theta q_i}^i(q_i, \theta) = 1 - \gamma \frac{\partial \tilde{q}_{-i}(q_i, \theta)}{\partial q_i} \geq 0.$$

Consider the two branches of the indirect payoff function in turn. When $\tilde{q}_{-i}(q_i, \theta) \leq \bar{q}_{-i}(\bar{\theta})$, then

$$v_{\theta q_i}^i(q_i, \theta) = 1 + \frac{\partial \tilde{q}_{-i}(q_i, \theta)}{\partial \theta}(-\gamma)^2 = \frac{3-6\gamma}{3-5\gamma} > 0.$$

When instead $\tilde{q}_{-i}(q_i, \theta) \geq \bar{q}_{-i}(\bar{\theta})$, the sorting condition is immediately verified since

$$v_{\theta q_i}(q_i, \theta) = \frac{1-\alpha-2\gamma}{1-\alpha-\gamma} \geq 0.$$

Because of the type-dependent participation constraint, following Jullien (2000) we define the modified virtual surplus function:

$$\sigma^i(g, q_i, \theta) = v^i(q_i, \theta) - (g-\theta)\, v_\theta^i(q_i, \theta)$$

where the "weight" $g \in [0,1]$ accounts for the possibility that the participation constraint may bind for a whole set of types. Let

$$\ell_i(g, \theta) = \arg\max_{q_i \geq 0} \sigma^i(g, q_i, \theta)$$

be the maximiser of the modified virtual surplus function. This solution is still in implicit form, as it depends on the value of g, which is still to be determined. This can be done by exploiting Proposition 5.5 of Jullien (2000).

To apply that Proposition, we first prove the following lemma.

Lemma 6 *Problem (OA.10) satisfies the conditions of Potential Separation, Homogeneity and Weak Convexity.*

Proof. Potential Separation requires that $\ell_i(g,\theta)$ is non-decreasing in θ. This follows from the fact that the modified virtual surplus function has increasing differences. To show this, consider each branch of the indirect payoff function separately. First, when $\tilde{q}_{-i}(q_i,\theta) \leq \bar{q}_{-i}(\bar{\theta})$ we have

$$\sigma^i_{q_i\theta}(q_i,\theta) = v^i_{q_i\theta}(q_i,\theta) - \left[1 + \frac{\partial \tilde{q}_{-i}(q_i,\theta)}{\partial q_i}\right] \frac{d}{d\theta}(g-\theta).$$

The first term is positive, as we have just shown. The second term is positive because $\frac{d}{d\theta}(g-\theta) < 0$ and

$$1 + \frac{\partial \tilde{q}_{-i}}{\partial q_i} = \frac{1-2\gamma}{3-5\gamma} > 0.$$

Second, when $\tilde{q}_{-i}(q_i,\theta) \geq \bar{q}_{-i}(\bar{\theta})$ the indirect payoff function coincides, *modulo* a constant, with the one arising in the equilibrium with non-linear pricing. In this case, it is immediate to show that $\sigma^i_{q_i\theta}(q_i,\theta) > 0$. This completes the proof that problem (OA.10) satisfies the condition of Potential Separation.

Homogeneity requires that $U_i^R(\theta)$ can be implemented by a continuous and non decreasing quantity. This is obvious, since $U_i^R(\theta)$ is implemented by $q^E(\theta)$, where $q^E(\theta)$ is the optimal buyer's quantity given the exclusive price schedule $P^E_{-i}(q)$:

$$q^E(\theta) = \begin{cases} q^e(\theta) & \text{if } \theta \leq \hat{\theta} \\ \bar{q}^E(\theta) & \text{if } \theta > \hat{\theta}. \end{cases}$$

To prove Weak Convexity, we first show that $\ell_i(0,\theta)+\tilde{q}_{-i}(\ell_i(0,\theta),\theta) \geq q^E(\theta)$ for all $\theta \in [0,1]$. By definition,

$$\ell_i(0,\theta) = \arg\max_{q_i} \left[v^i(q_i,\theta) + \theta v^i_\theta(q_i,\theta)\right].$$

Thus, $\ell_i(0,\theta)$ is implicitly defined by the first order condition

$$v^i_{q_i}(q_i,\theta) + \theta v^i_{\theta q_i}(q_i,\theta) = 0.$$

Since $v^i_{\theta q_i}(q_i,\theta) > 0$, this implies that $v^i_{q_i}(q_i,\theta) < 0$, or $u_{q_i}(q_i,\tilde{q}_{-i}(q_i,\theta),\theta) < 0$. In other words, $\ell_i(0,\theta)$ exceeds the satiation quantity $u_{q_i}(q_i,\tilde{q}_{-i}(q_i,\theta),\theta) = 0$. The quantity $q^E(\theta)$, on the contrary, is lower than the satiation quantity. It follows that $\ell_i(0,\theta) + \tilde{q}_{-i}(\ell_i(0,\theta),\theta) \geq q^E(\theta)$.

In addition, Weak Convexity requires that the curve $q^E(\theta)$ intersects the curve $\ell_i(1,\theta)+\tilde{q}_{-i}(\ell_i(1,\theta),\theta) = q_A^{cr}(\theta)+q_B^{cr}(\theta)$ from above. Noting that $\ell_i(1,\theta) = q_i^{cr}(\theta)$, this is indeed the case as

$$\frac{d[q_A^{cr}(\theta) + q_B^{cr}(\theta)]}{d\theta} \geq \frac{dq^E(\theta)}{d\theta},$$

irrespective of whether $q^E(\theta)$ is $q^e(\theta)$ or $\bar{q}^E(\theta)$. This finally proves Weak Convexity and hence the lemma. ■

With these preliminary results at hand, let us now consider the dominant firm's problem. The solution when $q_B(\theta) = 0$ has been already characterised. If $q_B(\theta) > 0$, Proposition 5.5 in Jullien (2000) guarantees that generally speaking the solution partitions the set of states of demand into three sets: states where buyers are excluded, states where buyers obtain their reservation utility $U_A^R(\theta)$, and states where buyers' payoff is strictly greater than $U_A^R(\theta)$. Clearly, the first set is always empty: if $q_B(\theta) > 0$, we always have $q_A(\theta) > 0$.

Next consider the second set of states of demand. When the participation constraint binds, firm A can guarantee the buyer his reservation utility $U_A^R(\theta)$ in two ways. First, it can offer an exclusive price schedule that just matches that of firm B. Alternatively, it can implement, *via* non-exclusive prices, the quantities that satisfy the condition

$$\bar{q}_A^{cr}(\theta) + \bar{q}_B^{cr}(\theta) = \bar{q}^E(\theta),$$

which by the Envelope Theorem guarantees that the participation constraint is met as an equality. The maximum payment that firm A can request for $\bar{q}_A^{cr}(\theta)$ is

$$\bar{P}_A^{cr}\left(\bar{q}_A^{cr}(\theta)\right) = -c\bar{q}_A^{cr}(\theta) + (1-2\gamma)\left[\bar{q}_A^{cr}(\theta)\right]^2 + cq^e(\hat{\theta}).$$

The second strategy is at least as profitable as the first one if

$$\bar{P}_A^{cr}\left(\bar{q}_A^{cr}(\theta)\right) \geq \bar{P}^E(\bar{q}^E(\theta)),$$

which is precisely the no-undercutting condition (OA.9), which holds by construction. This shows that offering $\bar{P}_A^{cr}(q_A)$ is indeed a best response for firm A when the participation constraint is binding.

Finally, when the participation constraint does not bind, the solution to firm A's program is simply obtained by setting $g = 1$. Assume that $\ell_A(1,\theta) \geq \bar{q}_A^{cr}(\bar{\theta})$ when $\theta > \hat{\theta}$ (this will be proven shortly). Since the modified virtual surplus function $\sigma_A(1, q_A, \theta)$ is exactly the same as in the non-linear pricing equilibrium, *modulo* a constant, the maximisers of the virtual surplus functions must coincide and the optimal quantity is

$$\ell_A(1,\theta) = q_A^{cr}(\theta).$$

Finally, the cutoff $\bar{\theta}$ is implicitly given by the condition

$$\bar{q}^E(\bar{\theta}) = \ell_A(1,\bar{\theta}) + \tilde{q}_B(\ell_A(1,\bar{\theta}),\bar{\theta}).$$

This also establishes that $\ell_A(1,\theta) \geq \bar{q}_A^{cr}(\bar{\theta})$ when $\theta > \hat{\theta}$.

To complete the verification of the best response property for firm A, it remains to consider the switch from exclusive to non-exclusive contracts. By the no-deviation condition (OA.9), which in the most cooperative equilibrium holds as an equality, firm A is just indifferent between imposing an exclusivity clause or not for $\theta \leq \bar{\theta}$. Exclusive dealing arises just when $\bar{q}_B^{cr}(\theta) \leq 0$, which is equivalent to $\theta \leq \hat{\theta}$. Because firm A is indifferent between the exclusive and non-exclusive regimes, at the switching point a smooth-pasting condition must now hold, which implies that aggregate quantities must be continuous, and hence that $P_A^{NE}(\bar{q}_A^{cr}(\hat{\theta})) = P_A^E(\bar{q}_A^E(\hat{\theta}))$.

The problem faced by firm B is similar, except that firm B can never make a profit by selling under an exclusivity clause. Thus, we can focus on problem (OA.10). Proceeding as for firm A, one can show that the optimal quantity is $\bar{q}_B^{cr}(\theta)$ when the participation constraint $U(\theta) \geq U_A^R(\theta)$ is binding, and $q_B^{cr}(\theta)$ when it is not.

These arguments complete the proof that the solution to the problem of firm i coincides with $q_i(\theta)$ as shown in the text of the proposition. By construction, this solution can be implemented by firm i using the equilibrium price schedules $\left(P_i^E(q_i), P_i^{NE}(q_i)\right)$.

Clearly, the solution is well defined when the three intervals $[c,\hat{\theta})$, $[\hat{\theta},\bar{\theta}]$ and $(\bar{\theta},1]$ are non-empty. This requires $c \leq \hat{\theta}$, $\hat{\theta} \leq \bar{\theta}$ and $\bar{\theta} \leq 1$. It is immediate to show that the first and the last of these inequality always hold. Thus, the solution is well defined if and only if $\hat{\theta} \leq \bar{\theta}$, which is equivalent to

$$c \leq \breve{c} \equiv \frac{2(1-2\gamma)}{5(1-\gamma)+\sqrt{1-2\gamma+9\gamma^2}}. \quad \blacksquare$$

We can now proceed to the proof of Lemma 4.

Proof. As usual, we start by reporting the equilibrium quantities, which are:

- when $\breve{c} \leq c \leq c^m$,

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq c \\ q^e(\theta) & \text{for } c \leq \theta \leq \theta^m \\ q_A^m(\theta) & \text{for } \theta^m \leq \theta \leq \hat{\theta} \\ q_A^{cr}(\theta) & \text{for } \hat{\theta} \leq \theta \leq 1 \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \leq \hat{\theta} \\ q_B^{cr}(\theta) & \text{for } \hat{\theta} \leq \theta \leq 1; \end{cases}$$

- when $c > c^m$

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq \frac{1}{2} \\ q_A^m(\theta) & \text{for } \frac{1}{2} \leq \theta \leq \hat{\theta} \\ q_A^{cr}(\theta) & \text{for } \hat{\theta} \leq \theta \leq 1 \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \leq \hat{\theta} \\ q_B^{cr}(\theta) & \text{for } \hat{\theta} \leq \theta \leq 1. \end{cases}$$

The strategy of the proof is the same as for Lemma 5. Many of the arguments are indeed the same as in previous proofs and thus need not be repeated here. In particular, notice that:

- first, when $c > \check{c}$, there is no longer any scope for coordinating exclusive prices (this was shown in the proof of Lemma 5). Hence, firm B always sets exclusive prices at the competitive level $P_B^E(q_B) = cq_B$. This implies that when firm A imposes an exclusivity clause, the buyers' reservation utility is exactly the same as in the competitive fringe model. It follows that the solution to problem (A.2) is still given by Lemma 1;
- second, without exclusivity the problems that are faced by the firms are exactly the same as in the proof of Lemma 3 when the participation constraint does not bind.

These remarks imply that Lemma 4 can be proved simply by combining arguments already presented in the proofs of Lemma 3 and Lemma 5. The only difference is that now the switch from the exclusive to the non-exclusive regime is the result of the interaction between the pricing choices of firm A and firm B. This point, however, has already been discussed above, where we have shown that the equilibrium switching point and the constants Φ_A and Φ_B must satisfy conditions (OA.3)-(OA.5). The explicit expressions for Φ_A and Φ_B are complicated and are reported in a Mathematica file that is available from the authors upon request. ■

Having characterised the equilibria with non-linear pricing and exclusive contracts, we can now proceed to a comparison of the two. First, it can be shown that when $c \leq \check{c}$ the equilibrium quantities with exclusive contracts are everywhere at least as large as under non-linear pricing, and are strictly larger over some non-vanishing interval. This implies that when $c \leq \check{c}$ social welfare is larger under exclusive contracts. Second, when $c \geq c^m$ the equilibrium quantities with exclusive contracts are nowhere larger than under non-linear pricing, and are strictly lower over some non-vanishing interval. This implies that when $c \geq c^m$ social welfare is higher under non-linear pricing.

When instead $\check{c} < c < c^m$, the equilibrium quantities are larger with exclusive contracts in the low-demand segment of the market, and with non-linear pricing in an intermediate-demand segment (quantities are the same in both regimes in the high-demand segment of the market). However, the benefit from exclusive contracts in the low-demand segment is decreasing in c, whereas the loss in the intermediate-demand segment is increasing in c. This implies that there exists a unique threshold, c^*, such that exclusive contracts increase social welfare when $c < c^*$ and decrease social welfare when $c > c^*$. (Numerical calculations show that c^* decreases with γ.)

4. Proof of Proposition 4

The non-linear pricing equilibrium can be found proceeding as in the baseline model. The competitive fringe always prices at cost, i.e. $P_B(q_B) = 0$. The indirect payoff function,

$$v(q_A, \theta) = \max_{q_B \geq 0} u\left(q_A, q_B, \theta\right),$$

has two branches, depending on whether $\tilde{q}_B(\theta) = \arg\max u\left(q_A(\theta), q_B, \theta\right)$ is strictly positive or is nil, and a kink in between. It can be easily checked that the single crossing condition holds, and that the indirect virtual surplus

$$s(q_A(\theta), \theta) = v(q_A(\theta), \theta) - (1-\theta) v_\theta(q_A, \theta),$$

is globally concave and has increasing difference in θ and q_A. Notice that we have a type-dependent participation constraint $U(\theta) \geq u(0, q_B^e(\theta), \theta)$ where the right hand side is the net payoff that the buyer could obtain by trading with the competitive fringe only, and

$$q_B^e(\theta) = \frac{1+\theta b}{1-\gamma}$$

is the quantity that he would buy in that case.

Let us start from the relaxed problem in which the participation constraint is $U(\theta) \geq 0$. In this case, the solution can be found by pointwise maximisation of the indirect virtual surplus. The maximum can occur on the branch where $\tilde{q}_B(\theta) = 0$, in which case it is $q^m(\theta)$, on the branch where $\tilde{q}_B(\theta) > 0$, in which case it is $q_A^{cr}(\theta)$, or at the kink $q_A(\theta) = q^{\lim}(\theta)$, where $q^{\lim}(\theta)$ is implicitly defined by the condition $u_{q_B}\left(q^{\lim}(\theta), 0, \theta\right) = 0$. These quantities are

$$\begin{aligned} q^m(\theta) &= \frac{2\theta}{1-\gamma} \\ q_A^{cr}(\theta) &= \frac{2[1-\gamma(1+b)]\theta - (1-b)\gamma}{1-2\gamma} \\ q^{\lim}(\theta) &= \frac{1+b\theta}{\gamma}. \end{aligned}$$

The corresponding quantity of product B is $q_B(\theta) = \arg\max u\left(q_A(\theta), q_B, \theta\right)$. By construction, this is positive only when $q_A(\theta) = 0$ or if $q_A(\theta) = q_A^{cr}(\theta)$, in which case it is

$$q_B^{cr}(\theta) = \begin{cases} \frac{1+\theta b}{1-\gamma} & \text{for } q_A(\theta) = 0 \\ q_A^{cr}(\theta) + \frac{1-\gamma(1+b)-[2(1-\gamma)+b]\theta}{1-3\gamma+2\gamma^2} & \text{for } q_A(\theta) = q_A^{cr}(\theta). \end{cases}$$

It is immediate to check that the $q_A(\theta)$ schedules are monotone, increasing, and hence implementable. The price schedules that implement those quantities are, respectively

$$
\begin{aligned}
P^m(q_A) &= q_A - \frac{1-\gamma}{4} q_A^2 \\
P_A^{cr}(q_A) &= \frac{2-(3+b)\gamma}{2(1-\gamma)} q_A - \frac{1-2\gamma}{4(1-\gamma)} q_A^2 \\
P^{\lim}(q_A) &= -\frac{1-b}{b} q_A + \frac{\gamma - b(1-\gamma)}{2b} q_A^2
\end{aligned}
$$

The schedule $q_B^{cr}(\theta)$, in contrast, is non-monotone. However, given firm A's pricing, it is obviously implemented by the price schedule $P_B(q_B) = 0$.

Let $\breve{\theta}_i$ be the solutions to $q_i^{cr}(\theta) = 0$. We have $\breve{\theta}_A = \frac{(1-b)\gamma}{2[1-\gamma(1+b)]} > 0$, whereas the condition $b \leq \bar{b}$ implies that $\breve{\theta}_B < 1$. In this case, therefore, the markets for each of the two products are uncovered: product A is not purchased in low-demand states, product B in high-demand ones. However, the market is covered in the sense that at least one good is bought in all states of demand. Both goods are bought when both common representation quantities are positive, i.e. for $\theta \in \left[\breve{\theta}_A, \breve{\theta}_B\right]$. This interval is not empty.

Notice that the condition $b \leq \bar{b}$ implies that $q^m(\theta)$ intersects $q^{\lim}(\theta)$ from below. The condition $q^m(\theta) > q^{\lim}(\theta)$ is then equivalent to $\theta > \theta^{\lim}$, where $\theta^{\lim}$ is the solution to $q^m(\theta) = q^{\lim}(\theta)$ and hence it is $\theta^{\lim} = \frac{1-\gamma}{(2+b)\gamma - b}$. The condition $b \leq \bar{b}$ guarantees precisely that $\theta^{\lim} \leq 1$. That is, the condition guarantees the existence of a monopoly region.

With these preliminaries at hand, we can now proceed to the maximisation of the virtual surplus. Since $s(q_A(\theta), \theta)$ is concave, it is clear that if $q^m(\theta) > q^{\lim}(\theta)$ then $s(q_A, \theta)$ is increasing at the kink and the maximum is achieved at $q^m(\theta)$. This solution then applies when $\theta \geq \theta^{\lim}$. If instead $q^m(\theta) < q^{\lim}(\theta)$, i.e. for $\theta < \theta^{\lim}$, then $s(q_A, \theta)$ is decreasing to the right of the kink, and one must further distinguish between two cases. If $q_A^{cr}(\theta) > q^{\lim}(\theta)$, then $s(q_A, \theta)$ is increasing to the left of the kink and so the maximum is achieved at the kink, $q^{\lim}(\theta)$. If instead $q_A^{cr}(\theta) < q^{\lim}(\theta)$, the maximum is achieved to the left of the kink and is $q_A^{cr}(\theta)$. Noting that by construction we have $q_A^{cr}(\theta) = q^{\lim}(\theta)$ when $\theta = \breve{\theta}_B$, we can conclude that for $\breve{\theta}_B \leq \theta \leq \theta^{\lim}$ we have the limit pricing solution, whereas for $\theta \leq \breve{\theta}_B$ we have common representation. However, for $\theta < \breve{\theta}_A$ we must have $q_A(\theta) = 0$, in which case only product B is sold, and the quantity is $q_B^e(\theta) > 0$.

Summarizing, the equilibrium quantities are

$$
q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq \breve{\theta}_A \\ q_A^{cr}(\theta) & \text{for } \breve{\theta}_A \leq \theta \leq \breve{\theta}_B \\ q^{\lim}(\theta) & \text{for } \breve{\theta}_B \leq \theta \leq \theta^{\lim} \\ q^m(\theta) & \text{for } \theta \geq \theta^{\lim} \end{cases}
\qquad
q_B(\theta) = \begin{cases} q_B^e(\theta) & \text{for } \theta \leq \breve{\theta}_A \\ q_B^{cr}(\theta) & \text{for } \breve{\theta}_A \leq \theta \leq \breve{\theta}_B \\ 0 & \text{for } \theta \geq \breve{\theta}_B. \end{cases}
$$

It is easy to see that the type-dependent participation constraint $U(\theta) \geq u(0, q_B^e(\theta), \theta)$ is always met, so the solution to the relaxed problem solves also the original problem, and hence is the non-linear pricing equilibrium.

For the case of exclusive contracts, the analysis also proceeds exactly as for the baseline model. The separation property guarantees that the solution to the dominant firm's problem is either $q^m(\theta)$ or the non-linear pricing solution that we have just characterised. (It can be confirmed that in the exclusive dealing region the type dependent participation constraint is met even if the dominant firm engages in monopoly pricing.) The analog of Lemma 2 in the proof of Proposition 2 guarantees that the switch among the two regimes can only occur in a region where in the non-linear pricing equilibrium both $q_A(\theta)$ and $q_B(\theta)$ are strictly positive.

The optimal switching point is still characterised by conditions (6) and (7). An argument identical to that proposed in the analysis of the baseline model confirms that at the switching point the average price jumps up when increasing demand, and hence so does the profit earned on the critical buyer $\hat{\theta}$.

Summarising, in equilibrium firm A offers the price schedules:

$$\begin{array}{ll} P_A^{NE}(q) = P_A^{cr}(q) & \text{for } 0 \leq q \leq q_A^m(\hat{\theta}) \\ P_A^{E}(q) = P^m(q) + \Phi_A & \text{for } q \geq q_A^m(\hat{\theta}). \end{array}$$

where $\hat{\theta}$ and Φ_A are determined by the equilibrium conditions (6) and (7). Now, however, Φ_A is negative. In other words, the dominant firm offers lump-sum subsidies to buyers who opt for exclusive dealing. (The explicit expression for $\hat{\theta}$ and Φ_A are complicated and are reported in a Mathematica file which is available from the authors upon request.) The competitive fringe will always price at cost. The associated equilibrium quantities are

$$q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq \breve{\theta}_A \\ q_A^{cr}(\theta) & \text{for } \breve{\theta}_A \leq \theta \leq \hat{\theta} \\ q^m(\theta) & \text{for } \theta > \hat{\theta} \end{cases} \qquad q_B(\theta) = \begin{cases} q_B^e(\theta) & \text{for } \theta \leq \breve{\theta}_A \\ q_B^{cr}(\theta) & \text{for } \breve{\theta}_A \leq \theta \leq \hat{\theta} \\ 0 & \text{for } \theta > \hat{\theta}. \end{cases}$$

5. Market-share contracts

Besides offering exclusive contracts, the dominant firm might employ other contracts that reference rivals' volume. For example, it may offer market-share discounts, i.e. discounts that depend on its share of a buyer's total purchases. To account for this possibility, we allow the dominant firm to freely condition its payment request on its market share, or, equally, on the competitive fringe's volume: $P_A = P_A(q_A, q_B)$.

Naturally, this pricing strategy requires that the dominant firm observe with some precision not only whether q_B is positive or nil, but also the exact value of q_B. Thus, market-share contracts are observationally demanding.[3] However, we find that, if market-share contracts are feasible, they will be offered, and signed, in certain states of demand.

[3] In practice, market-share contracts are often cast in terms of critical market-share thresholds. This may facilitate verification and hence enforcement.

For simplicity, we focus on the uniform-quadratic specification (2). Using the Revelation Principle, the pricing problem can be translated into a direct mechanism problem in which the dominant firm controls both $q_A(\theta)$ and $q_B(\theta)$. Firm A thus must now solve a problem of multi-product monopolistic screening with type-dependent participation constraints. Standard arguments then show that the problem can be formulated as follows:

$$
\begin{aligned}
& \max_{q_A(\theta)\geq 0, q_B(\theta)\geq 0} \int_{\theta_{\min}}^{\theta_{\max}} \left[u\left(q_A(\theta), q_B(\theta), \theta\right) - cq_B - U(\theta)\right] f(\theta) d\theta \\
\text{s.t. } \frac{dU}{d\theta} &= u_\theta(q_A(\theta), q_B(\theta), \theta) \\
U(\theta) &\geq u(0, q^e(\theta), \theta) - cq^e(\theta),
\end{aligned}
$$

where $u(0, q^e(\theta), \theta) - cq^e(\theta)$ is the net payoff that the buyer could obtain by trading with the competitive fringe only.

Let us consider first the relaxed problem in which the participation constraint is simply $U(\theta) \geq 0$. The associated virtual surplus function is

$$
s(q_A, q_B, \theta) = v(q_A, q_B, \theta) - \frac{1-F(\theta)}{f(\theta)} v_\theta(q_A, q_B, \theta).
$$

With the uniform-quadratic specification, the virtual surplus function is strictly concave in q_A and q_B, and it has increasing differences in θ and q_A and q_B, respectively. Thus, the solution can be obtained by pointwise maximisation of the virtual surplus. The necessary and sufficient conditions for a maximum are

$$
\begin{aligned}
u_{q_A} - \frac{1-F(\theta)}{f(\theta)} u_{\theta q_A} &\leq 0 \qquad \left[u_{q_A} - \frac{1-F(\theta)}{f(\theta)} u_{\theta q_A}\right] q_A = 0 \\
u_{q_B} - c - \frac{1-F(\theta)}{f(\theta)} u_{\theta q_B} &\leq 0 \qquad \left[u_{q_B} - c - \frac{1-F(\theta)}{f(\theta)} u_{\theta q_B}\right] q_B = 0.
\end{aligned}
$$

From these we easily obtain the optimal quantities in the relaxed problem,

$$
q_A(\theta) = \begin{cases} 0 & \text{for } \theta \leq \frac{1}{2} \\ q^m(\theta) & \text{for } \frac{1}{2} \leq \theta \leq \hat{\theta}^{ms} \\ q_A^{ms}(\theta) & \text{for } \hat{\theta}^{ms} \leq \theta \leq 1 \end{cases} \qquad q_B(\theta) = \begin{cases} 0 & \text{for } \theta \leq \hat{\theta}^{ms} \\ q_B^{ms}(\theta) & \text{for } \hat{\theta}^{ms} \leq \theta \leq 1, \end{cases}
$$

where

$$
q_A^{ms}(\theta) = 2\theta - 1 + c\frac{\gamma}{1-2\gamma}; \quad q_B^{ms}(\theta) = 2\theta - 1 - c\frac{1-\gamma}{1-2\gamma},
$$

and $\breve{\theta}^{ms}$ is implicitly defined by the condition $q_B^{ms}(\hat{\theta}^{ms}) = 0$; the explicit expression is $\hat{\theta}^{ms} = \frac{1+c-(2+c)\gamma}{2(1-2\gamma)}$.

Notice that $q_A^{ms}(\theta) \geq q_B^{ms}(\theta)$. Comparing the above quantities with those of the baseline model (see Section 2 of this Online Appendix) we observe that $q_A^{ms}(\theta) = q_A^{cr}(\theta)$, whereas $q_B^{ms}(\theta) \leq q_B^{cr}(\theta)$ with a strict inequality except at $\theta = 1$ (no distortion at the top). Thus, the marginal type $\breve{\theta}^{ms}$ is smaller than the marginal type $\breve{\theta}_B$ in the non-linear pricing equilibrium. The fact that $q_A^{ms}(\theta) = q_A^{cr}(\theta)$ is due to the linearity of the marginal payoff functions. (Majumdar and Shaffer (2009) obtain the same property in a linear demand function example with two types.)

When $c \geq c^m (= \frac{1}{2})$, it can be easily checked that the solution to the relaxed problem satisfies the type-dependent participation constraint, and thus is the optimum. When $c < c^m$, however, the participation constraint binds in low-demand states. We must then apply Proposition 5.5 in Jullien (2000). Proceeding as in the proof of Proposition 2, it is easy to show that the conditions of Homogeneity, Potential Separation and Weak Convexity are met. Jullien's result then guarantees that the solution partitions the set of types into three sets: buyers who are excluded, buyers who obtain their reservation utility $U^E(\theta)$, and buyers whose net payoff is strictly greater than $U^E(\theta)$.

This means that when $c < c^m$ the exclusive dealing branch of the $q_A(\theta)$ schedule is formed by two sub-branches, i.e. $q^e(\theta)$ for low types and $q^m(\theta)$ for intermediate types. This is similar to the equilibrium pattern that arises in the non-linear pricing equilibrium. In any case, equilibrium quantities are never greater than in the non-linear pricing equilibrium, and are strictly lower for a range of values of θ. This implies that market-share contracts are definitely anticompetitive with respect to that benchmark.

Summarising, in low-demand states *de facto* or contractual exclusivity still prevails. However, the transition from exclusive dealing to unconstrained common representation is now smoother. In other words, as demand rises exclusive dealing is no longer imposed on buyers, who therefore start purchasing both products. However, the dominant firm now uses market-share discounts to reduce the incentive to purchase product B. The share of product B increases with demand. However, it is always lower than in the non-linear pricing equilibrium, except when demand is highest (this is, once again, a no-distortion-at-the-top property).

Compared to the non-linear pricing equilibrium, market-share contracts are clearly anticompetitive.[4] The comparison with exclusive contracts is less clear: quantities are more heavily distorted in high-demand states, but less so in intermediate ones.

References

[1] Jullien, Bruno. 2000. "Participation Constraints in Adverse Selection Models." *Journal of Economic Theory* 93 (1): 1–47.

[4] This result differs from Majumdar and Shaffer (2009), who show that in a two-states model the first-best solution may be achieved with market-share contracts, but not with simple non-linear prices. This is because the reservation payoff is type dependent. However, with a continuum of types it is generally impossible to reproduce the full information solution even with type-dependent participation constraints (see, for instance, Jullien, 2000).

[2] Majumdar, Adrian, and Greg Shaffer. 2009. "Market-Share Contracts with Asymmetric Information." *Journal of Economics and Management Strategy* 18 (2): 393–421.

[3] Martimort, David, and Lars Stole. 2009. "Market Participation in Delegated and Intrinsic Common-Agency Games." *Rand Journal of Economics* 40 (1): 78–102.