Online Appendix: "Optimal taxation and debt with uninsurable risks to human capital accumulation"

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1 Proofs

1.1 Proof of Lemma 1

The proof of this lemma uses an argument similar to Epstein and Zin (1991) and Angeletos (2007). Since the idiosyncratic shocks, $\theta_{i,t}$, are i.i.d. across individuals and across periods, the utility maximization problem of each individual can be expressed as:

$$V_t(x) = \max_{c,\eta_h} \left\{ (1-\beta)c^{1-\frac{1}{\psi}} + \beta \left(E_t[V_{t+1}(x')^{1-\gamma}] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\psi}}}$$

$$\text{s.t.} \qquad x' = (x-c)\left[R_{k,t+1}(1-\eta_h) + R_{h,t+1}\theta'\eta_h \right] \geq 0,$$

$$c \in [0,x], \qquad \eta_h \in [0,1].$$

Here, $V_t(x)$ is the value function for the utility maximization problem of an individual whose total wealth is x at the beginning of period t. We conjecture that there exists a (deterministic) sequence $\{v_t\}_{t=0}^{\infty}$, with $v_t \in \mathbb{R}_+$ for all t, such that

$$V_t(x) = v_t x$$

Using this conjecture and the budget constraint, we obtain

$$\left(E_t[V_{t+1}(x')^{1-\gamma}] \right)^{\frac{1}{1-\gamma}} = v_{t+1}(x-c) \left\{ E_t \left[\left(R_{k,t+1}(1-\eta_h) + R_{h,t+1}\theta'\eta_h \right)^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}}$$

It follows that in the above maximization problem the individual chooses the portfolio η_h so as to solve the following maximization problem:

$$\eta_h = \arg\max_{\eta_h' \in [0,1]} \left\{ E_t \left[\left(R_{k,t+1}(1-\eta_h') + R_{h,t+1}\theta'\eta_h' \right)^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}}$$

Let ρ_{t+1} denote the maximized value in this problem. Note that neither η_h nor ρ_{t+1} depends on the initial state x. That is, under the conjectured value function, all individuals would choose the same portfolio and the same certainty-equivalent rate of return.

Given the certainty-equivalent rate of return, ρ_{t+1}, the level of consumption is chosen so as to solve

$$\max_{c \in [0,x]} \left\{ (1-\beta)c^{1-\frac{1}{\psi}} + \beta \left[v_{t+1}\rho_{t+1}(x-c) \right]^{1-\frac{1}{\psi}} \right\}^{\frac{1}{1-\frac{1}{\psi}}}$$

The first-order condition for this problem is

$$(1-\beta)c^{-\frac{1}{\psi}} = \beta v_{t+1}^{1-\frac{1}{\psi}} \rho_{t+1}^{1-\frac{1}{\psi}} (x-c)^{-\frac{1}{\psi}}$$

which leads to

$$\eta_c = \left\{1 + \left(\frac{\beta}{1-\beta}\right)^{\psi} (v_{t+1}\rho_{t+1})^{\psi-1}\right\}^{-1}$$

where $\eta_c = \frac{c}{x}$.

On the other hand, the Bellman equation implies

$$v_t^{1-\frac{1}{\psi}} = (1-\beta)\eta_c^{1-\frac{1}{\psi}} + \beta\left(v_{t+1}\rho_{t+1}\right)^{1-\frac{1}{\psi}} (1-\eta_c)^{1-\frac{1}{\psi}}$$

This equation and the above first-order condition for c imply that

$$v_t^{\psi-1} = (1-\beta)^{\psi} + \beta^{\psi} v_{t+1}^{\psi-1} \rho_{t+1}^{\psi-1}$$

The bounded solution to this difference equation is

$$v_t = (1-\beta)^{\frac{\psi}{\psi-1}} \left\{1 + \sum_{s=0}^{\infty} \prod_{j=0}^{s} \left(\beta^{\psi} \rho_{t+1+j}^{\psi-1}\right)\right\}^{\frac{1}{\psi-1}}$$

Also, the consumption rate η_c is

$$\eta_{c,t} = (1-\beta)^{\psi} v_t^{1-\psi}$$

It is straightforward to verify that, constructed in this way, $\{V_t(x), \eta_c, \eta_h\}$ indeed characterizes the solution to the utility maximization problem. The rest of the lemma follows immediately.

1.2 Proof of Proposition 3

Totally differentiating constraint (36) of problem (35), we obtain

$$(\tilde{r} - F_k + F_h - \tilde{w})\, d\eta_h - (1-\eta_h)\, d\tilde{r} - \eta_h\, d\tilde{w} = 0.$$

Evaluating this expression at the benchmark equilibrium, where $G_t = B_t = 0$, $\tilde{r}_t = \hat{F}_k$ and $\tilde{w}_t = \hat{F}_h$, for all t, yields

$$(1-\hat{\eta}_h)\, d\tilde{r} + \hat{\eta}_h\, d\tilde{w} = 0.$$

Thus, to satisfy the balanced budget, $\tilde{r}$ and $\tilde{w}$ must satisfy the following relationship around $(\tilde{r}, \tilde{w}) = (\hat{F}_k, \hat{F}_h)$:

$$\frac{d\tilde{w}}{d\tilde{r}} = -\frac{1-\hat{\eta}_h}{\hat{\eta}_h}.$$

Hence the effect of a marginal change in $\tilde{r}$, taking into account the induced change in $\tilde{w}$ via the government budget constraint, is given by $\frac{\partial}{\partial \tilde{r}} - \frac{1-\hat{\eta}_h}{\hat{\eta}_h}\frac{\partial}{\partial \tilde{w}}$ and will be denoted by $\frac{d}{d\tilde{r}}$. Since the lifetime utility is increasing in ρ_t for each t, it suffices to show that $\frac{d\rho}{d\tilde{r}} > 0$.

The envelope theorem implies that $\frac{\partial \rho}{\partial \eta_h} = 0$ at the benchmark equilibrium. It follows that

$$\begin{aligned}\frac{d\rho}{d\tilde{r}} &= \hat{\rho}^{\gamma} E\left[\hat{R}_x(\theta)^{-\gamma}\left\{(1-\hat{\eta}_h) + \theta\hat{\eta}_h\frac{d\tilde{w}}{d\tilde{r}}\right\}\right],\\ &= \hat{\rho}^{\gamma} E\left[\hat{R}_x(\theta)^{-\gamma}(1-\theta)\right](1-\hat{\eta}_h),\end{aligned}$$

where $\hat{R}_x(\theta) \equiv (1-\delta_k+\hat{F}_k)(1-\hat{\eta}_h) + (1-\delta_h+\hat{F}_h)\theta\hat{\eta}_h$. Since $E(\theta) = 1$, we have

$$E\left[\hat{R}_x(\theta)^{-\gamma}(1-\theta)\right] = \mathrm{Cov}(\hat{R}_x(\theta)^{-\gamma}, 1-\theta) > 0,$$

where the inequality follows from the fact that both $\hat{R}_x(\theta)^{-\gamma}$ and $1-\theta$ are decreasing functions of θ. Given that $\hat{\eta}_h < 1$, this proves that $\frac{d\rho}{d\tilde{r}} > 0$.

It remains to show that the after-tax rental rate of capital, $\tilde{r}$, and the tax rate on capital income, τ_k, move in the opposite directions around the benchmark equilibrium. Since $\tau_k = 1 - \frac{\tilde{r}}{F_k}$, we have

$$\frac{d\tau_k}{d\tilde{r}} = \frac{-\hat{F}_k + (-\hat{F}_{kk} + \hat{F}_{kh})\frac{d\eta_h}{d\tilde{r}}}{\hat{F}_k^2}. \tag{43}$$

Differentiating the individual first order conditions (15) yields

$$\left\{\Phi_{\tilde{r}} - \frac{1-\hat{\eta}_h}{\hat{\eta}_h}\Phi_{\tilde{w}}\right\} d\tilde{r} + \Phi_{\eta_h} d\eta_h = 0,$$

so that

$$\frac{d\eta_h}{d\tilde{r}} = \frac{\frac{1-\hat{\eta}_h}{\hat{\eta}_h}\Phi_{\tilde{w}} - \Phi_{\tilde{r}}}{\Phi_{\eta_h}}. \tag{44}$$

Thus we obtain

$$\frac{d\tau_k}{d\tilde{r}} = \frac{1}{\hat{F}_k^2}\frac{-\hat{F}_k\Phi_{\eta_h} + (-\hat{F}_{kk} + \hat{F}_{kh})\left(\frac{1-\hat{\eta}_h}{\hat{\eta}_h}\Phi_{\tilde{w}} - \Phi_{\tilde{r}}\right)}{\Phi_{\eta_h}} < 0,$$

since by Assumption 1 we have $\Phi_{\tilde{w}} > 0$, $\Phi_{\tilde{r}} < 0$, while $\Phi_{\eta_h} < 0$ follows from the strict concavity of $\rho(\tilde{r}, \tilde{w}, \eta_h)$ and $F_{kh} = (1-\alpha)\alpha k^{\alpha-1}h^{-\alpha} > 0$. This completes the proof.

1.3 Proof of Proposition 4

We are interested in the welfare effect of a marginal variation of $\bar{b}_{T+1}$ evaluated at $\bar{b}_{T+1} = 0$, that is the sign of $dv_0/d\bar{b}_{T+1}\big|_{\bar{b}_{T+1}=0}$. Denote the variables solving the Ramsey problem under (37) as $v_t(\bar{b}_{T+1})$, $\rho_t(\bar{b}_{T+1})$, etc.. It is immediate to see that its solution is the same as under (34) for all periods except two,

$$\rho_t(\bar{b}_{T+1}) = \rho^o, \quad \forall t \neq T+1, T+2 \tag{45}$$

Hence from (12) we get $v_t(\bar{b}_{T+1}) = v^o, \quad \forall t \geq T+2,$ and $dv_0/dv_T > 0$, so that

$$\left.\frac{dv_0}{db_{T+1}}\right|_{\bar{b}_{T+1}=0} \gtreqless 0 \qquad \Longleftrightarrow \qquad \left.\frac{dv_T}{db_{T+1}}\right|_{\bar{b}_{T+1}=0} \gtreqless 0.$$

We have so[27] $\rho_{T+2}(\bar{b}_{T+1}) = \rho^R(\bar{b}_{T+1}, 0, \eta_{c,T+1}(\bar{b}_{T+1}))$. Recalling again (12), we obtain

$$v_{T+1}(\bar{b}_{T+1}) = \left\{(1-\beta)^\psi + \beta^\psi \rho_{T+2}(\bar{b}_{T+1})^{\psi-1} v_{T+2}(\bar{b}_{T+1})^{\psi-1}\right\}^{\frac{1}{\psi-1}}. \tag{46}$$

Here, note that (45) implies $\partial v_{T+2}/\partial \bar{b}_{T+1} = 0$. In addition, $\partial \rho^R(0,0,\eta_c)/\partial \eta_c = 0$.[28] Differentiating then $v_{T+1}(\bar{b}_{T+1})$ with respect to $\bar{b}_{T+1}$ and evaluating it at $\bar{b}_{T+1} = 0$ yields

$$\left.\frac{dv_{T+1}}{d\bar{b}_{T+1}}\right|_{\bar{b}_{T+1}=0} = \beta^\psi (\rho^{Ro})^{\psi-2} \rho_1^o v^o, \tag{47}$$

where $\rho_1^{Ro} \equiv \partial \rho^R(b, b', \eta_c^o)/\partial b$, evaluated at $b = b' = 0$.[29]

Next, consider the expression analogous to (46) for date T:

$$v_T(\bar{b}_{T+1}) = \left\{(1-\beta)^\psi + \beta^\psi \left(\rho_{T+1}(\bar{b}_{T+1})\right)^{\psi-1} v_{T+1}(\bar{b}_{T+1})^{\psi-1}\right\}^{\frac{1}{\psi-1}}. \tag{48}$$

Its derivative with respect to $\bar{b}_{T+1}$, evaluated at $\bar{b}_{T+1} = 0$, using (47) and again the fact that $\partial \rho^R/\partial \eta_{c,T}\big|_{\bar{b}_{T+1}=b_T=0} = 0$, equals

$$\left.\frac{dv_T}{d\bar{b}_{T+1}}\right|_{\bar{b}_{T+1}=0} = \beta^\psi (\rho^o)^{\psi-2} v^o \left[\rho_2^{Ro} + \beta^\psi (\rho^o)^{\psi-1} \rho_1^{Ro}\right],$$

where $\rho_2^{Ro} \equiv \partial \rho^R(b, b', \eta_c^o)/\partial b'$ evaluated at $b = b' = 0$.

Let us denote then by $\lambda(b, b', \eta_c)$ the Lagrange multiplier on the flow budget constraint for the government in problem (32) and by $\eta_h(b, b', \eta_c)$, $\widetilde{r}(b, b', \eta_c)$, $\widetilde{w}(b, b', \eta_c)$, and $R_x(b, b', \eta_c)$ its solution. Using the envelope property and the fact that b, b' only appear in constraint (31) of the problem,

[27] Here and in what follows we omit the dependence of ρ^R on g whenever g_t is constant across periods.

[28] To see this, recall from the definition of $\rho^R(b, b', \eta_c)$ in (32) that η_c affects ρ^R only through the government budget constraint (31). Consider the associated function:

$$\begin{aligned} & f(b, b', \eta_c, \eta_h, \widetilde{r}, \widetilde{w}, R_x) \\ & \qquad \equiv g + (1 - \delta_k + \widetilde{r})b - (1-\eta_c) R_x b' - F\left[(1-\eta_c)(1-\eta_h) - b, (1-\eta_c)\eta_h\right] \\ & \qquad\quad + \widetilde{r}\left[(1-\eta_c)(1-\eta_h) - b\right] + \widetilde{w}(1-\eta_c)\eta_h \end{aligned}$$

We have $\left.\frac{\partial f}{\partial \eta_c}\right|_{b=b'=0} = 0$ and so, by the envelope theorem we get the claimed property.

[29] The superscript ^{o} indicates, as in the main text, variables evaluated at a solution of the Ramsey problem under the constraint $b_t = g_t = 0$ for all t.

we obtain, when $b_t = g_t = 0$ for all t:[30]

$$\rho_1^{Ro} = -\lambda^o(1-\delta_k+F_k^o),$$
$$\rho_2^{Ro} = \lambda^o\beta^\psi(\rho^o)^{\psi-1}R_x^o,$$

since

$$\eta_c^o = 1-\beta^\psi(\rho^o)^{\psi-1}.$$

Therefore,

$$\frac{dv_T}{d\bar{b}_{T+1}} = \xi\left[R_x^o - (1-\delta_k+F_k^o)\right], \tag{49}$$

where

$$\xi \equiv \beta^{2\psi}(\rho^o)^{2\psi-3}\lambda^o v^o$$

and $\xi > 0$ since $\lambda^o > 0$, as we show next. As argued in Section 3.1, when $b_t = g_t = 0$ for all t, problem (32) reduces to (35).

Let us write the solution to (10) as $\eta_h(\widetilde{r},\widetilde{w})$. Then the first order conditions for $\widetilde{r}$ and $\widetilde{w}$ in problem (35) are given by

$$0 = \frac{\partial\rho}{\partial\widetilde{r}} - (1-\eta_h^o)\lambda^o + \left[\frac{\partial\rho}{\partial\eta_h} + \lambda^o(-F_k^o+F_h^o+\widetilde{r}^o-\widetilde{w}^o)\right]\frac{\partial\eta_h}{\partial\widetilde{r}},$$
$$0 = \frac{\partial\rho}{\partial\widetilde{w}} - \eta_h^o\lambda^o + \left[\frac{\partial\rho}{\partial\eta_h} + \lambda^o(-F_k^o+F_h^o+\widetilde{r}^o-\widetilde{w}^o)\right]\frac{\partial\eta_h}{\partial\widetilde{w}}.$$

From the second equation, recalling that under Assumption 1 we have $\frac{\partial\eta_h}{\partial\widetilde{w}} > 0$ and $\frac{\partial\eta_h}{\partial\widetilde{r}} < 0$, we obtain

$$\lambda^o(-F_k^o+F_h^o+\widetilde{r}^o-\widetilde{w}^o) = \frac{-\frac{\partial\rho}{\partial\widetilde{w}}+\eta_h^o\lambda^o}{\frac{\partial\eta_h}{\partial\widetilde{w}}}.$$

Substituting then this equation into the first equation above, and solving for λ^o, we get

$$\lambda^o = \left(1-\eta_h^o - \frac{\eta_h^o\frac{\partial\eta_h}{\partial\widetilde{r}}}{\frac{\partial\eta_h}{\partial\widetilde{w}}}\right)^{-1}\left(\frac{\partial\rho}{\partial\widetilde{r}} - \frac{\frac{\partial\rho}{\partial\widetilde{w}}\frac{\partial\eta_h}{\partial\widetilde{r}}}{\frac{\partial\eta_h}{\partial\widetilde{w}}}\right) > 0,$$

where the sign of the inequality follows from the fact that $\eta_h^o \in (0,1)$, $\frac{\partial\rho}{\partial\widetilde{r}} > 0$ and $\frac{\partial\rho}{\partial\widetilde{w}} > 0$.

[30] To better understand the form of these expressions, notice that, as we see from (31), a marginal increase of $\bar{b}_{T+1}$ relaxes this constraint at $T+1$ yielding a gain of $\lambda^o\,(1-\eta_c^o)\,R_x^o$, while tightening this constraint at $T+2$ with a loss of $\lambda^o\beta^\psi(\rho^o)^{\psi-1}(1-\delta_k+F_k^o)$ (recall that ρ_1^{Ro} is multiplied by $\beta^\psi(\rho^o)^{\psi-1}$ in the expression of dv_T/db_{T+1}). Since $(1-\eta_c^o) = \beta^\psi(\rho^o)^{\psi-1}$, the comparison of these two reduce to the comparison between R_x^o and $(1-\delta_k+F_k^o)$.

1.4 Proof of Proposition 5

The Lagrangean for problem (33), using (12) and (14) to substitute for ρ_{t+1} and $\eta_{c,t}$, is

$$v_0 + \sum_{t=0}^{\infty} \lambda_t^v \left\{ (1-\beta)^\psi + \beta^\psi \rho^R(b_t, b_{t+1}, (1-\beta)^\psi v_t^{1-\psi})^{\psi-1} v_{t+1}^{\psi-1} - v_t^{\psi-1} \right\}.$$

The first-order condition with respect to b_{t+1} is then

$$\lambda_t^v \beta^\psi \rho_{t+1}^{\psi-2} \rho_{2,t+1}^R v_{t+1}^{\psi-1} + \lambda_{t+1}^v \beta^\psi \rho_{t+2}^{\psi-2} \rho_{1,t+2}^R v_{t+2}^{\psi-1} = 0, \tag{50}$$

where $\rho_{t+1} \equiv \rho^R(b_t, b_{t+1}, \eta_{c,t})$, $\rho_{2,t+1}^R \equiv \partial \rho^R(b_t, b_{t+1}, \eta_{c,t})/\partial b_{t+1}$, and $\rho_{1,t+2}^R \equiv \partial \rho^R(b_{t+1}, b_{t+2}, \eta_{c,t+1})/\partial b_{t+1}$. The first-order condition for v_{t+1} is

$$\lambda_t^v \beta^\psi \rho_{t+1}^{\psi-1} v_{t+1}^{\psi-2} + \lambda_{t+1}^v \beta^\psi \rho_{t+2}^{\psi-2} \rho_{\eta_c,t+2}^R (1-\beta)^\psi (1-\psi) v_{t+1}^{-\psi} v_{t+2}^{\psi-1} - \lambda_{t+1}^v v_{t+1}^{\psi-2} = 0, \tag{51}$$

where $\rho_{\eta_c,t+2}^R \equiv \partial \rho^R(b_{t+1}, b_{t+2}, \eta_{c,t+1})/\partial \eta_{c,t+1}$.

In a steady-state equilibrium, equation (50) reduces to

$$\rho_2^R + \frac{\lambda_{t+1}^v}{\lambda_t^v} \rho_1^R = 0 \tag{52}$$

and equation (51) to

$$\frac{\lambda_{t+1}^v}{\lambda_t^v} = \beta^\psi \rho^{\psi-1} \left(1 - \beta^\psi \rho^{\psi-1} (1-\beta)^\psi (1-\psi) \frac{\rho_{\eta_c}^R v^{1-\psi}}{\rho} \right)^{-1}, \tag{53}$$

where the term in parenthesis captures the effect on ρ of the change in the savings rate, given by the second term in (51), which only arises (as we saw in foonote 30) when debt is nonzero.

By a similar argument to the one in the proof of Proposition 4 above, at a steady state equilibrium the derivative of ρ^R with respect to b and b' satisfies

$$\begin{aligned} -\frac{\rho_1^R}{\rho_2^R} &= \frac{1-\delta_k+F_k}{(1-\eta_c)R_x} \\ &= \frac{1-\delta_k+F_k}{\beta^\psi \tilde{\rho}^{\psi-1} R_x}, \end{aligned} \tag{54}$$

where, for the second equality, we used again (14), $\eta_c = (1-\beta)^\psi v^{1-\psi}$, and constraint (12), $v^{\psi-1} = (1-\beta)^\psi + \beta^\psi \rho^{\psi-1} v^{\psi-1}$, of problem (33).

Combining (52)-(54) and using again (14), yields the claimed result:

$$R_x = (1-\delta_k+F_k) \left[1 - (1-\psi)\beta^\psi \rho^{\psi-2} \rho_{\eta_c}^R \eta_c \right]^{-1}.$$

2 Sufficient conditions for Assumption 1

Let us rewrite problem (9) more compactly as

$$\max_{\eta_h \geq 0} E\left[u\left(r\left(1-\eta_h\right)+\theta w \eta_h\right)\right],$$

where, with a slight abuse of notation, r denotes $1-\delta_k+\widetilde{r}$, w denotes $1-\delta_h+\widetilde{w}$, and the function $u(.)$ is increasing, concave and with a constant coefficient of relative risk aversion γ. Letting η_h^* be an interior solution of (9), the properties stated in Assumption 1 are equivalent to $\frac{\partial \eta_h^*}{\partial r}<0$ and $\frac{\partial \eta_h^*}{\partial w}>0$, as already noticed in the main text. Setting $R \equiv \theta w-\alpha$, problem (9) may also be written as

$$\max_{\eta_h \geq 0} E\left[u\left(r+R \eta_h\right)\right], \tag{55}$$

when $\alpha=r$. Problem (55) is often referred to as the standard portfolio choice problem. Hereafter, we shall use some results on such problem reported in Gollier (2004).[31]

From Proposition 9 in Gollier (2004) it follows that, when the coefficient of relative risk aversion γ is not larger than one, any first order stochastic improvement in R increases the optimal value of η_h. Since an increase in w induces such an improvement, we conclude that $\frac{\partial \eta_h^*}{\partial w}>0$ if $\gamma \leq 1$.

Note that an increase in r, keeping R (that is, α) constant, constitutes an increase in wealth and so from Proposition 8 in Gollier (2004) it follows that this change induces a decrease in η_h^* if u exhibits decreasing absolute risk aversion. With constant relative risk aversion, u indeed exhibits decreasing absolute risk aversion. There is then a second effect of the increase in r, given by the change in R : an increase in α induces a first order worsening on R and so reduces η_h^* if $\gamma \leq 1$. Hence we conclude that $\frac{\partial \eta_h^*}{\partial r}<0$ if $\gamma \leq 1$.

Having established that the stated properties always hold when $\gamma \leq 1$, we show next that, when $\gamma>1$, they hold for some family of distributions of θ. Assuming that θ is a continuous random variable with density function $g(t)$ differentiable almost everywhere, we shall show below that the stated comparative statics properties hold if both $t\frac{g'(t)}{t}$ and $\frac{g'(t)}{t}$ are non-increasing in t. The condition hold for example when θ is a uniform distribution over some interval, or a Pareto distribution (i.e., the density function is a power function).

To establish the result we build on Proposition 17 in Gollier (2004), stating that, if $u(.)$ is strictly increasing, then any improvement in R in monotone likelihood ratio (MLR) increases the optimal value η_h^* of problem (55). That is, if R and R' are distinct continuous random variables with density f_R and $f_{R'}$ respectively, the optimal value η_h^* under R' is larger than that under R if $f_{R'}(t)/f_R(t)$ is non decreasing in t.

Since $R=\theta w-\alpha$, $\Pr[R \leq z]=\Pr[\theta \leq(z+r)/w]$ and so the density function $f(z)$ of R is

[31] Gollier, C. (2004), "The Economics of Risk and Time," MIT Press.

given by

$$f(z) = \frac{d}{dz}\int_0^{(z+r)/w} g(t)\,dt = \frac{1}{w} g\left(\frac{z+r}{w}\right). \tag{56}$$

So in order to use the above proposition to establish the property $\frac{\partial \eta_h^*}{\partial w} > 0$, it suffices to show that for any $\hat{w} > w$ $\frac{1}{\hat{w}} g\left(\frac{z+r}{\hat{w}}\right) / \frac{1}{w} g\left(\frac{z+r}{w}\right)$ is non decreasing in z. Taking a monotone (logarithmic) transformation and differentiating with respect to z, this condition obtains when

$$\frac{1}{\hat{w}} \frac{g'\left(\frac{z+r}{\hat{w}}\right)}{g\left(\frac{z+r}{\hat{w}}\right)} - \frac{1}{w} \frac{g'\left(\frac{z+r}{w}\right)}{g\left(\frac{z+r}{w}\right)} \geq 0,$$

that is, when

$$\frac{1}{w} \frac{g'\left(\frac{z+r}{w}\right)}{g\left(\frac{z+r}{w}\right)} \text{ is non-decreasing in } w,$$

at any $w > 0$, for given z and r. Since the map $w \mapsto (z+r)/w$ is monotonic and decreasing, setting $t = (r+z)/w$, the condition above can be equivalently stated as

$$t \frac{g'(t)}{g(t)} \text{ is non-increasing in } t.$$

Next, we use the same proposition to derive a condition guaranteeing that $\frac{\partial \eta_h^*}{\partial r} < 0$. Recalling the argument above regarding the effect of increasing r keeping R constant, when $u(.)$ exhibits decreasing absolute risk aversion, it suffices to show that the optimal value of η_h^* decreases as α in $R = w\theta - \alpha$ increases, keeping r fixed. Hence we derive next a condition on $g(t)$ such that a decrease in α induces a MLR improvement: that is, for any $\hat{\alpha} < \alpha$ $\frac{1}{w} g\left(\frac{z+\hat{\alpha}}{w}\right) / \frac{1}{w} g\left(\frac{z+\alpha}{w}\right)$ is non decreasing in z. Arguing analogously as in the previous case, we can show that this property holds if $g'\left(\frac{z+\alpha}{w}\right) / g\left(\frac{z+\alpha}{w}\right)$ is non increasing in α at any $\alpha > 0$, where z and w are fixed. So changing variables we conclude that $\frac{\partial \eta_h^*}{\partial r} < 0$ holds if

$$\frac{g'(t)}{g(t)} \text{ is non-increasing in } t.$$

3 Exogenous government purchases

Here we extend our analysis to the case where the public expenditure policy is specified in terms of an exogenous sequence of absolute levels of expenditure $\{G_t\}_{t=0}^{\infty}$ (rather than per unit of total wealth). We will obtain conditions characterizing the Ramsey steady state which are analogous to those obtained in Proposition 5 and Corollary 6. Hence, also in the case of exogenous G_t, the capital income tax rate must be positive in the long run, as long as the effect on the saving rate is small enough.

When the sequence $\{G_t\}_{t=0}^{\infty}$ is exogenously given, we can no longer use the recursive approach followed in the paper to solve the Ramsey problem in the case where $\{g_t\}_{t=0}^{\infty}$ is exogenously given.

We solve instead the problem in a more direct way. Given X_0 and b_0, the Ramsey problem consists in the maximization of v_0 with respect to $\{b_{t+1},\ X_{t+1},\ v_{t+1},\ \widetilde{r}_{t+1},\ \widetilde{w}_{t+1}\}_{t=0}^{\infty}$ subject to

$$
\begin{aligned}
&v_t^{\psi-1} = (1-\beta)^{\psi} + \beta^{\psi}\rho_{t+1}^{\psi-1}v_{t+1}^{\psi-1} \\
&\frac{G_{t+1}}{X_t} + (1-\delta_k+\widetilde{r}_{t+1})b_t = (1-\eta_{c,t})R_{x,t+1}b_{t+1} + F(k_t,h_t) - \widetilde{r}_{t+1}k_t - \widetilde{w}_{t+1}h_t \\
&\frac{X_{t+1}}{X_t} = (1-\eta_{c,t})R_{x,t+1},
\end{aligned}
$$

where $\eta_{h,t}$, $\eta_{c,t}$, ρ_{t+1}, $R_{x,t+1}$, k_t, and h_t are the following functions of $\widetilde{r}_{t+1}$, $\widetilde{w}_{t+1}$, b_t, and v_t:

$$
\begin{aligned}
\eta_{h,t} &= \eta_h(\widetilde{r}_{t+1},\widetilde{w}_{t+1}) \equiv \arg\max_{\eta_h}\ \rho(\widetilde{r}_{t+1},\widetilde{w}_{t+1},\eta_h), \\
\rho_{t+1} &= \rho(\widetilde{r}_{t+1},\widetilde{w}_{t+1}) \equiv \max_{\eta_h}\ \rho(\widetilde{r}_{t+1},\widetilde{w}_{t+1},\eta_h), \\
R_{x,t+1} &= R_x(\widetilde{r}_{t+1},\widetilde{w}_{t+1}) \equiv (1-\delta_k+\widetilde{r}_{t+1})(1-\eta_h(\widetilde{r}_{t+1},\widetilde{w}_{t+1})) + (1-\delta_h+\widetilde{w}_{t+1})\eta_h(\widetilde{r}_{t+1},\widetilde{w}_{t+1}), \\
\eta_{c,t} &= \eta_c(v_t) \equiv (1-\beta)^{\psi}(v_t)^{1-\psi}, \\
k_t &= k(\widetilde{r}_{t+1},\widetilde{w}_{t+1},b_t,v_t) \equiv (1-\eta_c(v_t))(1-\eta_h(\widetilde{r}_{t+1},\widetilde{w}_{t+1})) - b_t, \\
h_t &= h(\widetilde{r}_{t+1},\widetilde{w}_{t+1},v_t) \equiv (1-\eta_c(v_t))\eta_h(\widetilde{r}_{t+1},\widetilde{w}_{t+1}),
\end{aligned}
$$

The Lagrangean for this problem is then:

$$
\begin{aligned}
v_0 + \sum_{t=0}^{\infty}\Bigg[&\lambda_{v,t}\Big\{(1-\beta)^{\psi} + \beta^{\psi}\rho(\widetilde{r}_{t+1},\widetilde{w}_{t+1})^{\psi-1}v_{t+1}^{\psi-1} - v_t^{\psi-1}\Big\} \\
&+ \lambda_{b,t}\Big\{[1-\eta_c(v_t)]R_x(\widetilde{r}_{t+1},\widetilde{w}_{t+1})b_{t+1} + F[k(\widetilde{r}_{t+1},\widetilde{w}_{t+1},b_t,v_t),h(\widetilde{r}_{t+1},\widetilde{w}_{t+1},v_t)] \\
&\quad - \widetilde{r}_{t+1}k(\widetilde{r}_{t+1},\widetilde{w}_{t+1},b_t,v_t) - \widetilde{w}_{t+1}h(\widetilde{r}_{t+1},\widetilde{w}_{t+1},v_t) - \frac{G_{t+1}}{X_t} - (1-\delta_k+\widetilde{r}_{t+1})b_t\Big\} \\
&+ \lambda_{x,t}\Big\{[1-\eta_c(v_t)]R_x(\widetilde{r}_{t+1},\widetilde{w}_{t+1}) - \frac{X_{t+1}}{X_t}\Big\}\Bigg].
\end{aligned}
$$

The first order conditions for v_t, b_t, and $\widetilde{r}_{t+1}$ are so, respectively,[32]

$$
\begin{aligned}
0 = &-\lambda_{v,t}\frac{v_t^{\psi-2}}{\psi-1} + \lambda_{v,t-1}\frac{\beta^{\psi}}{\psi-1}\rho_{t+1}^{\psi-1}v_{t+1}^{\psi-2} \\
&+ \lambda_{b,t}\eta_c'(v_t)\Big\{-R_{x,t+1}b_{t+1} - F_{k,t}(1-\eta_{h,t}) - F_{h,t}\eta_{h,t} + \widetilde{r}_{t+1}(1-\eta_{h,t}) + \widetilde{w}_{t+1}\eta_{h,t}\Big\} \\
&- \lambda_{x,t}\eta_c'(v_t)R_{x,t+1},
\end{aligned} \tag{57}
$$

$$
0 = \lambda_{b,t-1}(1-\eta_{c,t-1})R_{x,t} - \lambda_{b,t}(1-\delta_k+F_{k,t}), \tag{58}
$$

$$
\begin{aligned}
0 = &(\psi-1)\lambda_{v,t}\beta^{\psi}\rho_{t+1}^{\psi-2}\rho_{r,t+1}v_{t+1}^{\psi-2} \\
&+ \lambda_{b,t}\Big\{(1-\eta_{c,t})R_{x,r,t+1}b_{t+1} + F_{k,t}k_{r,t} + F_{h,t}h_{r,t} - k_t - \widetilde{r}_{t+1}k_{r,t} - \widetilde{w}_{t+1}h_{r,t} - b_t\Big\} \\
&+ \lambda_{x,t}(1-\eta_{c,t})R_{x,r,t+1},
\end{aligned} \tag{59}
$$

[32] To derive the steady state condition determining the tax rate on capital we do not have to use the first-order conditions with respect to $\widetilde{w}_{t+1}$ or X_{t+1}. But, of course, we would need those conditions to derive all the steady state equilibrium variables.

where $\eta_c'(v_t) \equiv d\eta_c(v_t)/dv_t$, $F_{k,t} \equiv \partial F(k_t, h_t)/\partial k_t$, $F_{h,t} \equiv \partial F(k_t, h_t)/\partial h_t$, $\rho_{r,t+1} \equiv \partial \rho(\widetilde{r}_{t+1}, \widetilde{w}_{t+1})/\partial \widetilde{r}_{t+1}$, $R_{x,r,t+1} \equiv \partial R_x(\widetilde{r}_{t+1}, \widetilde{w}_{t+1})/\partial \widetilde{r}_{t+1}$, $k_{r,t} \equiv \partial k(\widetilde{r}_{t+1}, \widetilde{w}_{t+1}, b_t, v_t)/\partial \widetilde{r}_{t+1}$, and $h_{r,t} \equiv \partial h(\widetilde{r}_{t+1}, \widetilde{w}_{t+1}, v_t)/\partial \widetilde{r}_{t+1}$.

Assuming that G_t grows at an exogenous, constant rate $\gamma_G > 0$, we focus again our attention on a steady state (balanced growth path) where all the variables in equations (57)-(59) remain constant, except for the Lagrange multipliers, $\lambda_{v,t}$, $\lambda_{b,t}$, and $\lambda_{x,t}$ that grow at the same rate:

$$\frac{\lambda_{v,t}}{\lambda_{v,t-1}} = \frac{\lambda_{b,t}}{\lambda_{b,t-1}} = \frac{\lambda_{x,t}}{\lambda_{x,t-1}} \equiv \gamma_\lambda.$$

Since ρ is constant we have $v = (1-\beta)^\psi/(1-\beta^\psi \rho^{\psi-1})$. Also, $\eta_c = (1-\beta)^\psi v^{1-\psi}$, and so

$$\beta^\psi \rho^{\psi-1} = 1 - \eta_c.$$

It then follows from equation (57) that, along a balanced growth path,

$$\frac{\lambda_{v,t}}{\lambda_{v,t-1}} = (1-\eta_c) + \Lambda \eta_c'(v),$$

where Λ is the term

$$\Lambda \equiv \frac{\psi - 1}{v^{\psi-2}} \left[\frac{\lambda_{b,t}}{\lambda_{v,t-1}} \left\{ -R_x b - F_k(1-\eta_h) - F_h \eta_h + \widetilde{r}(1-\eta_h) + \widetilde{w}\eta_h \right\} - \frac{\lambda_{x,t}}{\lambda_{v,t-1}} R_x \right],$$

a constant given the fact that all Lagrange multipliers grow at the same rate.

We can then use equation (58) to derive the following steady-state condition which is the counterpart of the one in Proposition 5:

$$R_x = (1 - \delta_k + F_k) \left[1 + \frac{\Lambda \eta_c'(v)}{1-\eta_c} \right]. \tag{60}$$

Just as in the case of a constant, exogenously given level of g, this condition implies that at a Ramsey steady state the average rate of return on consumers' portfolios, R_x, is equal to the before tax return on physical capital (or equivalently the cost of government debt), $1 - \delta_k + F_k$, augmented with the effect of public debt on the saving rate, $\Lambda \eta_c'/(1-\eta_c)$. As long as the latter effect is small, we get again $R_x \approx 1 - \delta_k + F_k$, which implies that the optimal capital tax rate is positive in the long run: $\tau_k > 0$.

When $\psi = 1$, again the effect on the saving rate valishes, so that condition (60) reduces to

$$R_x = 1 - \delta_k + F_k,$$

which is identical to the condition derived in Corollary 6.

4 Algorithm to solve the model numerically

The Ramsey equilibrium for our model can be computed in a straightforward way. The function $\rho^R(b, b', \eta_c)$ is computed as the solution to the maximization problem defined in (32). Then the steady state value of b is obtained by solving equation (39).

The transitional dynamics is computed for the calibrated economy where $\psi = 1$. In this case η_c is constant, so the function above can be written simply as $\rho^R(b, b')$ and (30) simplifies to

$$\ln(v_0) = \sum_{t=0}^{\infty} \beta^{t+1} \ln(\rho_{t+1}).$$

In the dynamic programming formulation, the Ramsey problem (33) can be written as

$$\ln v(b) = \max_{b'} \beta \ln \rho^R(b, b') + \beta \ln v(b').$$

This problem is solved by discretizing the state space and by the value function iteration.

5 Transitional dynamics

The Ramsey equilibrium converges to the steady state only in one period. Figure 1 in this appendix illustrates the transitional dynamics of the Ramsey equilibrium, starting from the "baseline equilibrium" in Table 2 in the main text.

Figure 1: Transitional dynamics of the Ramsey equilibrium starting from the baseline equilibrium.