Online Appendix to the paper

Efficient Firm Dynamics in a Frictional Labor Market

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Appendix A: Proofs

Proposition 1: *Consider recruitment cost functions satisfying property (C). The firm's value function $J^x(L,W)$ is strictly increasing and strictly concave in its workforce L, strictly increasing in productivity x, strictly supermodular in (x,L) and decreasing in the worker's search value ρ. The job-filling rate $m^x(L)$ is strictly increasing in productivity x and strictly decreasing in the workforce L. Posted vacancies $V^x(m,L)$ are increasing in L and strictly increasing in the desired job-filling rate m.*

Proof of Proposition 1:

Rewrite the firm's problem to express the dependence of the value function on x and on the workers' search value ρ as the solution to the dynamic programming problem

$$\begin{aligned} J(L,x;\rho) &= \max_{(m,V)\geq 0} xF(L) - C(V,L,x) - D(m;\rho)V + \beta(1-\delta)J(L_+,x;\rho) \\ & \quad \text{s.t. } L_+ = L(1-s) + mV \ , \end{aligned} \tag{1}$$

where function $D(m;\rho)$ is defined in the text. It is increasing, strictly convex in m and increasing in ρ. This problem is equivalently defined on a compact state space $L \in [0,\overline{L}]$ where $\overline{L}$ is so large that it never binds. This is possible because of the Inada condition $\lim_{L\to\infty} F'(L) = 0$. The RHS in problem (1) defines an operator T which maps a continuous function $J_0(L,x;\rho)$, defined on $\mathcal{S} = [0,\overline{L}] \times [0,\overline{x}] \times [0,\overline{\rho}]$ into a continuous function $J_1(L,x;\rho) = T(J_0)(L,x;\rho)$ defined on the same domain. Here $\overline{x}$ and $\overline{\rho}$ are arbitrary upper bounds on x and ρ. Operator T is a contraction, therefore there exists a unique fixed point J^* which is a continuous function and which is the limit of any sequence J_n defined by $J_n = T(J_{n-1})$.

Starting from a continuous J_0 that is differentiable and weakly increasing in L and x and weakly decreasing in ρ, successive application of T yields a sequence J_n where each element shares these properties. Since the subset of continuous functions on $\mathcal{S}$ that are weakly increasing in L and x and weakly decreasing in ρ is closed under the sup norm, the limit J^* of sequence J_n is in this set. Because $xF(L) - C(V,L,x)$ is strictly increasing in (L,x) and since $D(m;\rho)$ is strictly increasing in ρ, the limit J^* is strictly increasing in x and L and strictly decreasing in ρ.

We show in subsequent Lemmata A.1 and A.2 that T maps functions that are differentiable and concave in L and supermodular in L and x into functions with the same properties. Since the subset of concave and supermodular functions is closed, the same arguments as above imply that the unique fixed point J^* is concave in L and supermodular in (L, x). Since function $xF(L) - C(V, L, x)$ is strictly concave in L, J^* is also strictly concave in L. Concavity in L and differentiability of $xF(L) - C(V, L, x)$ together with the theorem of Benveniste and Scheinkman establishes differentiability of J^* in L.

Before we establish the remaining results, rewrite (1) in terms of hirings $H = mV$. Dropping argument ρ from J, we can equivalently write (1) as

$$J(L, x) = \max_H xF(L) - \mathcal{C}(H, L, x) + \beta(1-\delta)J(L(1-s) + H, x) \tag{2}$$

where

$$\mathcal{C}(H, L, x) \equiv \min_m C\Big(\frac{H}{m}, L, x\Big) + D(m)\frac{H}{m} \ . \tag{3}$$

The right hand side of (2) is an equivalent expression of the fixed-point operator T. As will become clear, the per period return $xF(L) - \mathcal{C}(H, L, x)$ is supermodular in (L, H), but when $C_{13} > 0$ (which arises in first specification in equation (1) of the main text for $h > 0$) the per period return is strictly submodular in (H, x) and in (L_+, x) when one writes $H = L_+ - (1 - s)L$, which renders standard tools to prove supermodularity (e.g., Amir (1996)) inapplicable. To proceed, the optimality condition for problem (3) is

$$C_1\Big(\frac{H}{m}, L, x\Big) = D'(m)m - D(m) \ . \tag{4}$$

Differentiate this equation to obtain

$$\frac{dm}{dH} = \frac{C_{11}}{C_{11}\frac{H}{m} + D''^2} > 0 \ , \tag{5}$$

$$\frac{dm}{dL} = \frac{C_{12}m}{C_{11}\frac{H}{m} + D''^2} = \frac{C_{12}m}{C_{11}}\frac{dm}{dH} \leq 0 \ , \tag{6}$$

$$\frac{dm}{dx} = \frac{C_{13}m}{C_{11}\frac{H}{m} + D''^2} = \frac{C_{13}m}{C_{11}}\frac{dm}{dH} \geq 0 \ . \tag{7}$$

Therefore, we can express the derivatives of cost function $\mathcal{C}$ as

$$\begin{aligned}
\mathcal{C}_1 &= D'(m) > 0 \ , \\
\mathcal{C}_2 &= C_2 \ , \\
\mathcal{C}_{11} &= D''(m)\frac{dm}{dH} > 0 \ , && (8) \\
\mathcal{C}_{12} &= D''(m)\frac{dm}{dL} \leq 0 \ , && (9) \\
\mathcal{C}_{22} &= C_{22} - C_{12}\frac{H}{m^2}\frac{dm}{dL} \ , && (10) \\
\mathcal{C}_{13} &= D''(m)\frac{dm}{dx} \geq 0 \ , && (11) \\
\mathcal{C}_{23} &= C_{23} - C_{12}\frac{H}{m^2}\frac{dm}{dx} \ . && (12)
\end{aligned}$$

Lemma A.1: *Suppose that J is twice differentiable and concave in L. Then $T(J)$ is twice differentiable and*

(a) *concave in L if the following condition holds:*

$$\mathcal{C}_{12}^2 + \mathcal{C}_{11}[xF'' - \mathcal{C}_{22}] \leq 0 \ . \qquad (13)$$

(b) *concave in L and supermodular in (L, x) if J is supermodular in (L, x) and if (13) and the following condition hold:*

$$\mathcal{C}_{12}\mathcal{C}_{13} + \mathcal{C}_{11}[F' - \mathcal{C}_{23}] \geq 0 \ . \qquad (14)$$

Lemma A.2:

(a) *Condition (13) holds under the following condition on the original cost function C:*

$$C_{12}^2 + C_{11}[xF'' - C_{22}] \leq 0 \ . \qquad (15)$$

(b) *Condition (14) holds under the following condition on the original cost function C:*

$$C_{12}C_{13} + C_{11}[F' - C_{23}] \geq 0 \ . \qquad (16)$$

Proof of Lemma A.1:
Consider $T(J)$ defined by the RHS of (2).
Part (a). Since J is a concave and twice differentiable function of L, $T(J)$ is also twice differentiable, and a policy function exists and is differentiable. Differentiate $T(J)$ twice with respect to L to obtain

$$\frac{d^2(T(J))}{dL^2} = xF'' - \mathcal{C}_{22} + \beta\varphi(1-s)J_{11} + \Big[-\mathcal{C}_{12} + \beta\varphi J_{11}\Big]\frac{dH}{dL} \ . \tag{17}$$

Differentiate the FOC $\mathcal{C}_1 = \beta(1-\delta)J_1$ with respect to L to obtain

$$\frac{dH}{dL} = \frac{\beta\varphi J_{11} - \mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)J_{11}} \ . \tag{18}$$

Substitute this into (17) to obtain

$$\frac{d^2(T(J))}{dL^2} = xF'' - \mathcal{C}_{22} + \frac{\beta\varphi(1-s)J_{11}\mathcal{C}_{11} + \mathcal{C}_{12}^2 - 2\beta\varphi J_{11}\mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)J_{11}} \ .$$

In the last term, the denominator is positive and larger than $\mathcal{C}_{11}$. In the numerator, all terms involving J_{11} are negative (due to (8) and (9)); hence the numerator is smaller than $\mathcal{C}_{12}^2$. Therefore,

$$\frac{d^2(T(J))}{dL^2} \le xF'' - \mathcal{C}_{22} + \frac{\mathcal{C}_{12}^2}{\mathcal{C}_{11}} \ ,$$

which is non-positive under (13). Hence, T maps a concave and twice differentiable function into a function with the same properties.
Part (b). Since J is a concave, supermodular and twice differentiable function of (L, x), $T(J)$ is twice differentiable and a differentiable policy function exists. Differentiate $T(J)$ twice with respect to L and x to obtain

$$\frac{d^2(T(J))}{dLdx} = F' - \mathcal{C}_{23} + \beta\varphi J_{12} + \Big[-\mathcal{C}_{12} + \beta\varphi J_{11}\Big]\frac{dH}{dx} \ . \tag{19}$$

Differentiate the FOC $\mathcal{C}_1 = \beta(1-\delta)J_1$ with respect to x to obtain

$$\frac{dH}{dx} = \frac{\beta(1-\delta)J_{12} - \mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)J_{11}} \ . \tag{20}$$

Substitute this into (19) to obtain

$$\frac{d^2(T(J))}{dLdx} = F' - \mathcal{C}_{23} + \frac{\beta\varphi J_{12}\mathcal{C}_{11} + \mathcal{C}_{12}\mathcal{C}_{13} - \beta(1-\delta)J_{12}\mathcal{C}_{12} - \beta\varphi J_{11}\mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)J_{11}} \ .$$

In the last term, the denominator is positive and larger than $\mathcal{C}_{11}$. In the numerator, all terms involving J_{11} and J_{12} are non-negative (due to (8), (9) and (11)); hence the numerator is greater than $\mathcal{C}_{12}\mathcal{C}_{13} \leq 0$. Therefore,

$$\frac{d^2(T(J))}{dLdx} \geq F' - \mathcal{C}_{23} + \frac{\mathcal{C}_{12}\mathcal{C}_{13}}{\mathcal{C}_{11}} ,$$

which is non-negative under (14). Hence, $T(J)$ is supermodular. □

Proof of Lemma A.2:
From (6), (7), (8), (9) and (11) follows that

$$\mathcal{C}_{12} = \frac{\mathcal{C}_{11}C_{12}m}{C_{11}} , \quad (21)$$

$$\mathcal{C}_{13} = \frac{\mathcal{C}_{11}C_{13}m}{C_{11}} . \quad (22)$$

Furthermore, substituting (9) into (6), and substituting (11) into (7) to eliminate $D''(m)$ imply that

$$\mathcal{C}_{22} = C_{22} - \frac{C_{12}^2}{C_{11}} + \frac{mC_{12}}{C_{11}}\mathcal{C}_{12} , \quad (23)$$

$$\mathcal{C}_{23} = C_{23} - \frac{C_{12}}{C_{11}}\Big[C_{13} - m\mathcal{C}_{13}\Big] . \quad (24)$$

Part (a): Rewrite (13) using (21) and (23) to obtain the equivalent condition

$$xF'' - C_{22} + \frac{C_{12}^2}{C_{11}} \leq 0 .$$

Because of $C_{11} > 0$, this condition is equivalent to (15).
Part (b): Rewrite (14) using (21), (22) and (24) to obtain the equivalent condition

$$F' - C_{23} + \frac{C_{12}C_{13}}{C_{11}} \geq 0 .$$

Because of $C_{11} > 0$, this condition is equivalent to (16). □

Proof of Proposition 1 (continued):
It follows from Lemma A.1 and A.2 that the value function $J(L, x)$ is concave in L and supermodular in (L, x) because property (C) together with the assumption that $xF(.) - C(.)$ is concave in (L, V) guarantee both (15) and (16).
Because of strict concavity of problem (1), policy functions $m^x(L)$ and $V^x(m^x(L), L)$ exist. To derive the first-order conditions (5) and (6) of the main text is straight-

forward: The first condition directly follows from (4); the second follows from the intertemporal optimality condition $\mathcal{C}_1(H,L,x) = \beta(1-\delta)J_1(L(1-s)+H,x)$ and from using the envelope theorem and the first condition.

The properties of V^x stated in Proposition 1 were already established in the main text. To see how $m^x(L)$ depends on L, use (6) and (18) to get

$$\frac{dm^x(L)}{dL} = \frac{dm(H,L,x)}{dL} + \frac{dm(H,L,x)}{dH}\frac{dH}{dL} = \frac{dm}{dH}\left[\frac{C_{12}m}{C_{11}} + \frac{\beta\varphi J_{11} - \mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)J_{11}}\right] .$$

Because of

$$\frac{C_{12}m}{C_{11}} = \frac{\mathcal{C}_{12}}{\mathcal{C}_{11}} \leq \frac{\mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)J_{11}} ,$$

the term in [.] is negative, and so is $dm^x/(dL)$.

To verify that m is increasing in x, use (7) and (20) to get

$$\frac{dm^x(L)}{dx} = \frac{dm(H,L,x)}{dx} + \frac{dm(H,L,x)}{dH}\frac{dH}{dx} = \frac{dm}{dH}\left[\frac{C_{13}m}{C_{11}} + \frac{\beta(1-\delta)J_{12} - \mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)J_{11}}\right] .$$

Because of

$$\frac{C_{13}m}{C_{11}} = \frac{\mathcal{C}_{13}}{\mathcal{C}_{11}} \geq \frac{\mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)J_{11}} ,$$

the term in [.] is positive, and so is $dm^x/(dx)$. □

Corollary 2: *If recruitment costs are given by either specification in (1) of the main text with parameter h sufficiently small, more productive firms have a higher growth rate, conditional on size; and larger/older firms have a lower growth rate, conditional on productivity.*

Proof of Corollary 2:

Because of exogenous separations, the growth rate of a firm, $[mV - sL]/L$ is perfectly correlated with the job-creation rate,

$$\text{JCR}(x,L) = m^x(L)\frac{V^x(m^x(L),L)}{L} .$$

Differentiation of the job-creation rate with respect to x implies

$$\frac{d\text{JCR}}{dx} = \frac{dm^x}{dx}\frac{V^x}{L} + \frac{m^x}{L}\frac{dV^x}{dx} + \frac{m^x}{L}\frac{dV^x}{dm}\frac{dm^x}{dx} .$$

In this expression, the first and the third term are strictly positive. Under the second cost function in (1) of the main text, the second term is zero. Under the first cost function, the second term is zero when $h = 0$, and negative but small if h is small. Thus, $d\text{JCR}/(dx)$ is positive if h is sufficiently small.

Differentiation of the job-creation rate with respect to L implies

$$\frac{d\text{JCR}}{dL} = \frac{dm^x}{dL}\frac{V^x}{L} + \frac{m^x}{L}\frac{dV^x}{dL} + \frac{m^x}{L}\frac{dV^x}{dm}\frac{dm^x}{dL} - m\frac{V^x}{L^2} \ .$$

In this expression, the first, the third and the fourth term are strictly negative. Under the second cost function in (1) of the main text, $\frac{dV^x}{dL} = \frac{V^x}{L}$, and the second and forth terms cancel out. Under the first cost function, the second term is zero when $h = 0$, and positive but small if h is small. Thus, $d\text{JCR}/(dL)$ is negative if h is sufficiently small. □

Lemma A.3: *In the stationary model with recruitment cost* $C(V, L, x) = xF(L) - xF(L-hV) + cV$*, job-filling rates in the optimal firm's problem follow the dynamic equation*

$$\rho\Big[m_{t+1}\lambda'(m_{t+1}) - \lambda(m_{t+1})\Big] - (b+\rho)h - c = \frac{\rho h}{\beta(1-\delta)}\Big[\lambda'(m_t) - \beta\varphi\lambda'(m_{t+1})\Big]. \quad (25)$$

It has a unique steady state solution $m^* > 0$ *if, and only if,*

$$h < \frac{\beta(1-\delta)\overline{m}}{1-\beta\varphi} \ , \quad (26)$$

with $\overline{m} \equiv \lim_{m\to 1} m - \dfrac{\lambda(m)}{\lambda'(m)} > 0$*. Under this condition, any sequence* $m_t > 0$ *satisfying this dynamic equation converges to* m^**.*

Proof of Lemma A.3:

It is straightforward to derive (25) by substitution of first-order condition (5) of the main text into condition (6) of the main text. A steady state m^* must satisfy the condition

$$\rho\Big[m - \frac{\lambda(m)}{\lambda'(m)}\Big] = \frac{\rho h(1-\beta\varphi)}{\beta(1-\delta)} + \frac{(b+\rho)h + c}{\lambda'(m)} \ . \quad (27)$$

The LHS is strictly increasing and goes from 0 to $\rho\overline{m}$ as m goes from 0 to 1. The RHS is decreasing in m with limit $\rho h(1-\beta\varphi)/[\beta(1-\delta)]$ for $m \to 1$. Hence, a unique steady state m^* exists iff (26) holds.[1] Furthermore, differentiation of (25) at m^* implies that

$$\left.\frac{dm_{t+1}}{dm_t}\right|_{m^*} = \frac{h}{\beta(1-\delta)m^* + h\beta\varphi} \ ,$$

[1] If this condition fails, firms cannot profitably recruit workers.

which is positive and smaller than one iff

$$h < \frac{\beta(1-\delta)m^*}{1-\beta\varphi} \ .$$

But this inequality must be true because (27) implies

$$h = \frac{\rho[m^*\lambda'(m^*) - \lambda(m^*)] - c}{\frac{\rho[1-\beta\varphi]}{\beta(1-\delta)}\lambda'(m^*)+b+\rho} < \frac{\beta(1-\delta)m^*}{1-\beta\varphi} \ .$$

Therefore, the steady state m^* is locally stable. Moreover, the dynamic equation defines a continuous, increasing relation between m_{t+1} and m_t which has only one intersection with the 45-degree line. Hence, $m_{t+1} > m_t$ for any $m_t < m^*$ and $m_{t+1} < m_t$ for any $m_t > m^*$, which implies that m_t converges to m^* from any initial value $m_0 > 0$. □

Proposition 2: *A stationary competitive search equilibrium exists and is unique. There is strictly positive firm entry provided that K is sufficiently small.*

Proof of Proposition 2:

It remains to prove existence and uniqueness. From Proposition 1 follows that the entrant's value function $J^x(0,0)$ is decreasing and continuous in ρ. Hence the expected profit prior to entry,

$$\Pi^*(\rho) \equiv \sum_{x\in\mathcal{X}} \pi(x) J^x(0,0)$$

is a decreasing and continuous function of ρ. Moreover, the function is strictly decreasing in ρ whenever it is positive. This also follows from the proof of Proposition 1 which shows that $J(0,x;\rho)$ is *strictly* decreasing in ρ when the new firm x recruits workers ($V^x(m^x(0),0) > 0$). If no new firm recruits workers, expected profit of an entrant cannot be positive. Hence, equation (7) of the main text can have at most one solution for any $K > 0$. This implies uniqueness, with entry of firms if equation (7) of the main text can be fulfilled or without entry of firms otherwise. A solution to this equation exists provided that K is sufficiently small. To see this, $\Pi^*(0)$ is strictly positive because of $F'(0) = \infty$: some entrants will recruit workers since the marginal product $J_1(mV, x; \rho)$ is sufficiently large relative to the cost of recruitment and relative to the wage cost which are, for $\rho = 0$, equal to mVb (see equation (1)). But when $\Pi^*(0) > 0$, a sufficiently small value of K guarantees that equation (7) of the main text has a solution since $\lim_{\rho\to\infty} \Pi^*(\rho) = 0$. □

Proposition 3: *The stationary competitive search equilibrium is socially optimal.*

Proof of Proposition 3:

The social planner decides at each point in time about firm entry, vacancy postings and job-filling rates for all firms. The planner takes as given the numbers of firms that entered in some earlier period, as well as the employment stocks of all these firms. Formally, the planner's state vector is $\sigma = (N_a, L_a^x)_{a\geq 1, x\in\mathcal{X}}$ where N_a is the mass of firms of age $a \geq 1$, and L_a^x is employment of a firm with productivity x and age a. The planner maximizes the present value of output net of opportunity costs of employment and net of the costs of entry and recruitment, subject to the economy's resource constraint. With $\sigma_+ = (N_{a,+}, L_{a,+}^x)_{a\geq 1, x\in\mathcal{X}}$ denoting the state vector in the next period, the recursive formulation of the social planning problem is

$$\begin{aligned} S(\sigma) \;=\; & \max_{N_0,(V_a^x,m_a^x)_{a\geq 0}} \left\{ \sum_{a\geq 0} N_a \sum_{x\in\mathcal{X}} \pi(x) \Big[xF(L_a^x) - bL_a^x - C(V_a^x, L_a^x, x) \Big] \right\} \\ & \quad -KN_0 + \beta S(\sigma_+) \qquad (28) \\ & \text{s.t. } L_0^x = 0,\; L_{a+1,+}^x = (1-s)L_a^x + m_a^x V_a^x,\; a\geq 0,\; x\in\mathcal{X}\;, \\ & N_{a+1,+} = (1-\delta)N_a\;,\; a\geq 0\;, \\ & \sum_{a\geq 0} N_a \sum_{x\in\mathcal{X}} \pi(x) \Big(L_a^x + \lambda(m_a^x) V_a^x \Big) \leq 1\;. \end{aligned}$$

We now show that the first-order conditions that uniquely characterize the decentralized allocation are also first order conditions to the planner's problem. The same argument that we use in the proof of Lemma A.4 part (b) (see the proof of Proposition 4) then establishes that the planner cannot improve upon this allocation. We denote by $S_{N,a}$ the derivative of S with respect to N_a and by $S_{L,a,x}$ the derivative of S with respect to L_a^x. The multiplier on the resource constraint is $\mu \geq 0$. First-order conditions with respect to N_0, V_a^x, and m_a^x, $a \geq 0$, are

$$\sum_{x\in\mathcal{X}} \pi(x) \Big[xF(0) - C(V_0^x, 0, x) \Big] - K + \beta(1-\delta)S_{N,1} - \mu \sum_{x\in\mathcal{X}} \pi(x)\lambda(m_0^x)V_0^x = 0 \;, \qquad (29)$$

$$-N_a\pi(x)\Big[C_1(V_a^x, L_a^x, x) + \mu\lambda(m_a^x) \Big] + \beta S_{L,a+1,x} m_a^x \leq 0\;,\; V_a^x \geq 0\;, \qquad (30)$$

$$\beta S_{L,a+1,x} - \mu N_a \pi(x) \lambda'(m_a^x) = 0\;. \qquad (31)$$

Here condition (30) holds with complementary slackness. The envelope conditions are, for $a \geq 1$ and $x \in \mathcal{X}$,

$$S_{L,a,x} = N_a \pi(x) \Big[xF'(L_a^x) - C_2'(V_a^x, L_a^x, x) - b - \mu \Big] + \beta(1-s) S_{L,a+1,x} \ , \quad (32)$$

$$S_{N,a} = \sum_{x \in \mathcal{X}} \pi(x) \Big[xF(L_a^x) - C(V_a^x, L_a^x, x) - bL_a^x \Big] - \mu \sum_{x \in \mathcal{X}} \pi(x) \Big(L_a^x + \lambda(m_a^x) V_a^x \Big) + \beta(1-\delta) S_{N,a+1} \ . \quad (33)$$

Use (31) to substitute $S_{L,a,x}$ into (32) to obtain

$$xF'(L_{a+1}^x) - C_2(V_{a+1}^x, L_{a+1}^x, x) - b - \mu = \frac{\mu}{\beta(1-\delta)} [\lambda'(m_a^x) - \beta\varphi\lambda'(m_{a+1}^x)] \ .$$

This equation is the planner's intertemporal optimality condition; it coincides with equation (6) of the main text for $\mu = \rho$. This is intuitive: when the social value of an unemployed worker μ coincides with the surplus value that an unemployed worker obtains in search equilibrium, the firm's recruitment policy is efficient. Next substitute (31) into (30) to obtain, for $a \geq 0$ and $x \in \mathcal{X}$,

$$C_1(V_a^x, L_a^x, x) \geq \mu[m_a^x \lambda'(m_a^x) - \lambda(m_a^x)] \ , \ V_a^x \geq 0 \ . \quad (34)$$

Again for $\mu = \rho$, this condition coincides with the firm's intratemporal optimality condition in competitive search equilibrium, equation (5) of the main text. Lastly, it remains to verify that entry is socially efficient when the value of a jobless worker is $\mu = \rho$. The planner's choice of firm entry, condition (29), together with the recursive equation for the marginal firm surplus $S_{N,a}$, equation (33), shows that

$$K = \sum_{a \geq 0} [\beta(1-\delta)]^a \sum_{x \in \mathcal{X}} \pi(x) \Big[xF(L_a^x) - bL_a^x - C(V_a^x, L_a^x, x) - \mu(L_a^x + \lambda(m_a^x) V_a^x) \Big] \ . \quad (35)$$

On the other hand, the expected profit value of a new firm is

$$\sum_{x \in \mathcal{X}} \pi(x) J^x(0,0) = \sum_{a \geq 0} [\beta(1-\delta)]^a \sum_{x \in \mathcal{X}} \pi(x) \Big[xF(L_a^x) - W_a^x - C(V_a^x, L_a^x, x) \Big] \ .$$

Hence, the free-entry condition in search equilibrium, equation (7) of the main text, coincides with condition (35) for $\mu = \rho$ if, for all $x \in \mathcal{X}$,

$$\sum_{a \geq 0} [\beta(1-\delta)]^a \Big[(b + \mu) L_a^x + \mu \lambda(m_a^x) V_a^x - W_a^x \Big] = 0 \ . \quad (36)$$

Now after substitution of

$$L_a^x = \sum_{k=0}^{a-1}(1-s)^{a-1-k} m_k^x V_k^x \ , \text{ and}$$
$$W_a^x = \sum_{k=0}^{a-1}(1-s)^{a-1-k} V_k^x \Big[\frac{\rho\lambda(m_k^x)(1-\beta\varphi)}{\beta(1-\delta)} + m_k^x(b+\rho)\Big]$$

into (36), it is straightforward to see that the equation is satisfied for $\mu = \rho$. □

Proposition 4:

(a) *Suppose that a solution of equations (14) and (15) of the main text exists with associated allocation $\mathbf{A} = (\mathbf{N}, \mathbf{L}, \mathbf{V}, \mathbf{m}, \mathbf{s}, \boldsymbol{\delta})$ satisfying $N(z^t) > 0$ for all z^t. Then $\mathbf{A}$ is a solution of the sequential planning problem (12) of the main text.*

(b) *If $K(z)$, f, and b are sufficiently small and if $z_1 = \ldots = z_n = \overline{z}$, equations (14) and (15) of the main text have a unique solution $(G, \mathbf{M})$. Moreover, if the transition matrix $\psi(z_j|z_i)$ is strictly diagonally dominant and if $|z_i - \overline{z}|$ is sufficiently small for all i, equations (14) and (15) of the main text have a unique solution.*

Proof of Proposition 4:

Part (a):

Let $\beta^t \psi(z^t)\mu(z^t) \geq 0$ be the multiplier on the resource constraint (13) of the main text in history node z^t. That is, $\mu(z^t)$ is the social value of a worker in history z^t. Write $\boldsymbol{\mu} = (\mu(z^t))$ for the vector of multipliers. Let $G_t(L, x, z^t)$ denote the social value of an existing firm with employment stock L, idiosyncratic productivity x and aggregate productivity history z^t. The sequence G_t obeys the recursive equations

$$G_t(L, x, z^t) = \max_{\delta,s,V,m} xz_t F(L) - bL - \mu(z^t)[L + \lambda(m)V] - C(V, L, xz_t) - f \quad (37)$$
$$+\beta(1-\delta)\mathrm{E}_{x,z^t} G_{t+1}(L_+, x_+, z^{t+1})$$
$$\text{s.t.} \quad L_+ = (1-s)L + mV \ ,$$
$$\delta \in [\delta_0, 1], \ s \in [s_0, 1], \ m \in [0,1], \ V \geq 0 \ .$$

We first prove the equivalence between problem (37) and the planner's problem

(12) of the main text (Lemma A.4). Then we show that the reduced problem (14) of the main text solves (37) if entry is positive in all states.

Lemma A.4:

(a) *For given multipliers* $\mu(z^t)$*, there exist value functions* $G_t : \mathbb{R}_+ \times \mathcal{X} \times \mathcal{Z}^{t+1} \to \mathbb{R}$*,* $t \geq 0$*, satisfying the system of recursive equations (37).*

(b) *If* $\mathbf{X} = (\mathbf{N}, \mathbf{L}, \mathbf{V}, \mathbf{m}, \mathbf{s}, \boldsymbol{\delta})$ *is a solution of the planning problem (12) of the main text with multipliers* $\boldsymbol{\mu} = (\mu(z^t))$*, then the corresponding firm policies also solve problem (37) and the complementary-slackness condition*

$$\sum_{x \in \mathcal{X}} \pi_0(x) G_t(0, x, z^t) \leq K(z_t) \ , \ N_0(z^t) \geq 0 \ , \tag{38}$$

is satisfied for all z^t*. Conversely, if* $\mathbf{X}$ *solves for every firm problem (37) with multipliers* $\boldsymbol{\mu}$*, and if condition (38) and the resource constraint (13) of the main text hold for all* z^t*, then* $\mathbf{X}$ *is a solution of the planning problem (12) of the main text.*

Proof of Lemma A.4:
Part (a): The RHS in the system of equations in (37) defines an operator T which maps a sequence of bounded functions $G = (G_t)_{t \geq 0}$, with $G_t : [0, \overline{L}] \times \mathcal{X} \times \mathcal{Z}^t \to \mathbb{R}$ such that $\|G\| \equiv \sup_t \|G_t\| < \infty$, into another sequence of bounded functions $\tilde{G} = (\tilde{G}_t)_{t \geq 0}$ with $\|\tilde{G}\| = \sup_t \|\tilde{G}_t\| < \infty$. Here $\overline{L}$ is sufficiently large such that the bound $L_+ \leq \overline{L}$ does not bind for any $L \in [0, \overline{L}]$. The existence of $\overline{L}$ follows from the Inada condition for F: the marginal product of an additional worker $xzF'(L_+) - b$ must be negative for any $x \in \mathcal{X}$, $z \in \mathcal{Z}$, for all $L_+ \geq \overline{L}$ with sufficiently large $\overline{L}$; hence no hiring will occur beyond $\overline{L}$. Because the operator satisfies Blackwell's sufficient conditions, it is a contraction in the space of bounded function sequences G. Hence, the operator T has a unique fixed point which is a sequence of bounded functions.
Part (b): Take first a solution $\mathbf{X}$ of the planning problem, and write $\beta^t \psi(z^t) \mu(z^t) \geq 0$ for the multipliers on constraints (13) of the main text. Then $\mathbf{X}$ maximizes the Lagrange function

$$\begin{aligned} \mathcal{L} \ = \ \max \sum_{t \geq 0, z^t} \beta^t \psi(z^t) \Bigg\{ & - K(z_t) N_0(z^t) + \sum_{a \geq 0, x^a} N(x^a, z^t) \Big[x_a z_t F(L(x^a, z^t)) - bL(x^a, z^t) \\ & - f - C(V(x^a, z^t), L(x^a, z^t), x_a z_t) - \mu(z^t) \Big[L(x^a, z^t) + \lambda(m(x^a, z^t)) V(x^a, z^t) \Big] \Big] \Bigg\} \end{aligned}$$

For each individual firm, this problem is the sequential formulation of the recursive problem (37) with multipliers $\mu(z^t)$. Hence, firm policies also solve the recursive problem; furthermore, the maximum of the Lagrange function is the same as the sum of the social values of entrant firms plus the social values of firms which already exist at $t = 0$, namely,

$$\begin{aligned}\mathcal{L} &= \max_{N_0(.)} \sum_{t,z^t} \beta^t \psi(z^t) N_0(z^t) \Big[-K(z_t) + \sum_x \pi_0(x) G_t(0, x, z^t) \Big] \\ &\quad + \sum_{z \in \mathcal{Z}} \psi(z^0) \sum_{a \geq 1, x^a} N(x^a, z^0) G_0(L(x^a, z^0), x_a, z^0) \ .\end{aligned}$$

This also proves that the complementary-slackness condition (38) describes optimal entry.
To prove the converse, suppose that $\mathbf{X}$ solves for every firm the recursive problem (37) with given multipliers $\mu(z^t)$, and that (38) and the resource constraints (13) of the main text are satisfied. We prove that $\mathbf{X}$ also solves the original planning problem (12) subject to (13) (main text) by contradiction: Suppose that there is an allocation $\mathbf{X}'$ is feasible for problem (12) under constraint (13) (main text) and strictly dominates $\mathbf{X}$. Write

$$O(x^a, z^t) \equiv x_a z_t F(L(x^a, z^t)) - bL(x^a, z^t) - f - C(V(x^a, z^t), L(x^a, z^t), x_a z_t)$$

for the net output created by firm (x^a, z^t) in allocation $\mathbf{X}$ and write $O'(x^a, z^t)$ for the same object in allocation $\mathbf{X}'$. Further, write S for the total surplus value in allocation $\mathbf{X}$ and write $S' > S$ for the surplus value in allocation $\mathbf{X}'$. Then

$$\begin{aligned}S' &= \sum_{t \geq 0, z^t} \beta^t \psi(z^t) \Big\{ -K(z_t) N_0'(z^t) + \sum_{a \geq 0, x^a} N'(x^a, z^t) O'(x^a, z^t) \Big\} \\ &= \sum_{t \geq 0, z^t} \beta^t \psi(z^t) \Big\{ -K(z_t) N_0'(z^t) + \mu(z^t) - \mu(z^t) + \sum_{a \geq 0, x^a} N'(x^a, z^t) O'(x^a, z^t) \Big\} \\ &\leq \sum_{t \geq 0, z^t} \beta^t \psi(z^t) \Big\{ -K(z_t) N_0'(z^t) + \mu(z^t) \\ &\quad + \sum_{a \geq 0, x^a} N'(x^a, z^t) \Big[O'(x^a, z^t) - \mu(z^t) \Big(L'(x^a, z^t) + \lambda(m'(x^a, z^t)) V'(x^a, z^t) \Big) \Big] \Big\} \\ &\leq \sum_{t \geq 0, z^t} \beta^t \psi(z^t) N_0'(z^t) \Big[-K(z_t) + \sum_x \pi_0(x) G_t(0, x, z^t) \Big]\end{aligned}$$

$$
\begin{aligned}
&+\sum_{z\in\mathcal{Z}}\psi(z^0)\sum_{a\geq 1,x^a}N(x^a,z^0)G_0(L(x^a,z^0),x_a,z^0)+\sum_{t,z^t}\beta^t\psi(z^t)\mu(z^t)\\
\leq\ &\sum_{t\geq 0,z^t}\beta^t\psi(z^t)N_0(z^t)\Big[-K(z_t)+\sum_x\pi_0(x)G_t(0,x,z^t)\Big]\\
&+\sum_{z\in\mathcal{Z}}\psi(z^0)\sum_{a\geq 1,x^a}N(x^a,z^0)G_0(L(x^a,z^0),x_a,z^0)+\sum_{t,z^t}\beta^t\psi(z^t)\mu(z^t)=S\ .
\end{aligned}
$$

Here the first equality just adds and subtracts $\mu(z^t)$. The subsquent inequality follows from resource constraint (13) of the main text. The second inequality follows since the discounted sum of surplus values for an individual firm which is of age a at time t, namely

$$
\sum_{\tau\geq t}\beta^{\tau-t}\sum_{x^{a+\tau-t}z^\tau}\psi(z^\tau|z^t)\pi(x^{a+\tau-t}|x^a)\prod_{k=t}^{\tau-1}[1-\delta(x^{a+k-t},z^k)]
$$

$$
\Big[O'(x^{a+\tau-t},z^\tau)-\mu(z^\tau)[L'(x^{a+\tau-t},z^\tau)+\lambda(m'(x^{a+\tau-t},z^\tau))V'(x^{a+\tau-t},z^\tau)]\Big]\ ,
$$

is bounded above $G_t(0,x_0,z_t)$ (for new firms, $a=0$) or by $G_0(L(x^a,z^0),x_a,z^0)$ (for firms of age $a>0$ existing at $t=0$) by definition of G_t. The third inequality follows from the complementary-slackness condition (38): either the term $-K(z_t)+\sum_x\pi_0(x)G_t(0,x,z^t)$ is zero in which case the first summand is zero on both sides of the inequality; or it is strictly negative in which case $N_0(z^t)=0$ and $N_0'(z^t)\geq 0$. The last equality follows from the definition of surplus value S and the assumption that allocation $\mathbf{X}$ solves problem (37) at the level of each individual firm. This proves $S'\leq S$ and hence contradicts the hypothesis $S'>S$. This completes the proof of Lemma A.4. □

Proof of Proposition 4 (continued):
To complete the proof of Prop. 3, part (a), let μ_i be the multiplier in aggregate state z_i, defined by (14) and (15) of the main text, and write $\mathbf{M}=(\mu_1,\ldots,\mu_n)$. With $\mu(z^t)\equiv\mu_i$ for $z_t=z_i$, the unique solution of (37) coincides with the one of (14) of the main text, i.e. $G_t(L,x,z^t)=G(L,x,i;\mathbf{M})$ for $z_t=z_i$, and also the firm-level policies coincide. If they give rise to an allocation $\mathbf{X}$ with positive entry in all aggregate states z^t, (15) of the main text implies that (38) holds for all z^t. Hence Lemma A.4(b) implies that $\mathbf{X}$ is a solution of the planning problem.
Part (b): Solving (14) of the main text in the stationary case $z=\overline{z}$ involves to find a single value function $G(L,x;\mathbf{M})$. Application of the contraction mapping theorem implies that such a solution exists, is unique, and is continuous and

non-increasing in $\mu \in \mathbb{R}$ and strictly decreasing in μ when $G(.) > 0$.
Therefore, the function $\Gamma(\mu) \equiv \sum_x \pi_0(x)G(0, x; \mu) \geq 0$ is continuous, strictly decreasing when positive, and zero for large enough μ. Furthermore, when f and b are sufficiently small, $\Gamma(0) > 0$; hence when $K > 0$ is sufficiently small, there exists a unique $\overline{\mu} \geq 0$ satisfying equation (15) of the main text.
In the stochastic case $z \in \{z_1, \ldots, z_n\}$ and for any given vector $\mathbf{M} = (\mu_1, \ldots, \mu_n) \in R^n_+$, the system of recursive equations (14) of the main text has a unique solution $G(.; \mathbf{M})$. Again this follows from the application of the contraction-mapping theorem. Furthermore, G is differentiable in $\mathbf{M}$, and all elements of the Jacobian $(dG(L, x, i; \mathbf{M})/(d\mu_j))_{i,j}$ are non-positive. The RHS of (14) of the main text defines an operator mapping a function $G(L, x, i; \mathbf{M})$ with a strictly diagonally dominant Jacobian matrix $(dG(L, x, i; \mathbf{M})/(d\mu_j))_{i,j}$ into another function $\tilde{G}$ whose Jacobian matrix $(d\tilde{G}(L, x, i; \mathbf{M})/(d\mu_j))_{i,j}$ is diagonally dominant. This follows since the transition matrix $\psi(z_j|z_i)$ is strictly diagonally dominant and since all elements of $(d\tilde{G}(L, x, i; \mathbf{M})/(d\mu_j))$ have the same (non-positive) sign. Therefore, the unique fixed point has a strictly diagonally dominant Jacobian. Now suppose that $(z_1, \ldots, z_n)$ is close to $(\overline{z}, \ldots, \overline{z})$ and consider the solution $\mu_1 = \ldots = \mu_n = \overline{\mu}$ of the stationary problem. Since the Jacobian matrix $(dG(0, x, i; \mathbf{M})/(d\mu_j))_{i,j}$ is strictly diagonally dominant, it is invertible. By the implicit function theorem, a unique solution $\mathbf{M}$ to equation (15) of the main text exits. □

For the **proof of Proposition 5**, see Appendix B.

Appendix B: Decentralization

The Workers' Search Problem
Let $U(z^t)$ be the utility value of an unemployed worker in history z^t, and let $E(\mathcal{C}_a, x^k, z^t)$ be the utility value of a worker hired by a firm of age a in contract $\mathcal{C}_a$ who is currently employed at that firm in history x^k, with $k > a$. The latter satisfies the recursive equation

$$E(\mathcal{C}_a, x^k, z^t) = w_a(x^k, z^t) + \beta\Big\{(1 - \varphi_a(x^k, z^t))\mathrm{E}_{z^t} U(z^{t+1}) \tag{39}$$
$$+\varphi_a(x^k, z^t)\mathrm{E}_{x^k,z^t} E(\mathcal{C}_a, x^{k+1}, z^{t+1})\Big\} \ .$$

An unemployed worker searches for contracts which promise the highest expected utility, considering that more attractive contracts are less likely to sign. The worker observes all contracts $\mathcal{C}_a$ and he knows that the probability to sign a contract is $m/\lambda(m)$ when m is the firm's matching probability at the offered contract. That is, potential contracts are parameterized by the tuple $(m, \mathcal{C}_a)$. Unemployed workers apply for those contracts where expected surplus is maximized:

$$\rho(z^t) = \max_{(m,\mathcal{C}_a)} \frac{m}{\lambda(m)}(1 - \delta(x^a, z^t))\beta\mathrm{E}_{x^a,z^t}\Big[E(\mathcal{C}_a, x^{a+1}, z^{t+1}) - U(z^{t+1})\Big] \ . \tag{40}$$

The Bellman equation for an unemployed worker reads as

$$U(z^t) = b + \rho(z^t) + \beta\mathrm{E}_{z^t} U(z^{t+1}) \ . \tag{41}$$

The Firms' Problem
A firm of age a in history (x^a, z^t) takes as given the employment stocks of workers hired in some earlier period, $(L_\tau)_{\tau=0}^{a-1}$, as well as the contracts signed with these workers, $(\mathcal{C}_\tau)_{\tau=0}^{a-1}$. For the contracts to be consistent with the firm's constraints on exit and separations, the retention probabilities must satisfy $\varphi_\tau(x^a, z^t) \leq (1 - s_0)(1 - \delta_0)$. The firm chooses an actual exit probability $\delta \geq \delta_0$ and cohort-specific layoff probabilities s_τ. For these probabilities to be consistent with separation probabilities specified in existing contracts, it must hold that $\delta \leq 1 - \varphi_\tau(x^a, z^t)$ for all $\tau \leq a - 1$, and $s_\tau = 1 - \varphi_\tau(x^a, z^t)/(1 - \delta)$ when $\delta < 1$, with arbitrary choice of s_τ when $\delta = 1$. The firm also decides new contracts $\mathcal{C}_a$ to be posted in V vacancies with desired matching probability m. It is no restriction to presuppose that the firm offers only one type of contract. When J_a is the value function of

a firm of age a, the firm's problem is written as

$$J_a\Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1},(L_\tau)_{\tau=0}^{a-1},x^a,z^t\Big] = \max_{(\delta,m,V,\mathcal{C}_a)} x_a z_t F(L) - W - C(V,L,x_a z_t) - f + \beta(1-\delta)\mathrm{E}_{x^a,z^t} J_{a+1}\Big[(\mathcal{C}_\tau)_{\tau=0}^{a},(L_{\tau+})_{\tau=0}^{a},x^{a+1},z^{t+1}\Big] \quad (42)$$

$$\text{s.t.}\quad L_{a+}=mV,\ m\in[0,1],\ V\geq 0,\ L_{\tau+}=L_\tau\frac{\varphi_\tau(x^a,z^t)}{1-\delta},\ \tau\leq a-1\ , \quad (43)$$

$$\delta\in[\delta_0,\min_{0\leq\tau\leq a-1}1-\varphi_\tau(x^a,z^t)],\ s_0\leq 1-\varphi_\tau(x^a,z^t)/(1-\delta)\ , \quad (44)$$

$$W=\sum_{\tau=0}^{a-1}w_\tau(x^a,z^t)L_\tau\ ,\ L=\sum_{\tau=0}^{a-1}L_\tau, \quad (45)$$

$$\rho(z^t)=\frac{m}{\lambda(m)}(1-\delta)\beta\mathrm{E}_{x^a,z^t}\Big[E(\mathcal{C}_a,x^{a+1},z^{t+1})-U(z^{t+1})\Big]\ \text{if } m>0 \quad (46)$$

The last condition is the workers' participation constraint; it specifies the minimum expected utility that contract $\mathcal{C}_a$ must promise in order to attract a worker queue of length $\lambda(m)$ per vacancy.

Definition: *A competitive search equilibrium is a list*

$$\Big[U(z^t),E(.),\rho(z^t),\mathcal{C}_a(x^a,z^t),m(x^a,z^t),V(x^a,z^t),\delta(x^a,z^t),J_a(.),L_\tau(x^a,z^t),N(x^a,z^t),N_0(z^t)\Big],$$

for all $t\geq 0$, $a\geq 0$, $x^a\in\mathcal{X}^{a+1}$, $z^t\in\mathcal{Z}^{t+1}$, $0\leq\tau\leq a$, and for a given initial firm distribution, such that

(a) *Firms' exit, hiring and layoff strategies are optimal. That is, J_a is the value function and $\mathcal{C}_a(.)$, $\delta(.)$, $m(.)$, and $V(.)$ are the policy functions for problem (42)-(46).*

(b) *Employment evolves according to*

$$L_\tau(x^a,z^t) = L_\tau(x^{a-1},z^{t-1})\frac{\varphi_\tau(x^a,z^t)}{1-\delta(x^a,z^t)}\ ,\ 0\leq\tau\leq a-1\ ,$$
$$L_a(x^a,z^t) = m(x^a,z^t)V(x^a,z^t)\ ,\ a\geq 0\ .$$

(c) *Firm entry is optimal. That is, the complementary slackness condition*

$$\sum_x \pi_0(x) J_0(x, z^t) \leq K(z_t) \ , \ \ N_0(z^t) \geq 0 \ , \tag{47}$$

holds for all z^t, and the number of firms evolves according to (9) and (11) of the main text.

(d) *Workers' search strategies are optimal, i.e. (ρ, U, E) satisfy equations (39), (40) and (41).*

(e) *Aggregate resource feasibility; for all z^t,*

$$\sum_{a \geq 0, x^a} N(x^a, z^t) \Big[\lambda(m(x^a, z^t)) V(x^a, z^t) + \sum_{\tau=0}^{a-1} L_\tau(x^a, z^t) \Big] = 1 \ . \tag{48}$$

Proposition 5: *A competitive search equilibrium is socially optimal.*
Proof of Proposition 5:
The proof proceeds in two steps. First, substitute the participation constraint (46) into the firm's problem and make use of the contracts' recursive equations (39) to show that the firms' recursive profit maximization problem is identical to the maximization of the social surplus of a firm. Second, show that a competitive search equilibrium is socially optimal.
First, define the social surplus of a firm with history (x^a, z^t) and with predetermined contracts and employment levels as follows:

$$G_a \Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t \Big] \equiv J_a \Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t \Big] + \sum_{\tau=0}^{a-1} L_\tau \Big[E(\mathcal{C}_\tau, x^a, z^t) - U(z^t) \Big] \ . \tag{49}$$

Using (39) and (41), the worker surplus satisfies

$$E(\mathcal{C}_\tau, x^a, z^t) - U(z^t) = w_\tau(x^a, z^t) - b - \rho(z^t) + \beta \varphi_\tau(x^a, z^t) \mathrm{E}_{x^a, z^t} \Big[E(\mathcal{C}_\tau, x^{a+1}, z^{t+1}) - U(z^{t+1}) \Big] \ .$$

Now substitute this equation and (42) into (49), and write

$$\sigma \equiv \Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t \Big] \text{ and } \sigma_+ \equiv \Big[(\mathcal{C}_\tau)_{\tau=0}^{a}, (L_{\tau+})_{\tau=0}^{a}, x^{a+1}, z^{t+1} \Big] \ ,$$

with $L_{\tau+}$ as defined in (43) and $L = \sum_{\tau=0}^{a-1} L_\tau$, to obtain

$$\begin{aligned} G_a(\sigma) &= \max_{\delta,m,V,\mathcal{C}_a} \Bigg\{ x_a z_t F(L) - C(V,L,x_a z_t) - f - \sum_{\tau=0}^{a-1} L_\tau w_\tau(x^a,z^t) \\ &\quad +\beta(1-\delta)\mathrm{E}_{x^a,z^t} J_{a+1}(\sigma_+) \Bigg\} + \sum_{\tau=0}^{a-1} L_\tau \Bigg[w_\tau(x^a,z^t) - b - \rho(z^t) \\ &\quad +\beta\varphi_\tau(x^a,z^t)\mathrm{E}_{x^a,z^t}\Big[E(\mathcal{C}_\tau,x^{a+1},z^{t+1}) - U(z^{t+1})\Big]\Bigg] \\ &= \max_{\delta,m,V,\mathcal{C}_a} \Bigg\{ x_a z_t F(L) - [b+\rho(z^t)]L - f - C(V,L,x_a z_t) + \beta(1-\delta)\mathrm{E}_{x^a,z^t} J_{a+1}(\sigma_+) \\ &\quad +\beta\sum_{\tau=0}^{a-1} L_\tau\varphi_\tau(x^a,z^t)\mathrm{E}_{x^a,z^t}\Big[E(\mathcal{C}_\tau,x^{a+1},z^{t+1}) - U(z^{t+1})\Big]\Bigg\} \\ &= \max_{\delta,m,V,\mathcal{C}_a} \Bigg\{ x_a z_t F(L) - bL - \rho(z^t)[L+\lambda(m)V] - f - C(V,L,x_a z_t) \\ &\quad +\beta(1-\delta)\mathrm{E}_{x^a,z^t} J_{a+1}(\sigma_+) \\ &\quad +\beta(1-\delta)\sum_{\tau=0}^{a} L_{\tau_+}\mathrm{E}_{x^a,z^t}\Big[E(\mathcal{C}_\tau,x^{a+1},z^{t+1}) - U(z^{t+1})\Big]\Bigg\} \\ &= \max_{\delta,m,V,\mathcal{C}_a} \Bigg\{ x_a z_t F(L) - bL - \rho(z^t)[L+\lambda(m)V] - f \\ &\quad -C(V,L,x_a z_t) + \beta(1-\delta)\mathrm{E}_{x^a,z^t} G_{a+1}(\sigma_+) \Bigg\} . \end{aligned} \tag{50}$$

Here maximization is always subject to (43) and (44), the third equation makes use of

$$(1-\delta)L_{\tau+} = \varphi_\tau(x^a,z^t)L_\tau \ ,$$

for $\tau \le a-1$, and

$$\rho(z^t)\lambda(m)V = \beta(1-\delta)L_{a+}\mathrm{E}_{x^a,z^t}\Big[E(\mathcal{C}_a,x^{a+1},z^{t+1}) - U(z^{t+1})\Big] \ ,$$

and the last equation makes use of (49) for G_{a+1}. This shows that the firm solves a surplus maximization problem which is identical to the one of the planner specified in (37) provided that $\rho(z^t) = \mu(z^t)$ holds for all z^t, where μ is the social value of an unemployed worker as defined in the proof of Proposition 4. The only difference between the two problems is that the firm commits to cohort-specific separation probabilities, whereas the planner chooses in every period an identical separation probability for all workers (and he clearly has no reason to do otherwise). Nonetheless, both problems have the same solution: they are dynamic optimization problems of a single decision maker in which payoff functions are the same and the decision sets are the same. Further, time inconsistency is not an issue since there is no strategic interaction and since discounting is exponential. Hence solutions to the two problems, with respect to firm exit, layoffs and hiring strategies, are identical. In both problems the decision maker could discriminate between different cohorts in principal. Because such differential treatment does not raise social firm value, there is also no reason for competitive search to produce such an outcome. Nonetheless, there can be equilibrium allocations where different cohorts have different separation probabilities, but these equilibria must also be socially optimal because they maximize social firm value.

It remains to verify that competitive search gives indeed rise to socially efficient firm entry. When $\mu(z^t) = \rho(z^t)$, $G_0(x, z^t) = J_0(x, z^t)$ as defined in (49) coincides with $G_0(0, x, z^t)$, as defined in (37). Hence, the free-entry condition (47) coincides with the condition for socially optimal firm entry (38). Because of aggregate resource feasibility (48), the planner's resource constraint (13) of the main text is also satisfied. Since the allocation of a competitive search equilibrium satisfies all the requirements of Lemma A.4(b), it is socially optimal. □

Appendix C: Calibration and Computation

Calibration

We choose the period length to be one week and set $\beta = 0.999$ so that the annual interest rate is about 5 percent. We assume a CES matching function $m(\lambda) = (1 + k\lambda^{-r})^{-1/r}$ (i.e. the inverse of the function $\lambda(m)$ used in the main text) and set the two parameters k and r to target a weekly job-finding rate of 0.129 and an elasticity of the job-finding rate with respect to the vacancy-unemployment ratio of 0.28 (Shimer (2005)).[2] By choosing parameter c of the recruitment technology (see below), we also target the (average) weekly job-filling rate at 0.3, which corresponds to a monthly vacancy yield of 1.3 (Davis, Faberman and Haltiwanger (2013)). Since in steady state the unemployment-vacancy ratio equals the ratio between the job-filling rate and the job-finding rate, we calculate the parameters k and r to attain the two targets at $\lambda = 0.3/0.129 = 2.326$.

The production technology is Cobb-Douglas with xL^{α} where the firm's idiosyncratic productivity $x = x_0 x_1$ contains a time-invariant component x_0 and a transitory component x_1 (cf. Elsby and Michaels (2013)). The time-invariant component is drawn upon firm entry from one of five values x_0^i, $i = 1, \ldots, 5$, with entry shares σ^i where (x_0^i, σ^i) are chosen to match the firm and employment shares within the five size classes 1-49, 50-249, 250-999, 1000-9999, and ≥ 10000, where data targets are taken from the Business Dynamics Statistics (BDS) of the U.S. Census Bureau. The transitory component x_1 is drawn from one of five equidistant values in the range $[1 - \overline{x}, 1 + \overline{x}]$ and is redrawn every period with probability π. Parameters π and $\overline{x}$ are chosen to match a monthly separation rate of 4.2 percent and the observation that about two thirds of employment is at firms with monthly employment growth rates in the range $[-0.02, 0.02]$ (see Davis et al. (2010)). Firm exit is exogenous; that is, we set the operating cost to $f = 0$ and choose exit probabilities specific for the five size classes δ^i, $i = 1, \ldots, 5$, to match annual firm exit rates from the BDS. Parameter α is set to 0.7 which gives rise to a labor share of roughly 2/3. Given that all capital is fixed at the level of a firm, this consideration does not take into account variable capital investment at the firm level which would suggest a higher value of α; see Appendix D for a robustness analysis regarding this parameter.

In the benchmark parameterization, we set unemployment income b at 0.7 of the average wage which is similar to the calibrated values of non-market work chosen

[2]Note that there is no third parameter in the CES matching function since we require that $\lim_{\lambda \to \infty} m(\lambda) = 1$.

by Hall and Milgrom (2008) and Pissarides (2009).[3] As mentioned in the main text, we also consider a much higher value of this parameter, namely 97.7 percent of the average wage which corresponds to the choice of Hagedorn and Manovskii (2008) and gives rise to much more amplification of aggregate shocks.[4] For this parameterization, we recalibrate all other parameters to hit the same targets as in the benchmark calibration.[5]

The exogenous quit rate is set at $s_0 = 0.0048$ to match a monthly quit rate of 2 percent. The entry cost parameter K can be normalized arbitrarily since all firm value functions (and thus the free-entry condition) are linearly homogeneous in the vector (x, b, c, K).

As mentioned in the main text, the recruitment technology has the form $c(V) = \frac{c}{1+\gamma}(\frac{V}{L})^{\gamma}V$, where we take a cubic function ($\gamma = 2$) for the benchmark calibration. When we compare the benchmark results with those for $\gamma = 0.1$ and for $\gamma = 8$, we recalibrate parameters c and b (equivalently, parameter K) to target the average unemployment-vacancy ratio $\lambda = 2.326$ which gives rise to an average weekly job-filling rate of 0.3 and the same b/w ratio as in the benchmark.[6] We note that recruitment costs per hire are reasonably low for all three parameterizations (below 1% of quarterly earnings).

In our business cycle analysis, we choose the aggregate state process z as described in the text and let the entry cost K vary with the aggregate state which stabilizes the volatility of job creation at opening firms. Specifically, K attains the values $(324.6, 327.2, 329.8, 332.4, 335.0)$ in the five productivity states in the calibration with $b/w \approx 0.7$. For $b/w \approx 0.977$, K attains the values $(214.1, 218.5, 222.6, 226.6, 230.5)$.

To complement the cross-sectional results in the main text, Table 1 reports quarterly job creation and job destruction rates in four different size classes taken from the Business Employment Dynamics dataset of the Bureau of Labor Statistics.[7] The model generates job creation and job destruction rates which are falling in

[3] Hall and Milgrom (2008) calibrate the flow value of unemployment at 0.71 of productivity (0.73 of wages). Their value includes a reasonably low value of unemployment benefits (0.25) and reflects some risk sharing within households.

[4] This alternative value of b corresponds to 96.8 percent of the average (employment-weighted) marginal product and 68 percent of labor productivity.

[5] Deviating from Table 1 of the main text, we set $c = 0.496$, $K = 236.3$ (which follows from $b = 0.1$ and the choice of $b/w \approx 0.977$), $\overline{x} = 0.104$, $(x_0^i) = (.273, .585, .913, 1.503, 3.060)$, $(\sigma^i) = (99.07, 0.80, .10, .025, .001)$, $\pi = 0.06$.

[6] Deviating from Table 1 of the main text, we set $c = 0.04092$, $K = 358.69$ for $\gamma = 0.1$ and $c = 1.274 \cdot 10^9$, $K = 287.46$ for $\gamma = 8$ (fixing $b = 0.1$ throughout).

[7] The largest size class in this dataset are firms with 1000 or more workers. Hence, we merge the two largest size classes for the reported model statistics.

firm size, which is qualitatively in line with the data. The relationship is less pronounced than in the data because we do not calibrate the transitory productivity processes separately for each size class. Negative relationships between size and job flows are also observed for entering and exiting firms.

Table 1: Firm size and quarterly job flows

Size class	1–49	50–249	250–999	$\geq$ 1000
Data				
Job creation	10.6	6.0	4.6	2.9
Job destruction	10.4	5.7	4.3	2.7
Job creation (openings)	3.0	0.3	0.1	0.01
Job destruction (closings)	2.9	0.4	0.2	0.02
Model				
Job creation	9.6	7.5	7.3	6.3
Job destruction	9.4	7.3	7.2	6.4
Job creation (openings)	0.6	0.02	0.0	0.0
Job destruction (closings)	2.1	0.3	0.2	0.0

Notes: Data statistics are from the Business Employment Dynamics (1992-2011) of the Bureau of Labor Statistics. Model statistics are from a cross section of $4.9 \cdot 10^6$ firms for the benchmark calibration ($\gamma = 2$).

Computation

To solve the model numerically, we implement the procedure as outlined in the main text. Given the discrete sets of idiosyncratic states $x \in \mathcal{X}$ and aggregate states $z \in \mathcal{Z}$ and the corresponding Markov transition matrices, as well as a grid for employment L, we solve recursive problems (14) of the main text for a given initial guess of multipliers $\mu(z)$, $z \in \mathcal{Z}$, by value-function iteration. To make sure that the free-entry conditions (15) of the main text are satisfied, multipliers must be adjusted accordingly. This yields firm value functions $G(L, x, z)$, as well as policy functions $\lambda(L, x, z)$, $V(L, x, z)$, $s(L, x, z)$.[8] Given the calibrated (exogenous) exit rates $\delta(x)$, this allows us to compute retention rates $\varphi(L, x, z) = (1 - \delta(x))(1 - s(L, x, z))$.

For the particular decentralization with flat-wage contracts mentioned in the text, we use the following procedure for the calculation. A flat-wage contract offered

[8]Deviating from the main text, we write firm policies in terms of worker-job ratios λ, so that matching rates $m(\lambda)$ follow from the matching function.

to a new hire specifies the wage w together with retention probabilities $\varphi(L,x,z)$ that are identical for all workers in firm (L,x,z). In a recursive equilibrium, this allows us to rewrite the identities for worker value functions $E(w,L,x,z)$, unemployment values $U(z)$, and the search surplus $\rho(z)=\mu(z)$, given by (39), (40) and (41) as follows:

$$\begin{aligned} E(w,L,x,z) &= w+\beta\Big\{(1-\varphi(L,x,z))\mathrm{E}_z U(z_+) \\ &\quad +\varphi(L,x,z)\mathrm{E}_{x,z}E(w,L_+,x_+,z_+)\Big\}\ , \end{aligned} \tag{51}$$

$$\mu(z) = \frac{m(\lambda(L,x,z))}{\lambda(L,x,z)}(1-\delta(z))\beta\mathrm{E}_{x,z}\Big[E(w,L_+,x_+,z_+)-U(z_+)\Big] \tag{52}$$

$$U(z) = b+\mu(z)+\beta\mathrm{E}_z U(z_+)\ . \tag{53}$$

Here $L_+ = L(1-s(L,x,z))+m(\lambda(L,x,z))V(L,x,z)$ is next period's employment which follows from the firms' policy functions. Equation (52) defines the flat wage $w=w^*(L,x,z)$ that firm (L,x,z) offers to new hires. Subtracting (53) from (51) gives

$$E(w,L,x,z)-U(z)=w-b-\mu(z)+\beta\varphi(L,x,z)\mathrm{E}_{x,z}\Big[E(w,L_+,x_+,z_+)-U(z_+)\Big]\ .$$

It follows that

$$E(w,L,x,z)-U(z)=A(L,x,z)(w-b)-B(L,x,z)\ , \tag{54}$$

where $A(L,x,z)$ and $B(L,x,z)$ are defined recursively by

$$A(L,x,z) = 1+\beta\varphi(L,x,z)\mathrm{E}_{x,z}A(L_+,x_+,z_+)\ , \tag{55}$$

$$B(L,x,z) = \mu(z)+\beta\varphi(L,x,z)\mathrm{E}_{x,z}B(L_+,x_+,z_+)\ . \tag{56}$$

From (52) and (54), we can compute the wage $w=w^*(L,x,z)$ that firm (L,x,z) offers to new hires:

$$w^*(L,x,z)=b+\left\{\frac{\mu(z)\lambda(L,x,z)}{m(\lambda(L,x,z))(1-\delta(x))\beta}+\mathrm{E}_{x,z}B(L_+,x_+,z_+)\right\}\frac{1}{\mathrm{E}_{x,z}A(L_+,x_+,z_+)}\ .$$

For the model computation, we solve (55) and (56) simultaneously with $G(L,x,z)$ in the value-function iteration. This allows us to compute the flat wages $w^*(L,x,z)$ offered to new hires.

After we solve the model for the firms' policy functions, we can first simulate a stationary cross-section of firms (in the absence of aggregate productivity shocks).

This is done by following a given number of entrant firms (according to their permanent productivity types and entry shares) along their lifecycles. Regarding business cycle dynamics, we start from a stationary firm distribution and follow those firms across time when aggregate shocks are active. The numbers of new entrants are determined each period residually so that all workers are either employed or search for work at any of the existing or entering firms.

Appendix D: Robustness

We explore the robustness of the main calibration results regarding different parameter choices for unemployment income b and for the returns-to-scale parameter α. Departing from the benchmark calibration with cubic vacancy costs we consider two variations. First, we consider the alternative of setting unemployment income to 97.7 percent of average wages (68% of labor productivity), instead of 70 percent as in the benchmark. Second, relative to the benchmark with $\alpha = 0.7$ which gives rise to a plausible labor share (with fixed capital at any individual firm) we consider the alternative of $\alpha = 0.95$ which is more in line with a model where capital can be adjusted at the firm level. In both variations, parameters c, $\overline{x}$ and (x_0^i) are readjusted so that the model hits the same calibration targets as in the benchmark calibration.[9]

Table 2 replicates Table 2 of the main text to show that both model variations are calibrated to match firm and employment shares in the five size classes. The bottom three rows show how the shares of younger firms are declining with firm size in the two versions. Relative to the benchmark calibration, the model with high production function elasticity generates considerably lower shares of very young firms in the larger size classes, which indicates slower firm growth for this parameterization.

Figure 1 shows that the cross-sectional behavior of vacancy rates, vacancy yields, hires rates and layoff rates is almost unchanged relative to the benchmark calibration. That is, irrespective of the parameter values for b and α, the model with cubic vacancy costs explains more than half of the cross-sectional variation in vacancy yields, although the vacancy yield curve for $\alpha = 0.95$ (green/dotted curve) flattens out at firm growth above 20 percent relative to the benchmark calibration (blue/solid curve).

[9] The calibration with $\alpha = 0.95$ requires $c = 15.64$, $K = 80.46$ (again $b = 0.1$ and $b/w \approx 0.7$), $\overline{x} = 0.11$, $(x_0^i) = (.164, .184, .198, .218, .245)$, $(\sigma^i) = (98.82, 1.00, .153, .025, .002)$, $\pi = 0.027$. Parameters for the the version with $b/w \approx 0.977$ are stated in footnote 5.

Table 2: Firm size and employment distribution (higher values of b and α)

Size class	1–49	50–249	250–999	1000–9999	$\geq$ 10000
Data					
Firm shares	95.62	3.64	0.54	0.17	0.02
Employment shares	29.31	16.23	10.88	17.64	25.93
% younger than 2 yrs.	24.68	7.24	4.38	2.26	1.08
% younger than 5 yrs.	39.71	16.88	10.19	5.35	3.65
% younger than 10 yrs.	57.76	31.30	20.23	12.01	7.14
Model (high b)					
Firm shares	95.78	3.42	0.61	0.17	0.02
Employment shares	30.65	15.87	12.62	18.20	22.67
% younger than 2 yrs.	16.19	3.62	2.55	2.26	1.79
% younger than 5 yrs.	35.62	9.17	6.82	6.06	4.91
% younger than 10 yrs.	58.44	17.91	13.90	12.40	10.13
Model (high α)					
Firm shares	96.27	3.06	0.50	0.15	0.02
Employment shares	29.23	16.36	11.60	18.27	24.53
% younger than 2 yrs.	16.13	1.65	1.30	0.95	0.42
% younger than 5 yrs.	35.41	6.79	5.75	4.78	3.50
% younger than 10 yrs.	58.12	15.28	12.24	11.66	7.59

Notes: The top two rows report firm and employment shares in five size classes (calibrated). The bottom rows are the shares of younger firms in these classes. Data statistics are from the Business Dynamics Statistics of the Census Bureau for the year 2005. Model statistics are from a cross section of $4.9 \cdot 10^6$ firms. The model with high b has $b \approx 0.977w$, $\gamma = 2$, $\alpha = 0.7$, and the model with high α has $b \approx 0.7w$, $\gamma = 2$, $\alpha = 0.95$.

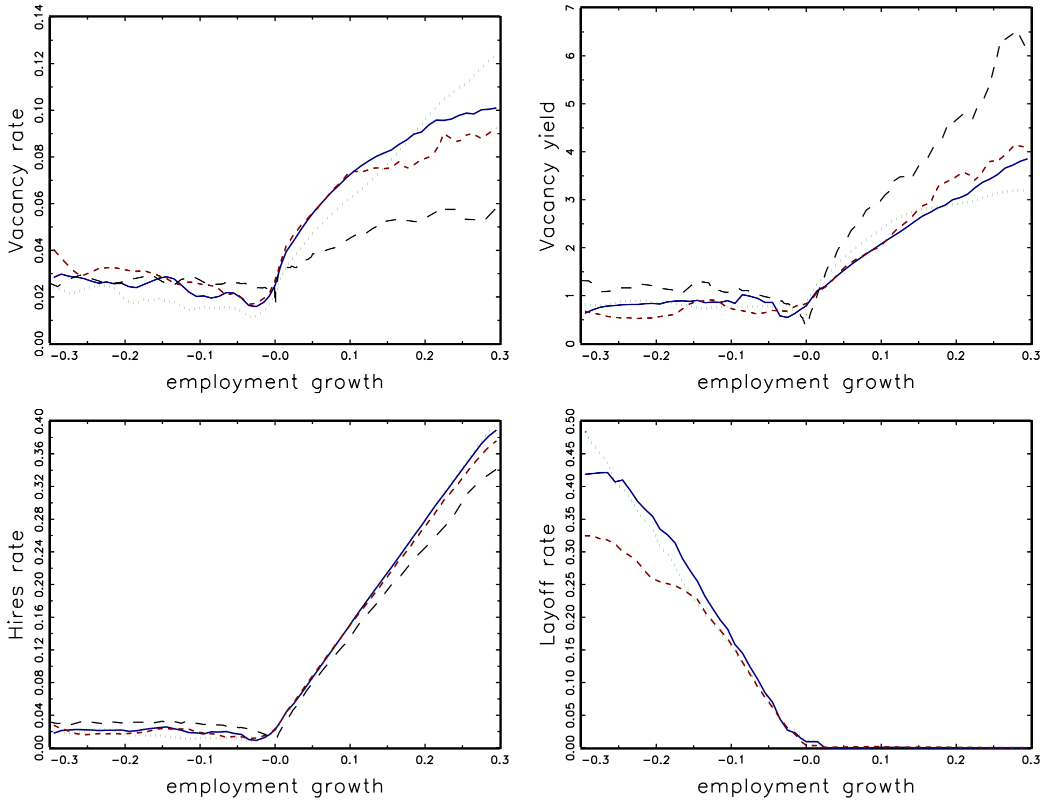

Figure 1: Cross-sectional relationships between monthly employment growth and the vacancy rate, the vacancy yield, the hires rate and the layoff rate. The dashed curves (in the first three graphs) are from the data used in Davis, Faberman and Haltiwanger (2013), the blue (solid) curves are for the benchmark parameterization ($b/w \approx 0.7$, $\alpha = 0.7$), the red (closely dashed) curves are for the calibration with $b/w \approx 0.977$ and the green (dotted) curves are for $\alpha = 0.95$.

Appendix E: Impulse response in a VAR model

We borrow the methodology for constructing the impulse responses in Figure 4 of the main text straight from Fujita and Ramey (2007) - except for the details discussed below.[10] We use data from 1951:Q1 to 2011:Q4. The data is real quarterly GDP from FRED; the number of vacancies is from the Help Wanted Index from Barnichon's Composite Help-Wanted Index series (Barnichon (2010)); the number of unemployed is from the CPS; employment is from the BLS total payroll employment; and population is from the BLS. The job-finding rate is then calculated in the same way as in Elsby, Michaels and Solon (2009). All data other than GDP are averaged over their monthly (seasonally adjusted) observations to obtain quarterly series. They are then logged and detrended by regressing each on a cubic polynomial in time.

To generate impulse responses of output, employment, labor market tightness and the job-finding rate to a permanent rise in productivity, we first identify *exogenous* productivity deviations in the data series and look at how the variables of interest respond to these. Let

$$\begin{aligned} p_t &\equiv \text{observed (detrended) output per worker }, \\ \theta_t &\equiv \text{observed (detrended) vacancy-unemployment ratio }, \\ e_t &\equiv \text{observed (detrended) employment-population ratio }, \\ \phi_t &\equiv \text{observed (detrended) job-finding rate }, \end{aligned}$$

and let z_t be the unobserved exogenous productivity deviation. To identify z_t, we first estimate (by OLS) the following system:

$$\begin{aligned} \ln p_t &= \begin{bmatrix} \ln p_t & \ln \theta_t & \ln e_t & \ln \phi_t \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix} \begin{bmatrix} L \\ L^2 \\ L^3 \end{bmatrix} + \varepsilon_t^p \\ &= \begin{bmatrix} \ln p_t & \ln \theta_t & \ln e_t & \ln \phi_t \end{bmatrix} A(L) + \varepsilon_t^p , \end{aligned}$$

where L is the lag operator. Given this estimation, $\left\{\hat{A}_{ij}\right\}$, we follow Fujita and Ramey (2007) by assuming that the exogenous productivity deviations, $\ln z_t$, can

[10]We are grateful to David Ratner for providing an initial code.

be identified by

$$
\begin{aligned}
\ln p_{t<0} &= \ln \theta_{t<0} = \ln e_{t<0} = \ln \phi_{t<0} = \ln z_{t<0} = 0 \ , \\
\hat{\varepsilon}_t^p &= \ln p_t - \left[\begin{array}{cccc} \ln p_t & \ln \theta_t & \ln e_t & \ln \phi_t \end{array} \right] \hat{A}(L) \ , \\
\ln z_t &= \hat{A}_{11} \ln z_{t-1} + \hat{A}_{12} \ln z_{t-2} + \hat{A}_{13} \ln z_{t-3} + \hat{\varepsilon}_t^p \ .
\end{aligned}
$$

Once a series for $\ln z_t$ has been identified from the data in this way, an $AR(3)$ process for $\ln z_t$ can be estimated,

$$
\begin{aligned}
\ln z_t &= C_{01} \ln z_{t-1} + C_{02} \ln z_{t-2} + C_{03} \ln z_{t-3} + \varepsilon_t^z \\
&= C_0(L) \ln z_t + \varepsilon_t^z \ ,
\end{aligned}
$$

and the relationship between the endogenous variables $\ln e_t, \ln \theta_t$, $\ln \phi_t$ and the exogenous process $\ln z_t$ can be calculated by estimating the following relationships (by OLS):

$$
\begin{aligned}
\ln e_t &= \left[\begin{array}{ccc} \ln e_t & \ln \theta_t & \ln \phi_t \end{array} \right] \left[\begin{array}{ccc} B_{111} & B_{112} & B_{113} \\ B_{121} & B_{122} & B_{123} \\ B_{131} & B_{132} & B_{133} \end{array} \right] \left[\begin{array}{c} L \\ L^2 \\ L^3 \end{array} \right] + C_1(L) \ln z_t + D_1 \hat{\varepsilon}_t^p + \varepsilon_t^e \\
&= \left[\begin{array}{ccc} \ln e_t & \ln \theta_t & \ln \phi_t \end{array} \right] B_1(L) + C_1(L) \ln z_t + D_1 \hat{\varepsilon}_t^p + \varepsilon_t^e \ ,
\end{aligned}
$$

$$
\begin{aligned}
\ln \theta_t &= \left[\begin{array}{ccc} \ln e_t & \ln \theta_t & \ln \phi_t \end{array} \right] B_2(L) + C_2(L) \ln z_t + D_2 \hat{\varepsilon}_t^p + \varepsilon_t^\theta \ , \\
\ln \phi_t &= \left[\begin{array}{ccc} \ln e_t & \ln \theta_t & \ln \phi_t \end{array} \right] B_3(L) + C_3(L) \ln z_t + D_3 \hat{\varepsilon}_t^p + \varepsilon_t^\phi \ .
\end{aligned}
$$

The impulse-response functions to a permanent increase in exogenous productivity of 1% are then constructed by simulating these estimated relationships forward:

$$
\begin{aligned}
\ln z_{t<0} &= \ln e_{t<0} = \ln \theta_{t<0} = \ln \phi_{t<0} = 0 \ , \\
\ln z_{t\geq 0} &= 0.01 \ , \\
\hat{\varepsilon}_{t\geq 0}^p &= \hat{\varepsilon}_{t\geq 0}^z = \ln z_{t\geq 0} - \hat{C}_0(L) \ln z_{t\geq 0} \ , \\
\hat{\varepsilon}_t^e &= \hat{\varepsilon}_t^\theta = \hat{\varepsilon}_t^\phi = 0 \ ,
\end{aligned}
$$

$$\begin{aligned}
\ln e_t &= \left[\begin{array}{ccc} \ln e_t & \ln \theta_t & \ln \phi_t \end{array}\right] \hat{B}_1(L) + \hat{C}_1(L) \ln z_t + \hat{D}_1 \hat{\varepsilon}_t^p \ , \\
\ln \theta_t &= \left[\begin{array}{ccc} \ln e_t & \ln \theta_t & \ln \phi_t \end{array}\right] \hat{B}_2(L) + \hat{C}_2(L) \ln z_t + \hat{D}_2 \hat{\varepsilon}_t^p \ , \\
\ln \phi_t &= \left[\begin{array}{ccc} \ln e_t & \ln \theta_t & \ln \phi_t \end{array}\right] \hat{B}_3(L) + \hat{C}_3(L) \ln z_t + D_3 \hat{\varepsilon}_t^p \ ,
\end{aligned}$$

$$\ln p_t = \left[\begin{array}{cccc} \ln p_t & \ln \theta_t & \ln e_t & \ln \phi_t \end{array}\right] \hat{A}(L) + \hat{\varepsilon}_t^p \ .$$

Our construction differs from Fujita and Ramey (2007) only in that their estimations are based on data to 2005, they use three variables (p_t, θ_t, e_t) for the VAR, and they show impulse responses for a one-time rather than a permanent shock. We replicated their settings and find their results, and we checked that adding the fourth variable does not qualitatively change the outcome for the three initial variables in their methodology.

References

Amir, Rabah. 1996. "Sensitivity Analysis of Multisector Optimal Economic Dynamics." *Journal of Mathematical Economics*, 25: 123–141.

Barnichon, Regis. 2010. "Building a Composite Help-Wanted Index." *Economics Letters*, 109: 175–178.

Davis, Steven J., R. Jason Faberman, and John C. Haltiwanger. 2013. "The Establishment–Level Behavior of Vacancies and Hiring." *Quarterly Journal of Economics*, 128: 581–622.

Davis, Steven J., R. Jason Faberman, John C. Haltiwanger, and Ian Rucker. 2010. "Adjusted Estimates of Worker Flows and Job Openings in JOLTS." In *Labor in the New Economy.* , ed. Katharine G. Abraham, James R. Spletzer and Michael J. Harper, 187–216. Chicago:University of Chicago Press.

Elsby, Michael W. L., and Ryan Michaels. 2013. "Marginal Jobs, Heterogenous Firms, and Unemployment Flows." *American Economic Journal: Macroeconomics*, 5: 1–48.

Elsby, Michael W. L., Ryan Michaels, and Gary Solon. 2009. "The Ins and Outs of Cyclical Unemployment." *American Economic Journal: Macroeconomics*, 1: 84–110.

Fujita, Shigeru, and Garey Ramey. 2007. "Job Matching and Propagation." *Journal of Economic Dynamics and Control*, 31: 3671–3698.

Hagedorn, Marcus, and Iourii Manovskii. 2008. "The Cyclical Behavior of Equilibrium Unemployment and Vacancies Revisited." *American Economic Review*, 98: 1692–1706.

Hall, Robert E., and Paul R. Milgrom. 2008. "The Limited Influence of Unemployment on the Wage Bargain." *American Economic Review*, 98: 1653–1674.

Pissarides, Christopher A. 2009. "The Unemployment Volatility Puzzle: Is Wage Stickiness the Answer?" *Econometrica*, 77: 1339–1369.

Shimer, Robert. 2005. "The Cyclical Behavior of Equilibrium Unemployment and Vacancies." *American Economic Review*, 95: 25–49.