Online Appendix to "On discrimination in auctions with endogenous entry" by Philippe Jehiel and Laurent Lamy

Generalized second-price auctions

Under exogenous participation, Myerson (1981) shows that the optimal auction can be implemented with a generalized second-price auction where bids are distorted in a very general (nonlinear) way. Similarly, bid distortions play a crucial role in presence of incumbents.

A generalized second-price auction with (general) non-linear distortion (or a bid preference program) is characterized by a reserve price $r \in \mathbb{R}_+$ and a set of right-continuous non-decreasing function, called next bid distortion functions, $A_k : \mathbb{R}_+ \to \mathbb{R}_+$ (for each group $k \in \mathcal{K}$ of potential entrants) and $A_i^I : \mathbb{R}_+ \to \mathbb{R}_+$ (for each incumbent $i \in \mathcal{I}$), which are assumed to be strictly increasing on the set $\{x \in \mathbb{R}_+ | A_k(x) \geq r\}$ for a group k entrant and $\{x \in \mathbb{R}_+ | A_i^I(x) \geq r\}$ for incumbent i. The rules of the generalized second-price auction are as follows:

1. The seller collects all the bids and computes a new (or distorted) bid $A_i^I(b)$ [resp. $A_k(b)$] for each bid b from an incumbent i [resp. from an entrant from group k].

2. The reserve price r is considered next as a bid from the seller.

3. One of the agents (including the seller) with the highest new bid is declared to be the winner and receives the good.[51]

4. Let p denote the maximum of the second highest new bid from participating bidders (if any) and the reserve price r. If the winner is an incumbent i [resp. an entrant from group k], he has to pay $\min\{b \in \mathbb{R}_+ | A_i^I(b) \geq p\}$ [resp. $\min\{b \in \mathbb{R}_+ | A_k(b) \geq p\}$]. The monetary transfer of a buyer who does not receive the good is null.[52]

The price paid by the winner corresponds to the lowest bid he would have to submit in order to be still declared the winner (with some positive probability). Note that the price paid by the winner can never be strictly above his bid. Compared to truthful bidding, bidding below its valuation involves only the loss of some profitable opportunities. Compared to truthful bidding, bidding above its valuation changes the final outcome only in the case where p is above his valuation, i.e. in the events where the final price would have been greater than his valuation. On the whole, we obtain that

LEMMA 11 *For any generalized second-price auction, truthful bidding is a (weakly) dominant strategy.*

[51] In case of multiple winning bids, we need also a tie-breaking rule to complete the description of a specific auction. Here, any rule would suit, e.g. the one consisting in picking the winner at random.

[52] When there are atoms the tie-breaking rule may matter in terms of the final assignment. Nevertheless, it does not matter in terms of final payoffs in equilibrium since the pricing rule under truthful bidding guarantees that the bidders involved in a tie obtain pay their valuation (this is because $\min\{b \in \mathbb{R}_+ | A_i^I(b) \geq p\} = x$ if $A_i^I(x) = p \geq r$ for any incumbent i while the same holds for entrants).

This further implies that in equilibrium, bidders should bid truthfully in generalized second-price auctions (since we assume that bidders use undominated strategies).

Let $\mathcal{M}_A^{GSP}$ denote the set of generalized second-price auctions with $A_k(b) = b$ for any $k \in \mathcal{K}$. For any $m \in \mathcal{M}_A^{GSP}$, we let $m[r]$ denote the reserve price in the auction and $m[A_i^I]$ the bid distortion of incumbent i, for any $i \in \mathcal{I}$.

Key remark: The virtual pivotal mechanism $m_{\beta,X}^{v\text{-}piv}$ corresponds to the generalized second-price auction in $\mathcal{M}_A^{GSP}$ characterized by $m_{\beta,X}^{v\text{-}piv}[r] = X$ and $m_{\beta,X}^{v\text{-}piv}[A_i^I] = x_i^{virt}(.)$ for any $i \in \mathcal{I}$. We stress that $m_{\beta,X}^{v\text{-}piv}[A_i^I]$ is strictly increasing on the set $\{x \in \mathbb{R}_+ | A_i^I(x) \geq X\}$ thanks to our regularity assumption.

From the perspective of potential entrants, a mechanism $m \in \mathcal{M}_A^{GSP}$ is equivalent to a standard second-price auction with the reserve $m[r]$ and where conditional on z and for any $i \in \mathcal{I}$, the valuation distributions of the incumbents are no longer $F_i^I(.|z)$ but are rather replaced by $F_i^I([m[A_i^I]]^{-1}(.)|z)$ which denotes the distribution of the variable $m[A_i^I](u)$ where the variable u is drawn according to $F_i^I(.|z)$. This results from the fact that their bids are not distorted for $m \in \mathcal{M}_A^{GSP}$. For a given $m \in \mathcal{M}_A^{GSP}$, let $\widetilde{F}_m^{(1:N\cup S)}(x) = E_Z\Big[\prod_{k=1}^K [F_k(x|z)]^{n_k} \cdot \prod_{i\in S} F_i^I([m[A_i^I]]^{-1}(x)|z)\Big]$ denote the CDF of the highest new (or distorted) bid among the bidders under truthful bidding given that the realization of the profile of entrants is N. When m is a (standard) second-price auction, then note we have $\widetilde{F}_m^{(1:N\cup S)} = F^{(1:N\cup S)}$.

With this change of perspective, we obtain on the whole that for any $m \in \mathcal{M}_A^{GSP}$, the expected ex ante utility of an entrant from group k is given by $u_k(q, q^I, m) = \sum_{N\in\mathbb{N}^K} \sum_{S\subseteq\mathcal{I}} P(N|q) \cdot P(S|q^I) \cdot V_{k,N_{+k},S}(m)$ where

$$V_{k,N_{+k},S}(m) = \int_{m[r]}^{\infty} (\widetilde{F}_m^{(1:N\cup S)}(x) - \widetilde{F}_m^{(1:N_{+k}\cup S)}(x))dx. \tag{50}$$

The problem is somehow the same as before from the perspective of entrants, up to the twist that the CDFs of the valuation of the incumbents are now possibly distorted. In particular, (35) still holds which implies that $M(m) \subseteq [0,1)^K \times [0,1]^I$ for any $m \in \mathcal{M}_A^{GSP}$ such that $m[r] = X$. Analogously to the expected welfare function $W_N(m, X; \sigma(m))$, we can define a notion of 'distorted welfare' for any $m \in \mathcal{M}_A^{GSP}$ (which guarantees truthful bidding), denoted by $\widetilde{W}_N(m, X)$, where incumbents' valuations have been substituted by their distorted valuations and the seller's reservation value X by the reserve price $m[r]$. Formally, we define

$$\widetilde{W}_{N,S}(m, X) := \int \max\left\{m[r], \max_{j=1,\dots,|N|}\{x_j^N\}, \max_{i\in S}\{m[A_i^I](x_i^{\mathcal{I}})\}\right\} d[G_{N,S}(y^{N,S})]. \tag{51}$$

We have also $\widetilde{W}_{N,S}(m, X) = m[r] \cdot \widetilde{F}_m^{(1:N\cup S)}(m[r]) + \int_{m[r]}^{\infty} x d[\widetilde{F}_m^{(1:N\cup S)}(x)]$.

The fundamental property of the pivotal mechanism (11) translates now into

$$\widetilde{W}_{N_{+k},S}(m, X) - \widetilde{W}_{N,S}(m, X) = V_{k,N_{+k},S}(m) \quad \text{for any} \quad k \in \mathcal{K} \quad \text{and} \quad m \in \mathcal{M}_A^{GSP} \tag{52}$$

In words, entrants obtain the incremental surplus they generate where the surplus is defined according to the distorted valuations.

For any $m \in \mathcal{M}_A^{GSP}$, we let $\widetilde{NW}(q, q^I, m, X) := \sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}} P(N|q) \cdot P(S|q^I) \cdot \widetilde{W}_{N,S}(m, X) - \sum_{k=1}^K q_k \mathcal{N}_k \cdot C_k - \sum_{i=1}^I \beta_i^I \cdot q_i^I \cdot C_i^I$ denote the total expected (ex ante) distorted welfare.

Comment: For the virtual pivotal mechanism, namely when $m = m_{\beta,X}^{v\text{-}piv}$, then the terms $\widetilde{W}_{N,S}(m, X)$, $\widetilde{NW}(q, q^I, m, X)$ are equal to $W_{N,S}^{virt}(m, X)$, $NW^{virt}(q, q^I, m, X)$.

As in (40), we get for each $k \in \mathcal{K}$ that

$$\frac{\partial \widetilde{NW}(q, q^I, m, X)}{\partial q_k} = \mathcal{N}_k \cdot \left(\sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}} P_k(N|q) \cdot P(S|q^I) \cdot [\widetilde{W}_{N_{+k},S}(m, X) - \widetilde{W}_{N,S}(m, X)] - C_k \right). \quad (53)$$

Combined with (52), we get then that

$$\frac{\partial \widetilde{NW}(q, q^I, m, X)}{\partial q_k} = \mathcal{N}_k \cdot \Big[u_k(q, q^I, m, X) - C_k \Big]. \quad (54)$$

Concerning incumbents, we have for any $i \in S$

$$V_{i,N,S}^I(m) = E_Z \Big[\int_{m[r]}^{\infty} \int_0^u ([m[A_i^I]]^{-1}(u) - [m[A_i^I]]^{-1}(\max\{y, m[r]\})) \cdot d[\widetilde{F}_m^{(1:N \cup S_{-i})}(x|z)] \widetilde{f}_i^I(u|z) du \Big]. \quad (55)$$

If the function $x - m[A_i^I](x)$ is decreasing, then we obtain that

$$\begin{aligned} V_{i,N,S}^I(m) &\leq E_Z \Big[\int_{m[r]}^{\infty} \int_0^u (u - \max\{y, m[r]\}) \cdot d[\widetilde{F}_m^{(1:N \cup S_{-i})}(x|z)] \widetilde{f}_i^I(u|z) du \Big] \\ &= \int_{m[r]}^{\infty} (\widetilde{F}_m^{(1:N \cup S_{-i})}(x) - \widetilde{F}_m^{(1:N \cup S)}(x)) dx \\ &= \widetilde{W}_{N,S}(m, X) - \widetilde{W}_{N,S_{-i}}(m, X). \end{aligned} \quad (56)$$

Thanks to the regularity assumption, the function $x - m[A_i^I](x)$ is decreasing for any $i \in \mathcal{I}$ in the virtual pivotal mechanism and we get thus (26).

As in (42), we get for each $i \in \mathcal{I}$ that

$$\frac{\partial \widetilde{NW}(q, q^I, m, X)}{\partial q_i^I} = \sum_{N \in \mathbb{N}^K} \sum_{\substack{S \subseteq \mathcal{I} \\ i \in S}} P(N|q) \cdot \prod_{i' \in S_{-i}} q_{i'}^I \cdot \prod_{i' \in \mathcal{I} \setminus S} (1 - q_{i'}^I) \cdot \Big[\widetilde{W}_{N,S}(m, X) - \widetilde{W}_{N,S_{-i}}(m, X) \Big] - C_i^I. \quad (57)$$

Combined with (26), we get then for the virtual pivotal mechanism that

$$\frac{\partial \widetilde{NW}(q, q^I, m^{v\text{-}piv}, X)}{\partial q_i^I} \geq u_i^I(q, q^I, m^{v\text{-}piv}, X) - C_i^I. \quad (58)$$

The Poisson model: Adapting in a straightforward way the notation we introduced in the binomial model when there are incumbents, we have in the Poisson model with incumbents that $\mu \to \widetilde{NW}(\mu, q^I, m, X)$ is concave on $\mathbb{R}_+^K$ for any $m \in \mathcal{M}_A^{GSP}$ with $m[r] = X$ and so in particular for either the virtual pivotal mechanism or the efficient second-price auction. This holds because:

$$\frac{\partial \widetilde{NW}(\mu, q^I, m, X)}{\partial \mu_k} = \sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}} P(N|\mu)\cdot P(N|q^I)\cdot V_{k,N_{+k},S}(m) - C_k \tag{59}$$

which implies then from (50) that

$$\frac{\partial^2 \widetilde{NW}(\mu, q^I, m, X)}{\partial \mu_k \partial \mu_l} = -\sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}} P(N|\mu)\cdot P(S|q^I)E_Z\left[\int_X^\infty \prod_{k=1}^K [F_k(x|z)]^{n_k}\cdot\prod_{i\in S} F_i^I([m[A_i^I]]^{-1}(x)|z)\cdot(1-F_l(x|z))(1-F_k(x|z))dx\right]. \tag{60}$$

Proof of Proposition 5

The proof is almost the same as the one of Proposition 2. The difference is that due to the rents of the incumbents, a calculation à la Myerson (1981) comes on the top of it such that we have to deal with the virtual net total welfare instead of the net total welfare.

From a classic calculation using the Envelope Theorem (see Myerson 1981)[53] and given that from (14) the distribution of $y_{-i}^{N,S}$ conditional on x_i^S coincides with the unconditional distribution $G_{-i,N,S}(y_{-i}^{N,S})$, we have that for any equilibrium

$$\frac{dV_i^I(x, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m))}{dx} = \sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}_{-i}} P(N|\widehat{q}(m))\cdot P_i(S|\widehat{q}^I(m))\cdot E_{y_{-i}^{N,S}|x_i^S=x}\left[Q_{i,N,S}^I(y^{N,S};\widehat{\sigma}(m))\right] \tag{61}$$

and then by integration

$$\begin{aligned} &V_i^I(x, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) = V_i^I(\underline{x}_i, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) \\ &+\textstyle\sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}_{-i}} P(N|\widehat{q}(m))\cdot P_i(S|\widehat{q}^I(m))\cdot\int_{\underline{x}_i}^x \int Q_{i,N,S}^I(y^{N,S};\widehat{\sigma}(m))d[G_{-i,N,S}(y_{-i}^{N,S})]dx_i^S \end{aligned} \tag{62}$$

for any $x \geq \underline{x}_i$ and any $m \in \mathcal{M}$ and then by integration over x and with an integration per parts

$$\begin{aligned} &u_i^I(\widehat{q}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m)) = V_i^I(\underline{x}_i, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) \\ &+\textstyle\sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}_{-i}} P(N|\widehat{q}(m))\cdot P_i(S|\widehat{q}^I(m))\cdot\int \frac{1-F_i^I(x_i^{\mathcal{I}})}{f_i^I(x_i^{\mathcal{I}})} Q_{i,N,S}^I(y^{N,S};\widehat{\sigma}(m))d[G_{N,S}(y^{N,S})]. \end{aligned} \tag{63}$$

Summing those rents, we get the expression (15) for the rents of the seller's objective.

For each $i \in \mathcal{I}$, note that the participation constraints at the auction stage reduce to

$$V_i^I(\underline{x}_i, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) \geq 0, \tag{64}$$

while the incentive compatibility constraints require that the function

[53] We stress that the fact that the incumbents may receive additional information other than just their private valuation, e.g. about the variable z, does not change the argument.

$$x \to \sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}_{-i}} P(N|\widehat{q}(m)) \cdot P_i(S|\widehat{q}^I(m)) \cdot \int Q^I_{i,N,S}((x, y^{N,S}_{-i}); \widehat{\sigma}(m)) d[G_{-i,N,S}(y^{N,S}_{-i})] \quad \text{is non-decreasing on} \quad [\underline{x}_i, \overline{x}_i], \tag{65}$$

a constraint that will not be binding next thanks to the 'regularity' assumption which guarantees that the virtual pivotal mechanism belongs to $\mathcal{M}^{GSP}_A$ as shown previously.

For a given $(\widetilde{q}, \widetilde{q}^I) \in J^{virt}(m^{v\text{-}piv}_{\beta,X}, X) \cap \mathcal{Q}^{ex}$ satisfying A5, we let $\widehat{m} = m^{v\text{-}piv}_{\beta,X}$, $(\widehat{q}(m), \widehat{q}^I(m)) = (\widetilde{q}, \widetilde{q}^I)$ and $\widehat{\sigma}(m)$ be the truthful strategy if $m = m^{v\text{-}piv}_{\beta,X}$ and we pick any undominated strategy $\sigma(m) \in \Sigma(m)$ and any participation rates $(\widehat{q}(m), \widehat{q}^I(m)) \in M(m; \sigma(m))$ for $m \in \mathcal{M} \setminus \{m^{v\text{-}piv}_{\beta,X}\}$ (which is possible since $M(m; \sigma(m)) \neq \emptyset$). From (28), we obtain that $(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m})) \in M(\widehat{m}; \widehat{\sigma}(\widehat{m}))$. To check that this is an equilibrium, we are left with (3). On the one hand we have $u(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)) \leq NW^{virt}(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m))$ for any $m \in \mathcal{M}$ with $u(\widehat{q}(m^{v\text{-}piv}_{\beta,X}), \widehat{q}^I(m^{v\text{-}piv}_{\beta,X}), m^{v\text{-}piv}_{\beta,X}, X)$
$= NW^{virt}(\widehat{q}(m^{ESP}_X), \widehat{q}^I(m^{ESP}_X), m^{ESP}_X, X)$ since $\widehat{q}(m^{ESP}_X) \in [0,1)^K$ (see eq. (19) and the sequel). On the other hand, we have $NW^{virt}(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)) \leq NW^{virt}(\widehat{q}(m), \widehat{q}^I(m), m^{v\text{-}piv}_{\beta,X}, X)$
$\leq NW^{virt}(\widetilde{q}, \widetilde{q}^I, m^{v\text{-}piv}_{\beta,X}, X)$ for any $m \in \mathcal{M}$ (see eq. (25) and the definition of $J^{virt}(m^{v\text{-}piv}_{\beta,X}, X)$). On the whole, we obtain that $m^{v\text{-}piv}_{\beta,X} \in \operatorname{Arg\,max}_{m \in \mathcal{M}} u(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m))$ which guarantees that this is an equilibrium and thus completes the proof of the existence of an equilibrium where the seller proposes the virtual pivotal mechanism and which implements the virtual first-best.

We next show that any equilibrium that implements the virtual first-best (and thus a fortiori any equilibrium with $\widehat{q}^I(m^{v\text{-}piv}_{\beta,X}) = \widetilde{q}^I$ under the PG-refinement) is equivalent to such an equilibrium as derived above.

Consider a given equilibrium $(\widehat{m}, \widehat{q}, \widehat{q}^I, \widehat{\sigma})$ that implements the virtual first best. We have thus that $u(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m}), \widehat{m}, X; \widehat{\sigma}(\widehat{m})) = NW^{virt}(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m}), \widehat{m}, X; \widehat{\sigma}(\widehat{m})) = NW^{virt}(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m}), m^{v\text{-}piv}_{\beta,X}, X)$
$= \max_{(q,q^I) \in [0,1]^{K+I}} NW^{virt}(q, q^I, m^{v\text{-}piv}_{\beta,X}, X)$. The last equality implies that $(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m})) \in J^{virt}(m^{v\text{-}piv}_{\beta,X}, X)$. From (2), the second equality implies that $W^{virt}_{N,S}(\widehat{m}, X; \widehat{\sigma}(\widehat{m})) = W^{virt}_{N,S}(m^{v\text{-}piv}_{\beta,X}, X)$ for any pair (N, S) that occurs with positive probability, or equivalently that the good is assigned with probability one to the agent with the highest virtual valuation. Besides, any assignment where the good is assigned to the agent with the highest valuation can be implemented with the virtual pivotal mechanism provided that the breaking rule is well-specified (remember that we do not exclude that ties occur with a positive probability). On the whole we have shown that the equilibrium $(\widehat{m}, \widehat{q}, \widehat{q}^I, \widehat{\sigma})$ is equivalent to an equilibrium where the seller proposes the virtual pivotal mechanism and with the entry rates $(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m})) \in J^{virt}(m^{v\text{-}piv}_{\beta,X}, X)$ (an equilibrium which exists thanks to the first part of our proof).

Extension of Proposition 5.2 to multi-object auctions

The analysis without incumbents in Section II and IV extends in a straightforward way with multiple heterogenous objects when buyers' valuations and the seller's reservation values are both additive across objects, a model where the "pivotal" mechanism corresponds to using the efficient second-price auction for each objects. The key element in the argument still consists in showing that the net total welfare function is concave as a function of the vector of entry in the Poisson model. The net first part of the total welfare (i.e. the one which comes only from the assignment of the goods) can be viewed as a sum of the expressions where each object would have been treated separately. In particular, the only thing that matters are the marginal distributions for each object. As a sum of concave functions, this first term remains concave with multiple objects. The second part of the total welfare (i.e. the one capturing the entry costs) is linear with respect to μ which concludes the argument for concavity.

Remark: At first glance, such a setup seems to resume to a sum of various setups with a single good for sale. This is not completely the case since we allow implicitly some economies of scale through the entry costs. However those terms in the net total welfare function are linear in the vector of entry which does not alter the concavity property.

The analysis extends also to multi-unit auctions when buyers have unit-demand and the seller has flat reservation values. The key element in the argument still consists in showing that the net total welfare function is concave as a function of the vector of entry as it is established below. Formally, consider that the seller has L identical units of a good and assume that her reservation value for each unit equals X. The generalization of the standard second-price auction with the reserve price r is the $L+1^{th}$-price auction with the reserve price r. When $r = X$, this corresponds precisely to the "pivotal mechanism", i.e. the mechanism that match bidders' rents with their contribution to the welfare. To alleviate the notation, we show this point without incumbents and when entrants' valuations are drawn independently. But the result generalizes in a straightforward way to a setup with incumbents and with conditionally independent valuations.

A buyer with valuation $u \geq r$ who participates in the $L+1^{th}$-price auction with the reserve price r against the profile N when the reserve price is r will receive the expected payoff of $\int_r^u F^{(L:N)}(x)dx$. Sticking to our previous notation, the expected payoff of a group k buyer from entering such an auction (i.e. before knowing the realization of his valuation) is thus given by

$$V_{k,N_{+k}}(X) = \int_X^\infty F^{(L:N)}(x)(1 - F_k(x))dx \tag{66}$$

which is a generalization of (34). We have then

$$\begin{aligned} V_{k,[N_{+k}]_{+l}}(X) - V_{k,N_{+k}}(X) &= \int_X^\infty (F^{(L:N_{+l})}(x) - F^{(L:N)}(x))(1 - F_k(x))dx \\ &= \int_X^\infty (F_l(x) \cdot F^{(L:N)}(x) + (1 - F_l(x)) \cdot F^{(L-1:N)}(x) - F^{(L:N)}(x))(1 - F_k(x))dx \\ &= -\int_X^\infty \underbrace{(F^{(L:N)}(x) - F^{(L-1:N)}(x))}_{\geq 0} \cdot (1 - F_l(x))(1 - F_k(x))dx \end{aligned} \tag{67}$$

We obtain then a kind of generalized version of (46)

$$Y^\top \mathbf{H}_X^\mu Y = -\sum_{N\in\mathbb{N}^K} P(N|\mu)\cdot\left[\int_X^\infty \left[F^{(L:N)}(x)-F^{(L-1:N)}(x)\right]\cdot Y^\top\cdot(Q(x)^\top Q(x))\cdot Y dx\right]\leq 0, \tag{68}$$

where $\mathbf{H}_X^\mu$ is the Hessian matrix of the net total welfare function at μ in the pivotal mechanism and where $Q(x) := [(1-F_1(x)),\dots,(1-F_K(x))]$.

With respect to this setup, the allocation problem in the sponsored search auction setup (that has been presented in Section IV) can be decomposed as assigning first L homogenous units of size s_L where each bidder can receive at most one unit, second $L-1$ homogenous units of size $s_{L-1}-s_L$, and so on the last stage being a single unit of size s_1-s_2. Sticking to our previous notation, we let $V_{k,N_{+k}}(X)$ denote the expected payoff of a group k buyer from entering the pivotal mechanism associated to the reservation value X per unit of good for the seller and facing the profile N of entrants. The generalization of (34) is now

$$V_{k,N_{+k}}(X) = \sum_{l=1}^{L} s_l\cdot\int_X^\infty F^{(l:N)}(x)(1-F_k(x))dx. \tag{69}$$

We see thus that the problem shrinks to a linear combination of the previous one. The generalized version of (68) is then

$$Y^\top \mathbf{H}_X^\mu Y = -\sum_{N\in\mathbb{N}^K} P(N|\mu)\big[\int_X^\infty \sum_{l=1}^{L} s_l\cdot\left[F^{(l:N)}(x)-F^{(l-1:N)}(x)\right]\cdot Y^\top\cdot(Q(x)^\top Q(x))\cdot Y dx\big]\leq 0, \tag{70}$$

with $Q(x) := [(1-F_1(x)),\dots,(1-F_K(x))]$ and with the convention $F^{(0:N)}(x)=0$.

The Poisson model and its foundation

To define the equilibrium formally in the Poisson model, we have to slightly adapt our notation. We let

- $\mu = (\mu_1,\dots,\mu_K)\in[0,\infty)^K$ denote the profile of entry rates of potential entrants, namely when the Poisson distribution of group k buyer has mean μ_k for any $k\in\mathcal{K}$.
- $P(N|\mu) = P_k(N|\mu) = e^{-\sum_{k'=1}^K \mu_{k'}}\cdot\prod_{k'=1}^K \frac{[\mu_{k'}]^{n_{k'}}}{n_{k'}!}$ denote the probability of both the realization $N\in\mathbb{N}^K$ for the set of entrants and the realization $N\in\prod_{k=1}^K[0,\mathcal{N}_k]$ for the set of opponents of a given entrant from group k, when the profile of entry probabilities for potential entrants is μ.

Then all the expressions of the expected ex ante utilities of the various agents extend by replacing q and $P(N|q)$ with μ and $P(N|\mu)$ respectively. We have e.g. that $u(\mu,q^I,m,X;\sigma(m)) = \sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}} P(N|\mu)\cdot P(S|q^I)\cdot\Lambda_{N,S}(m,X;\sigma(m))$ denote the expected (ex ante) utility of the seller with valuation X in the mechanism m when the profile of entry rates is μ for potential entrants and q_I for incumbents and when buyers follow the bidding profile $\sigma(m)$.

For technical reasons, we add an additional constraint for the set of possible mechanisms $\mathcal{M}^*$ in the Poisson model: we assume that the monetary transfers of all agents (both the buyers and the seller) are bounded by some amount $\overline{T} > 0$. This restriction is a purely technical trick to define equilibria properly, in particular to avoid problems that could arise in unbounded mechanisms.

To define the equilibrium formally, for each $k \in \mathcal{K}$ we introduce as a counterpart to the binomial parameter functions $\widehat{q}_k : \mathcal{M} \to [0,1]$ a Poisson parameter function $\widehat{\mu}_k : \mathcal{M} \to R_+$, where $\widehat{\mu}_k(m)$ characterizes the distribution of participation of buyers of type k in the mechanism m. An equilibrium in the Poisson model is then defined as:

DEFINITION 4 *For a given set of possible mechanisms $\mathcal{M} \subseteq \mathcal{M}^*$, an equilibrium in the Poisson model of entry is defined as a strategy profile $(\widehat{m}, (\widehat{\mu}_k)_{k\in\mathcal{K}}, (\widehat{q}_i^I)_{i\in\mathcal{I}}, \widehat{\sigma})$, where $\widehat{m} \in \mathcal{M}$ stands for the seller's chosen mechanism, $\widehat{\mu}_k : \mathcal{M} \to R_+$ [resp. $\widehat{q}_i^I : \mathcal{M} \to [0,1]$] describes the Poisson entry rates of group k buyers [resp. the entry probability of incumbent i] in the various possible mechanisms $m \in \mathcal{M}$, and $\widehat{\sigma}(m) \in \Sigma(m)$ describes the bidding profile of the bidders in $m \in \mathcal{M}$ such that*

1. *(Utility maximization for the seller)*

$$\widehat{m} \in Arg \max_{m\in\mathcal{M}} \ u(\widehat{\mu}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)). \tag{71}$$

2. *(Utility maximization for group k buyers at the entry stage, for any $k \in \mathcal{K}$) for any $m \in \mathcal{M}$,*

$$\widehat{\mu}_k(m) \underset{resp.\ =0}{>0} \Longrightarrow u_k(\widehat{\mu}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m)) \underset{resp.\ \leq}{=} C_k. \tag{72}$$

3. *(Utility maximization for incumbent i at the entry stage, for any $i \in \mathcal{I}$) for any $m \in \mathcal{M}$,*

$$\widehat{q}_i^I(m) \underset{resp.\ =\binom{0}{1}}{\in (0,1)} \Longrightarrow u_i^I(\widehat{\mu}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m)) \underset{resp.\ \binom{\leq}{\geq}}{=} C_i^I. \tag{73}$$

4. *(Equilibrium conditions at the bidding stage) in any mechanism $m \in \mathcal{M}$, bidders are using undominated strategies. Furthermore, when the seller chooses the mechanism $\widehat{m}$, the bidding profile $\widehat{\sigma}(\widehat{m})$ forms a Bayes-Nash equilibrium given the entry profile $(\widehat{\mu}(\widehat{m}), \widehat{q}^I(\widehat{m}))$.*

The notion of equivalence between two strategy profiles become:

DEFINITION 5 *In the Poisson model, we say that two strategy profiles $(m, \{\mu_k\}_{k\in\mathcal{K}}, \{q_i^I\}_{i\in\mathcal{I}}, \{\sigma(m)\}_{m\in\mathcal{M}})$ and*
$(\widetilde{m}, \{\widetilde{q}_k\}_{k\in\mathcal{K}}, \{\widetilde{q}_i^I\}_{i\in\mathcal{I}}, \{\widetilde{\sigma}(m)\}_{m\in\mathcal{M}})$ are equivalent if the profile of entry rates/probabilities at the mechanism proposed by the seller are the same, namely $\mu(m) = \widetilde{\mu}(\widetilde{m})$ and $q^I(m) = \widetilde{q}^I(\widetilde{m})$, and if for any profile of bidders (N, S) that occurs with positive probability,[54] then the good is assigned in the same way with probability one (which implies in particular that $W_{N,S}(m, X; \sigma(m)) = W_{N,S}(\widetilde{m}, X; \widetilde{\sigma}(m))$).

[54] Formally, $P(N|\mu(m)) \cdot P(S|q^I(m)) > 0$.

From the same arguments as in Proposition 3, we have:

LEMMA 12 *Assume that $m_X^{ESP} \in \mathcal{M}$ and A2. Any equilibrium in the Poisson model that implements the first-best is equivalent to an equilibrium where the seller proposes the efficient second-price auction.*

Comment: Under A3 and A4 and if $m_{\beta,X}^{v-piv} \in \mathcal{M}$, then we have the result analogous to Proposition 4.1: Any equilibrium in the Poisson model that implements the virtual first-best is equivalent to an equilibrium where the seller proposes the virtual pivotal mechanism (and with the same entry profile for the incumbents).

The analysis with a finite set of potential entrants extends to the Poisson model: as detailed in Section IV, the revenue of the seller is equal to the first-best when the seller proposes the efficient second-price auction and for any entry profile $(\widehat{\mu}, \widehat{q}^I) \in J(m_X^{ESP}, X)$ which are equilibrium profiles since we still have $J(m_X^{ESP}, X) \subseteq M(m_X^{ESP})$. What remains to be shown in order to guarantee that it is an equilibrium is that $M(m; \widehat{\sigma}(m)) \neq \emptyset$ for any $m \in \mathcal{M}^*$. Furthermore, we will delineate the status of the equilibria in the Poisson model by showing that any limit (in a sense that will be made precise in the sequel) of a sequence of equilibria in the binomial model should implement the first-best and thus be equivalent to an equilibrium in which the seller proposes the efficient second-price auction.

Comment: All our arguments for environments without incumbents (where the efficient second-price auction implements the first-best) extend straightforwardly to environments with incumbents as in Section III and where the virtual pivotal mechanism implements the virtual first-best. In particular, we would obtain that any limit of a sequence of equilibria implements the virtual first-best.

A preliminary mathematical lemma Consider two sequences $(q_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}} \in [-K, K]^{\mathbb{N}}$ with $K \in R_+$. For any $n \in \mathbb{N}$, we define

$$S[n] := \sum_{i=0}^{n} \binom{n}{i} [q_n]^i [1-q_n]^{n-i} \cdot A_i. \tag{74}$$

LEMMA 13 *If $n \cdot q_n$ converges to $\mu < \infty$ when n goes to infinity, then*

$$\lim_{n\to\infty} S[n] = \sum_{i=0}^{\infty} e^{-\mu} \frac{\mu^i}{i!} \cdot A_i \in [-K, K]. \tag{75}$$

Proof Let $\overline{\mu} = \max_{n\in\mathbb{N}} n \cdot q_n < \infty$ which is well defined since $n \cdot q_n$ converges to $\mu < \infty$. For any $z \in \mathbb{N}$ with $z \leq n$, we define $S_1[n,z] := \sum_{i=0}^{z-1} \binom{n}{i} [q_n]^i [1-q_n]^{n-i} \cdot A_i$ and $S_2[n,z] := \sum_{i=z}^{n} \binom{n}{i} [q_n]^i [1-q_n]^{n-i} \cdot A_i$. We have thus $S[n] = S_1[n,z] + S_2[n,z]$.

For any pair $i, z \in \mathbb{N}$, we have $(i+z)! \geq i!z!$, which further implies that $\binom{n}{i+z} \leq \frac{n^z}{z!}\binom{n-z}{i}$ for any $n \geq i+z$. After some calculation we have then for any pair $n, z \in \mathbb{N}$ with $z \leq n$, $|S_2[n,z]| = |\sum_{i=0}^{n-z} \binom{n}{i+z} \cdot [q_n]^{i+z}[1-q_n]^{n-i-z} \cdot A_i| \leq \sum_{i=0}^{n-z} \frac{[n \cdot q_n]^z}{z!} \binom{n-z}{i} \cdot [q_n]^i [1-q_n]^{n-i-z} \cdot |A_i| \leq \frac{\overline{\mu}^z}{z!} \cdot K \cdot \sum_{i=0}^{n-z} \binom{n-z}{i} \cdot [q_n]^i [1-q_n]^{n-i-z} = \frac{\overline{\mu}^z}{z!} \cdot K$. It is well-known from the properties of the factorial that

$\lim_{n\to\infty} \frac{\overline{\mu}^z}{z!} = 0$. We have also $|\sum_{i=z}^{\infty} e^{-\mu}\frac{\mu^i}{i!}\cdot A_i| \leq \frac{\mu^z}{z!}\cdot|\sum_{i=0}^{\infty} e^{-\mu}\frac{\mu^i}{i!}\cdot A_{i+z}| \leq \frac{\mu^z}{z!}\cdot K$. For any $\epsilon > 0$, we can thus pick z large enough (say $z = z^*$) such that $\max_{n\in\mathbb{N}:n\geq z^*}|S_2[n,z^*]| \leq \frac{\epsilon}{3}$ and $|\sum_{i=z^*}^{\infty} e^{-\mu}\frac{\mu^i}{i!}\cdot A_i| \leq \frac{\epsilon}{3}$.

For any $i < z^*$, we have $\lim_{n\to\infty}[1-q_n]^{n-i} = e^{-\mu}$ and then we can easily check that $\lim_{n\to\infty}\binom{n}{i}[q_n]^i[1-q_n]^{n-i} = e^{-\mu}\frac{\mu^i}{i!}$.

Consequently, if n is large enough, $|S_1[n,z^*]-\sum_{i=0}^{z^*-1} e^{-\mu}\frac{\mu^i}{i!}\cdot A_i|$ can be bounded by $\frac{\epsilon}{3}$ and then finally $|S[n]-\sum_{i=0}^{\infty} e^{-\mu}\frac{\mu^i}{i!}\cdot A_i| \leq |S_1[n,z^*]-\sum_{i=0}^{z^*-1} e^{-\mu}\frac{\mu^i}{i!}\cdot A_i|+|S_2[n,z^*]|+|\sum_{i=z^*}^{\infty} e^{-\mu}\frac{\mu^i}{i!}\cdot A_i|$ is bounded by ϵ. Since this is true for any $\epsilon > 0$, we have established (75). **Q.E.D.**

For a given vector of potential entrants $\mathcal{N}$, let $\widehat{m}[\mathcal{N}]$, $\widehat{q}[\mathcal{N}]$ and $\widehat{\sigma}[\mathcal{N}]$ denote the equilibrium played. We let also $\overline{NW}[\mathcal{N}]$ denote the corresponding equilibrium net total welfare, namely $\overline{NW}[\mathcal{N}] := \sum_{N\in\mathbb{N}^K} P(N|\widehat{\mu}[\mathcal{N}])\cdot W_N(\widehat{m}[\mathcal{N}],X;\widehat{\sigma}(m)[\mathcal{N}]) - \sum_{k=1}^{K}\widehat{q}_k[\mathcal{N}]\cdot\mathcal{N}_k\cdot C_k$.

To develop "limit" results, we consider sequences of the form $(\mathcal{N}[l])_{l\in\mathbb{N}}$ where $\mathcal{N}[l] = (\mathcal{N}_1[l],\ldots,\mathcal{N}_K[l]) \in \mathbb{N}^K$ such that for any $k\in\mathcal{K}$, $\mathcal{N}_k[l]$ goes to infinity when l goes to infinity. When l goes to infinity, it is our way to formalize that the number of potential entrants goes large in each group.

The following proposition gives the status of considering the Poisson model: any limit of equilibria in the binomial model (under the conditions of Proposition 2) implements the first-best in the Poisson model.

PROPOSITION 14 *Assume that $m_X^{ESP}\in\mathcal{M}$ and A2. When l goes to infinity, $\overline{NW}[\mathcal{N}[l]]$ goes to $\max_{\mu\in R_+^K} NW(\mu, m_X^{ESP}, X)$, namely the equilibrium net total welfare goes to the first-best total welfare in the Poisson model when the number of potential participants goes to infinity in the binomial model.*

Proof As a preliminary, we show the following lemma for any given $m\in\mathcal{M}^*$:

LEMMA 15 *Consider that a given strategy σ^* is played in equilibrium in the mechanism m for any $\mathcal{N}$.*[55] *If for any $k\in\mathcal{K}$, $\mathcal{N}_k[l]\cdot\widehat{q}_k(m)[\mathcal{N}[l]]$ goes to $\mu_k(m)\in R_+\cup\{\infty\}$ when l goes to infinity, then $\mu_k(m)<\infty$ for each k and $\mu(m)\in M(m;\sigma^*)$.*

In words, any limit of equilibrium entry probabilities in the binomial model is an equilibrium profile in the Poisson model.

Proof of Lemma 15 Below, we use the following notation for each $k\in\mathcal{K}$: $\mathcal{N}_j^k = \mathcal{N}_k$ for $j\in\mathcal{K}\setminus\{j\}$ and $\mathcal{N}_k^k = \mathcal{N}_k - 1$. Consider a given k. We first establish that $\mu_k(m)<\infty$. Suppose by contradiction that $\mu_k(m)=\infty$. Then there exists l^* such that $l\geq l^*$ implies that $\widehat{q}_k(m)[\mathcal{N}[l]]>0$ and then from (4), we obtain

$$\sum_{n_1=0}^{\mathcal{N}_1^k[l]}\cdots\sum_{n_K=0}^{\mathcal{N}_K^k[l]}\prod_{j=1}^{K}\binom{\mathcal{N}_j^k[l]}{n_j}[\widehat{q}_j(m)[\mathcal{N}[l]]]^{n_j}[1-\widehat{q}_j(m)[\mathcal{N}[l]]]^{\mathcal{N}_j^k[l]-n_j}\cdot V_{k,N_{+k}}(m;\sigma^*) \underset{(resp.\ \geq)}{=} C_k \text{ if } \widehat{q}_k(m)[\mathcal{N}[l]] \underset{(resp.\ =)}{<} 1,$$

[55] This is a very strong assumption. However, in generalized second-price auctions, we have in mind that σ^* corresponds to truthful bidding in the sequel.

for any $l \geq l^*$. This is also equivalent to

$$\sum_{n_1=0}^{\mathcal{N}_1^k[l]} \cdots \sum_{n_K=0}^{\mathcal{N}_K^k[l]} \prod_{j=1}^{K} \binom{\mathcal{N}_j^k[l]}{n_j} [\widehat{q}_j(m)[\mathcal{N}[l]]]^{n_j} [1-\widehat{q}_j(m)[\mathcal{N}[l]]]^{\mathcal{N}_j^k[l]-n_j} \cdot n_k \cdot V_{k,N_{+k}}(m;\sigma^*) \underset{(resp.\ \geq)}{=} \mathcal{N}_k[l] \cdot \widehat{q}_k(m)[\mathcal{N}[l]] \cdot C_k \quad (76)$$

if $\widehat{q}_k(m)[\mathcal{N}[l]] < 1$ [resp. $\widehat{q}_k(m)[\mathcal{N}[l]] = 1$], for any $l \geq l^*$. Since transfers and valuations are bounded we have then that the sum of the payoffs of any subset of agents is bounded by $\overline{x} + \overline{T}$ for any realization of the set of entrants. We have thus in particular that $n_k \cdot V_{k,N_{+k}}(m) \leq \overline{x} + \overline{T}$ for any N. The left-hand term in (76) is the expectation over N of an expression that is uniformly bounded by $\overline{x} + \overline{T}$ is also uniformly bounded by $\overline{x} + \overline{T}$ while the right-hand term goes to infinity, which leads to a contradiction. We have thus $\mu_k(m) < \infty$.

By repeated use of Lemma 13 for each of the K sums, we obtain then for each $k \in \mathcal{K}$ that the fact that the inequality

$$\sum_{n_1=0}^{\mathcal{N}_1^k[l]} \cdots \sum_{n_K=0}^{\mathcal{N}_K^k[l]} \prod_{j=1}^{K} \binom{\mathcal{N}_j^k[l]}{n_j} [\widehat{q}(m)[\mathcal{N}[l]]]^{n_j} [1-\widehat{q}(m)[\mathcal{N}[l]]]^{\mathcal{N}_j^k[l]-n_j} \cdot V_{k,N_{+k}}(m;\sigma^*) \leq C_k$$

holds for any $l \in \mathbb{N}$ implies that
$\sum_{N\in\mathbb{N}^K} e^{-\sum_{j=1}^K \mu_j(m)} \prod_{j=1}^K \frac{[\mu_j(m)]^{n_j}}{n_j!} V_{k,N_{+k}}(m;\sigma^*) \leq C_k$. Suppose now that $\mu_k(m) > 0$. Then there exists l^* such that $l \geq l^*$ implies that $\widehat{q}_k(m)[\mathcal{N}[l]] > 0$ and then from (4), we obtain then that the equality
$\sum_{n_1=0}^{\mathcal{N}_1^k[l]} \cdots \sum_{n_K=0}^{\mathcal{N}_K^k[l]} \prod_{j=1}^K \binom{\mathcal{N}_j^k[l]}{n_j} [\widehat{q}_j(m)[\mathcal{N}[l]]]^{n_j} [1-\widehat{q}_j(m)[\mathcal{N}[l]]]^{\mathcal{N}_j^k[l]-n_j} \cdot V_{k,N_{+k}}(m;\sigma^*) = C_k$ holds for any $l \geq l^*$ which further implies that
$\sum_{N\in\mathbb{N}^K} e^{-\sum_{j=1}^K \mu_j(m)} \prod_{j=1}^K \frac{[\mu_j(m)]^{n_j}}{n_j!} V_{k,N_{+k}}(m;\sigma^*) = C_k$. We have then established that $\mu_k(m)$ satisfies the equilibrium equation (72) for any k. On the whole we have $\mu(m) \in M(m;\sigma^*)$. **End of the proof of Lemma 15**

The rest of the proof goes as follows. In the limit Poisson model, we know all equilibria implement the first-best. Thus, we cannot have accumulation points of the sequence of welfare for the finite economy that are away from the first-best, as otherwise it would imply that in the Poisson model, some net welfare other than the first-best could be achieved. Formally, consider the following sequence of entry rates: in equilibrium in the binomial model with $\mathcal{N}$, when the seller proposes the efficient second-price auction, the entry probabilities are given by $\widehat{q}(m_X^{ESP})[\mathcal{N}]$ and let $NW^*[\mathcal{N}]$ be the corresponding total welfare in the binomial model (which may be strictly lower than the first-best solution given by (10). Next the first-best welfare is denoted by $NW^{opt}[\mathcal{N}]$). Note that we necessarily have $\widehat{q}_k(m_X^{ESP})[\mathcal{N}] \cdot \mathcal{N}_k \leq \frac{\overline{x}}{C_k}$ because the expected revenue of the seller should be larger than X (and so must be the net total welfare in equilibrium). We show below that the sequence $NW^*[\mathcal{N}[l]]$ converges to $\max_{\mu\in R_+^K} NW(\mu, m_X^{ESP}, X)$. To establish this, we show that every subsequence has a subsequence that converges to $\max_{\mu\in R_+^K} NW(\mu, m_X^{ESP}, X)$.

Every subsequence has a subsequence such that $\widehat{q}_k(m_X^{ESP})[\mathcal{N}[l]] \cdot \mathcal{N}_k[l]$ converges to some μ_k^*

for each k (because $\widehat{q}_k(m_X^{ESP})[\mathcal{N}[l]] \cdot \mathcal{N}_k[l]$ stays in the compact set $[0, \frac{\overline{x}}{C_k}]$). Then we can apply Lemma 15 (where the strategy σ^* in Lemma 15 corresponds to truthful bidding here) and we obtain that $\mu^* \in M(m_X^{ESP})$ and then $\mu^* \in J(m_X^{ESP}, X)$ (because $M(m_X^{ESP}) = J(m_X^{ESP}, X)$ in the Poisson model). Then we can apply Lemma 13 iteratively for each k to the corresponding subsequence of the net welfare $(NW^*[\mathcal{N}[l']])$,[56]

and we obtain that there exist a subsequence whose total welfare has the limit $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$, namely the first-best in the Poisson model. With the same argument, we have also that the sequence $(NW^{opt}[\mathcal{N}[l]])_{l \in \mathbb{N}}$ converges to $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$ when l goes to infinity.

Let us come back to the core of the proof of Proposition 14. Since the expected net total welfare is bounded in equilibrium ($\overline{NW}[\mathcal{N}] \in [X, \overline{x}]$), it is sufficient to show that the sequence $(\overline{NW}[\mathcal{N}[l]])_{l \in \mathbb{N}}$ cannot have another accumulation point.

Suppose that an accumulation point of the sequence $\overline{NW}[\mathcal{N}[l]]$ lies strictly below $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$, then it would raise a contradiction since it would imply that we can pick a large enough l such that the revenue of the seller is strictly below the one it would have raised with the efficient second-price auction (because we have shown above that the net total welfare, or equivalently the revenue of the seller in equilibrium when the number of potential in each is large enough (such that A1 is satisfied), converges to $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$) when l goes to infinity.

We have that $\overline{NW}[\mathcal{N}] \leq \overline{NW}^{opt}[\mathcal{N}]$ for any $\mathcal{N}$. Since we have shown above that the sequence $(NW^{opt}[\mathcal{N}[l]])_{l \in \mathbb{N}}$ converges to $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$) when l goes to infinity, then we obtain as corollary that any accumulation point of the sequence $(\overline{NW}[\mathcal{N}[l]])_{l \in \mathbb{N}}$ is below $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$. On the whole, the only possible accumulation point of the sequence $(\overline{NW}[\mathcal{N}[l]])_{l \in \mathbb{N}}$ is the first-best welfare in the Poisson model. **Q.E.D.**

To conclude this section, let us show that there exist an equilibrium within the Poisson model. For a given mechanism $m \neq m_X^{ESP}$, fix a given undominated strategy $\widehat{\sigma}(m)$. Then for any $\mathcal{N}$, we can define corresponding equilibrium entry profiles in the binomial model, say $\widehat{q}_k(m; \widehat{\sigma}(m))[\mathcal{N}]$. Note that we have shown that $\mathcal{N}_k \cdot \widehat{q}_k(m; \widehat{\sigma}(m))[\mathcal{N}] \in [0, \frac{\overline{X}+T}{C_k}]$ for any binomial model $\mathcal{N}$. For any sequence $(\mathcal{N}[i])_{i \in \mathbb{N}}$, there is a subsequence $(\mathcal{N}[i])_{\sigma(i) \in \mathbb{N}}$ ($\sigma : \mathbb{N} \to \mathbb{N}$ is an increasing function) such that $\mathcal{N}_k[\sigma(i)] \cdot \widehat{q}_k(m; \widehat{\sigma}(m))[\mathcal{N}[\sigma(i)]]$ goes to $\mu(m) \in [0, \frac{\overline{X}+T}{C_k}]^K$. From Lemma 15, we obtain that $\mu(m) \in M(m; \widehat{\sigma}(m))$ and thus that $M(m; \widehat{\sigma}(m)) \neq \emptyset$. As a corollary, the construction we did to build an equilibrium that implements the firsts-best in the binomial model carries over for the Poisson model. In particular, the set of equilibria according to Definition 4 is not empty.

Discrimination with linear distortions

Let $\mathcal{M}_X^{linear} \subseteq \mathcal{M}_A^{GSP}$ denote the set of generalized second-price auctions with $r = X$, $A_k(b) =$

[56] Note that we have:

$$NW^*[\mathcal{N}[l]] = \sum_{n_1=0}^{\mathcal{N}_1[l]} \cdots \sum_{n_K=0}^{\mathcal{N}_K[l]} \prod_{k=1}^{K} \binom{\mathcal{N}_k[l]}{n_k} [\widehat{q}_k(m_X^{ESP})[\mathcal{N}[l]]]^{n_k} [1 - \widehat{q}_k(m_X^{ESP})[\mathcal{N}[l]]]^{\mathcal{N}_k[l]-n_k} \cdot W_N(m_X^{ESP}, X) - \sum_{k=1}^{K} \widehat{q}_k(m_X^{ESP})[\mathcal{N}[l]] \cdot \mathcal{N}_k[l] \cdot C_k.$$

$\alpha \cdot b$ if $b \geq r$ and $A_k(b) = 0$ otherwise for any $k \in \mathcal{K}$, and $A_i^I(b) = \alpha^I \cdot b$ if $b \geq r$ and $A_i^I(b) = 0$ otherwise for any $i \in \mathcal{I}$, with both $\alpha \geq 1$ and $\alpha^I \geq 1$. It is straightforward to check that both the allocation rule and the payment rule depend solely on the ratio $r^\alpha = \frac{\alpha}{\alpha^I} \in (0, \infty)$ and so that any generalized second-price auction in $\mathcal{M}_X^{linear}$ is characterized solely by this ratio. We use next the notation $r_\alpha \in (0, \infty)$ to denote a generic mechanism in $\mathcal{M}_X^{linear}$. The mechanism $r_\alpha = 1$ corresponds to the efficient second-price auction.

We consider the Poisson model throughout this Section. We will also consider that the entry costs of the incumbents are null so that they enter with probability one for any $m \in \mathcal{M}_X^{linear}$ and that their rents are not fully internalized by the seller, i.e. A3. We also assume homogenous entrants ($K = 1$). Last we assume that if the efficient second-price auction is posted, then some potential entrants will enter with positive probability. To sum up,

ASSUMPTION A 7 $K = 1$, $C_i^I = 0$ *for each* $i \in \mathcal{I}$, *and* $u(0, (1, \cdots, 1), m_X^{ESP}) > C$.

To alleviate notation, for any $r_\alpha \in \mathcal{M}_X^{linear}$, we let then $V_{1,N,\mathcal{I}}(r_\alpha; \sigma(r_\alpha)) \equiv V_n(r_\alpha)$, $V_{i,N,\mathcal{I}}^I(r_\alpha; \sigma(r_\alpha)) \equiv V_{i,n}^I(r_\alpha)$, $u_k(\mu, (1, \cdots, 1), r_\alpha; \sigma(r_\alpha)) \equiv u_k(\mu, r_\alpha)$ and $u_i^I(\mu, (1, \cdots, 1), r_\alpha; \sigma(r_\alpha)) \equiv u_i^I(\mu, r_\alpha)$.

To alleviate the proof, we also add this technical assumption:

ASSUMPTION A 8 *The CDFs* $F(.|z)$ *and* $F_i^I(.|z)$ *do not depend on* z *and are continuously differentiable on their (common) support* $[\underline{x}, \overline{x}]$ *with* $X \in (\underline{x}, \overline{x})$.

In the auction $r_\alpha \in \mathcal{M}_X^{linear}$, the probability of an entrant [resp. the incumbent $i \in \mathcal{I}$] with valuation $x \geq X$ to win the good when he faces $n-1$ competing entrants [resp. n entrants] and when the set of incumbents that participate is $\mathcal{I}$ is equal to $[F(x)]^{n-1} \cdot \prod_{i=1}^{I} F_i^I(\max\{r_\alpha \cdot x, X\}) \equiv F_{r_\alpha}^{1:(n-1)\cup\mathcal{I}}(x)$ [resp. $[F(\max\{\frac{x}{r_\alpha}, X\})]^n \cdot \prod_{\substack{i'=1 \\ i' \neq i}}^{I} F_{i'}^I(x) \equiv F_{r_\alpha}^{1:n\cup\mathcal{I}_{-i}}(x)$]. Using standard results from auction theory, we obtain that the (interim) payoff of an entrant [resp. the incumbent i] from participating in the auction r_α when facing $n-1$ competing entrants [resp. n entrants] and when the set of incumbents that participate is $\mathcal{I}$ is given by

$$V_n(r_\alpha) = \int_X^{\overline{x}} F_{r_\alpha}^{1:(n-1)\cup\mathcal{I}}(x) \cdot (1 - F(x))dx \ \ \text{[resp. } V_{i,n}^I(r_\alpha) = \int_X^{\overline{x}} F_{r_\alpha}^{1:n\cup\mathcal{I}_{-i}}(x) \cdot (1 - F_i^I(x))dx \ \text{]}. \tag{77}$$

The ex ante expected payoff of an entrant [resp. the incumbent i] from participating in the auction r_α with the participation rate μ is then given by

$$u(\mu, r_\alpha) = \int_X^{\overline{x}} e^{-\mu(1-F(x))} \prod_{i=1}^{I} F_i^I(\max\{r_\alpha \cdot x, X\})(1 - F(x))dx \tag{78}$$

[resp. $u_i^I(\mu, r_\alpha) = \int_X^{\overline{x}} e^{-\mu(1-F(\max\{\frac{x}{r_\alpha}, X\}))} \prod_{\substack{i'=1 \\ i' \neq i}}^{I} F_{i'}^I(x)(1 - F_i^I(x))dx$]. Note that $u_i^I(\mu, r_\alpha) > 0$ for any μ and r_α which justifies that we assumed above that incumbents participate to the auction with probability 1 (more generally any bidders with a valuation strictly above X has always a strictly positive expected profit because he may face no competitors with a valuation above r). We have then

$$\frac{\partial u(\mu, r_\alpha)}{\partial r_\alpha} = r_\alpha \cdot \sum_{i=1}^{I} \int_{\max\{\frac{X}{r_\alpha}, X\}}^{\overline{x}} e^{-\mu(1-F(x))} \prod_{\substack{i'=1 \\ i'\neq i}}^{I} F_i^I(r_\alpha \cdot x)(1-F(x)) f_i^I(r_\alpha \cdot x)dx \geq 0 \quad (79)$$

where the inequality is strict if $r_\alpha \in (\frac{X}{\overline{x}}, \frac{\overline{x}}{X})$, and

$$\frac{\partial u_i^I(\mu, r_\alpha)}{\partial r_\alpha} = -\frac{\mu}{[r_\alpha]^2} \int_{\max\{r_\alpha \cdot X, X\}}^{\overline{x}} e^{-\mu(1-F(\frac{x}{r_\alpha}))} \prod_{\substack{i'=1 \\ i'\neq i}}^{I} F_{i'}^I(x)(1-F_i^I(x)) f(\frac{x}{r_\alpha})dx \leq 0 \quad (80)$$

where the inequality is strict if $r_\alpha \in (\frac{X}{\overline{x}}, \frac{\overline{x}}{X})$ and $\mu > 0$. We have similarly:

$$\frac{\partial u(\mu, r_\alpha)}{\partial \mu} = -\int_X^{\overline{x}} e^{-\mu(1-F(x))} \prod_{i=1}^{I} F_i^I(\max\{r_\alpha \cdot x, X\})(1-F(x))^2 dx < 0 \quad (81)$$

and

$$\frac{\partial u_i^I(\mu, r_\alpha)}{\partial \mu} = -\int_X^{\overline{x}} e^{-\mu(1-F(\max\{\frac{x}{r_\alpha}, X\}))} \prod_{\substack{i'=1 \\ i'\neq i}}^{I} F_{i'}^I(x)(1-F_i^I(x))(1-F(\max\{\frac{x}{r_\alpha}, X\}))dx \leq 0 \quad (82)$$

where the last inequality is strict if $r_\alpha > \frac{X}{\overline{x}}$. From (81), the equilibrium condition (72) has a unique solution when the posted mechanism is r_α: the solution corresponds thus to $\widehat{\mu}(r_\alpha)$ the equilibrium entry rate for potential entrants when the posted mechanism is r_α.

LEMMA 16 *We have $\frac{d\widehat{\mu}(r_\alpha)}{dr_\alpha} \geq 0$ for any point $r_\alpha \in (0, \infty)$ such that $\widehat{\mu}(r_\alpha) > 0$. As a corollary, the function $r_\alpha \to \widehat{\mu}(r_\alpha)$ is nondecreasing.*

Proof Deriving the equilibrium condition (72) at r_α such that $\widehat{\mu}(r_\alpha) > 0$, we have $\frac{d\widehat{\mu}(r_\alpha)}{dr_\alpha} = -\frac{\frac{\partial u(\widehat{\mu}(r_\alpha), r_\alpha)}{\partial r_\alpha}}{\frac{\partial u(\widehat{\mu}(r_\alpha), r_\alpha)}{\partial \mu}} \geq 0$. We conclude with the inequalities (79) and (81). **Q.E.D.**

Favoring entrants with respect to incumbents has two impacts on the informational rents of the incumbents: on the one hand, raising the ratio r_α reduces their informational rents ceteris paribus (eq. (80)); on the other hand, raising r_α increases the incentives of the potential entrants to enter the auction which is detrimental indirectly to the incumbents because they face more competition from new entrants (eq. (82)). On the whole, it is thus not ambiguous that increasing r_α is detrimental to the incumbents.

LEMMA 17 *For each incumbent $i \in \mathcal{I}$, we have $\frac{du_i^I(\widehat{\mu}(r_\alpha), r_\alpha)}{dr_\alpha} \leq 0$ and the inequality is strict if $r_\alpha = 1$.*

Proof We note first that $\frac{du_i^I(\widehat{\mu}(r_\alpha),r_\alpha)}{dr_\alpha} = \underbrace{\frac{\partial u_i^I(\widehat{\mu}(r_\alpha),r_\alpha)}{\partial r_\alpha}}_{\substack{\leq 0 \text{ with} \\ <0 \text{ if } r_\alpha \in (\frac{X}{\overline{x}}, \frac{\overline{x}}{X})}} + \underbrace{\frac{d\widehat{\mu}(r_\alpha)}{dr_\alpha}}_{\geq 0} \cdot \underbrace{\frac{\partial u_i^I(\widehat{\mu}(r_\alpha),r_\alpha)}{\partial \mu}}_{\leq 0} \leq 0.$

We conclude by noting that $1 \in (\frac{X}{\overline{x}}, \frac{\overline{x}}{X})$. **Q.E.D.**

Next proposition formalizes that incumbents should be discriminated against entrants.

PROPOSITION 18 *Assume A3, A7, A8, and $\mathcal{M} = \mathcal{M}_X^{linear}$. In equilibrium, the chosen mechanism r_α satisfies $r_\alpha > 1$.*

Proof of Proposition 18 In equilibrium, the revenue of the seller is given by

$$u(\widehat{\mu}(r_\alpha), r_\alpha) = \left[\sum_{n=0}^{\infty} e^{-\widehat{\mu}(r_\alpha)} \frac{[\widehat{\mu}(r_\alpha)]^n}{n!} \cdot W_{N,\mathcal{I}}(r_\alpha, X) - \widehat{\mu}(r_\alpha) \cdot C\right] - \sum_{i=1}^{I} (1 - \beta_i^I) \cdot u_i^I(\widehat{\mu}(r_\alpha), r_\alpha). \quad (83)$$

Furthermore, the term in the bracket corresponds to the total net welfare which is maximized at $r_\alpha = 1$. Combined with Lemma 17, we conclude that $\operatorname{Arg\,max}_{r_\alpha \in (0,\infty)} u(\widehat{\mu}(r_\alpha), r_\alpha) \subseteq (1, \infty)$. **Q.E.D.**