

HOW TO CONTROL CONTROLLED SCHOOL CHOICE ONLINE APPENDIX

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APPENDIX B. CONTROLLED SCHOOL CHOICE IN THE US

The legal background on diversity in school admissions is complicated. Since the landmark 1954 *Brown v. Board of Education* supreme court ruling, which ended school segregation, many school districts have attempted to achieve more integrated schools. The current legal environment is summarized in the 2011 guidelines issued by the US departments of justice and education: “Guidance on the voluntary use of race to achieve diversity and avoid racial isolation in elementary and secondary schools.” (There is a separate set of guidelines for college admissions.) We shall not summarize these guidelines here, but suffice it to say that they are perfectly compatible with the theory developed in this paper.

In particular, the “race neutral” approaches described in the guidelines can be carried out through our methods (race neutrality goes into the definition of types). We proceed to briefly describe some of the best-known programs in the US.

B.1. Chicago. We illustrate the implications of our results by discussing the case of public high schools in Chicago (Pathak and Sönmez, 2013; Kominers and Sönmez, 2012). The point of this exercise is to illustrate why our results, and more generally the axiomatic method adopted here, are useful in practical mechanism design.

Chicago uses a rule that is very close to reserves, and can therefore be understood in terms of some of our axioms. In fact, the relatively minor difference between reserves and the Chicago rule provides an opportunity for

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discussing how our axioms can be used to evaluate a system that is already in place, and how it can be modified.

Selective Enrollment High Schools in Chicago have an affirmative action policy using socioeconomic status that divides students into four types: Tier 1, Tier 2, Tier 3, and Tier 4. Any given high school has 30% of its seats open; assigned purely based on students' priority. The remaining seats are divided into four equal categories, each reserved for a particular tier. Students are ordered in terms of priorities, and they are selected in that order to apply to schools. A student submits a preference over schools, and gets assigned to her most preferred school that still has space for her. When she is assigned to a school, she occupies an open seat, if one is available. If no seat is available, then she occupies a reserved seat.¹

The Chicago model can be thought of as our reserves system, with the difference that open seats are allocated before reserved seats. In terms of our results, we can trace the difference between Chicago and reserves to the fact that the *Chicago rule violates the axiom of saturated \succ -compatibility*. We can illustrate the situation using a simple example, in which there are four students and one school: $\mathcal{S} = \{s_1, s_2, s_3, s_4, s_5\}$. The first three students are of one type ($t = 1$) and the second two are of another type ($t = 2$). The school has a capacity of $q = 3$; so that one seat is open, one is reserved for $t = 1$, and one is reserved for $t = 2$. Students' priorities are

$$s_1 \succ s_4 \succ s_5 \succ s_2 \succ s_3.$$

When $S = \{s_1, s_2, s_4, s_5\}$ apply, the reserves rule would assign students s_1 , s_4 and s_5 to the school, leaving out the weakest student in S , s_2 . The Chicago rule would instead assign s_1 , s_2 and s_4 to the school. This is a violation of saturated \succ -compatibility because the lower-priority student s_2 is accepted over the higher-priority student s_5 , even though type $t = 1$ is saturated in $\{s_1, s_2, s_4, s_5\}$.²

¹This mechanism is a special case of SOSM.

²For the Chicago rule, type $t = 1$ is saturated in $S = \{s_1, s_2, s_4, s_5\}$ because $S' = \{s_2, s_3, s_4, s_5\}$ has the same number of type $t = 1$ students as in S and from S' only one type $t = 1$ student is chosen.

In practice, the system in Chicago seems to seek diversity goals over and above those expressed by the number of reserved seats. The simulations in Kominers and Sönmez (2012) suggest that the Chicago rule favors low-socioeconomic tiers more than a system of reserves based on the same number of reserved seats. A recent article in the Chicago Tribune gives anecdotal evidence in the same direction.³ Such results, favoring one type over another, depend on the details of the actual score (priority) distributions for different types. The details of the score distributions are complicated to both families and policy makers. It may be more transparent and predictable, and therefore better policy, to achieve those goals through increased reserves for low-socioeconomic tiers. The recent paper by Dur, Kominers, Pathak, and Sönmez (2014) makes a similar point regarding the order in which open seats, and seats reserved for “walk zone” students, are allocated in Boston.

There is another, less relevant, difference between reserves and the Chicago rule. In Chicago, if any seats are left unfilled then they are again split equally into reserves for the four ties in a second stage assignment. Our reserves rule instead opens up the unassigned seats for any type to compete for based on priorities. The reason why this difference does not matter is that Selective Enrollment Schools in Chicago are overdemanded: In practice there typically are no vacant seats.

B.2. Jefferson County. The Jefferson County (KY) School District is prominent in promoting diversity among its schools, and the litigation surrounding its admissions policies serves partly as a basis for the 2011 US government guidelines mentioned above. The rules proposed by the county violate the GS axiom, and therefore would be incompatible with an assignment mechanism based on the deferred acceptance algorithm, like that in use in Boston, New York City, or Chicago. We believe that Jefferson County’s objectives could be satisfied by using one of the rules we propose in this paper—for example, reserves.

Starting from the early 1970s, the student assignment plan used in Jefferson County went through major changes. First, in order to avoid segregation, a racial assignment plan was used and students were bussed to their schools. In

³“A really brutal admissions year,” Chicago Tribune March 3rd, 2012.

the early 1990s a school choice system was implemented, allowing parents to state their preferences over schools. In 1996, schools were required to have between 15 and 50 percent of African American students. In 2002 a lawsuit was filed against the Jefferson County School District because it had a racial admissions policy. After a litigation process, the case came before the US Supreme Court. The Supreme Court in 2007 ruled in favor of the plaintiffs, and decided that race cannot be the only factor to use for admissions.

Following this ruling, Jefferson County switched to an assignment plan that considered the socioeconomic status of parents: Using census data, the school district divided the county into two regions and required all schools to have between 15 and 50 percent of their students to be from the first region. This rule violates GS.

Jefferson County is undergoing yet another change at the moment. The new assignment plan, which was accepted by the school district to be implemented in 2013/14 admissions cycle, divides students into three types: Type 1, Type 2, and Type 3. These types are determined by educational attainment, household income, and percentage of white residents in the census block group that the student lives in. Then each school is assigned a diversity index, defined as the average of student types. The new admissions policy requires each school to have a diversity index between 1.4 and 2.5.

These two assignment policies are in conflict with the GS axiom, so they are incompatible with a school choice plan that would seek to install a stable (or fair) matching. It should be clear, however, that the rules proposed in our paper can achieve similar objectives to the ones in the current policies, while satisfying GS.

APPENDIX C. CONTROLLED SCHOOL CHOICE IN OTHER COUNTRIES

Policies to enhance diversity can be found in many countries around the world (Sowell, 2004). Some of these policies implement preferential policies, whereas some of them implement policies based on quotas. The former resemble the reserves rule that we have studied, while the latter are similar to the quotas model (with regional variations in actual implementation). Many countries have similar policies, including, but not limited to, Brazil, China,

Germany, Finland, Macedonia, Malaysia, Norway, Romania, Sri Lanka, and the United States. Below we discuss two particular examples: college admissions in India and high school admissions in French-speaking Belgium.

C.1. Indian College Admissions. In India the caste system divides society into hereditary groups or castes (“types” in our model). Historically, it has enforced a particular division of labor and power in society, and placed severe limits on socioeconomic mobility. To overcome this, the Indian Constitution has since 1950 implemented affirmative action. It prescribes that the “scheduled castes” (SCs) and “scheduled tribes” (STs) be represented in government jobs and public universities proportional to their population percentage in the state that they belong to. These percentages change from state to state. For example, in Andhra Pradesh, each college allocates 15 percent of its seats for SCs, 6 percent for STs, 35 percent for other “backward classes,” and the remaining 44 percent is left open for all students.

The college admissions to these public schools is administered by the state, and it works as follows. Students take a centralized exam that determines their ranking. Then students are called one by one to make their choices from the available colleges. In each college, first the open seats are filled. Afterward the reserved seats are filled only by students for whom the seats are reserved. This model corresponds to the situation described above for Chicago. Therefore, this affirmative action policy fits into our quotas-ideal-point model in which we replace each school with two copies, the first representing the open seats and the second representing the rest. For the first copy of the college, each student is treated the same and the choice rule picks the best available students regardless of their caste. For the second copy, a choice model based on quotas-ideal-point model is used. Similarly to the Chicago school district, if a soft quota policy were used, all students would be weakly better off.

The Indian choice rule can also be generated by reserves in which each bound is greater than the school’s capacity. But the second copy of the college implements a choice rule that is generated by the quotas rule described in the previous paragraph.⁴

⁴For an empirical study of affirmative action policies in Andhra Pradesh see Bagde, Epple, and Taylor (2011).

C.2. High Schools in French-Speaking Belgium. In French-speaking Belgium, high school admissions are set up to promote diversity. However, in contrast with many examples we have seen thus far, the target of affirmative action policy is the set of students who have attended “disadvantaged primary schools.” The administration announces these primary schools, which may change each year depending on supply and demand. Each school is required to reserve at least 15 percent of its seats for students from disadvantaged primary schools, and also some seats for students living in the neighborhood of the school. If a reserved seat for either group cannot be filled then it can also be allocated to other students so long as there is no student from the privileged group willing to take that seat. This choice corresponds to the reserves model described above.

APPENDIX D. ENDOGENOUS PRIORITIES.

The discussion in Section IIResultssection.2 takes the priority as exogenously given. It may be questionable to have an axiom depending on particular priorities. Particular priorities depend on particular circumstances (such as where students live, and their results on test scores), and one may not want an axiom that is sensible one year, under a particular set of circumstances, but ceases to be reasonable the following year, when circumstances have changed.

Here we present general characterization results, using axioms that do not depend on a particular priority and ensure the existence of a priority under which the axioms of Section IIResultssection.2 hold true.

Type-weak axiom of revealed preference (t-WARP): *For any s, s', S , and S' such that $\tau(s) = \tau(s')$ and $s, s' \in S \cap S'$,*

$$s \in C(S) \text{ and } s' \in C(S') \setminus C(S) \text{ imply } s \in C(S').$$

Theorem D.1. *A choice rule is generated by an ideal point for some priority if and only if it satisfies gross substitutes, monotonicity, and type-weak axiom of revealed preference.*

Proof. Suppose that C satisfies the axioms. We shall prove that it is generated by an ideal point for some priority. To this end, we show that there exist an

ideal point z^* and a strict priority \succ such that the choice function created by these coincides with C . We start with the following lemma.

Lemma D.1. *If C satisfies GS and Mon, then it also satisfies IRS.*

Proof. Let $C(S') \subseteq S \subseteq S'$. By GS, $C(S) \supseteq C(S')$. Since $S \subseteq S'$, we have $\xi(S) \leq \xi(S')$ and by Mon, $\xi(C(S)) \leq \xi(C(S'))$. This together with $C(S) \supseteq C(S')$ imply that $C(S') = C(S)$, so C satisfies IRS. \square

Define f as in the proof of Theorem 1Diversity Firsttheorem.1 and z^* be as in the proof of Lemma 2Matching Marketslemma.2. Since $f(z^*) = z^*$, we have that $\|z^*\| \leq q$. As in the proof of Theorem 1Diversity Firsttheorem.1, f is generated by ideal point z^* .

Define a binary relation R by saying that $s R s'$ if $\tau(s) = \tau(s')$ and there is some $S \ni s, s'$ such that $s \in C(S)$ and $s' \notin C(S)$. We shall prove that R is transitive.

Lemma D.2. *If C satisfies GS, t-WARP and IRS, then R is transitive.*

Proof. Let $s R s'$ and $s' R s''$; we shall prove that $s R s''$. Let S' be such that $s', s'' \in S'$, $s' \in C(S')$, and $s'' \notin C(S')$. Consider the set $S' \cup \{s\}$. First, note that $s \in C(S' \cup \{s\})$. The reason is that if $s \notin C(S' \cup \{s\})$ then $C(S' \cup \{s\}) = C(S') \ni s'$, by IRS. Thus $s' R s$, in violation of t-WARP. Second, note that $s'' \notin C(S' \cup \{s\})$, as $s'' \notin C(S')$ and C satisfies gross substitutes. \square

The relation R is transitive. Thus it has an extension to a linear order \succ over S .

C is generated by z^* for \succ because 1) f is generated by the ideal point z^* implies $\xi(C(S))$ is the closest vector to z^* in $B(\xi(S))$ for every S and 2) type-WARP implies that any type- t student in $C(S)$ has a higher priority than any student in $S \setminus C(S)$ for every S .

Conversely, let C be generated by ideal point z^* for \succ . It is immediate that C satisfies t-WARP. The rest of the proof is exactly the same as in the proof of Theorem1Diversity Firsttheorem.1. \square

Saturated strong axiom of revealed preference (S-SARP): *There are no sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, such that, for all k*

- (1) $s_{k+1} \in C(S_{k+1})$ and $s_k \in S_{k+1} \setminus C(S_{k+1})$;
(2) $\tau(s_{k+1}) = \tau(s_k)$ or $\tau(s_{k+1})$ is saturated at S_{k+1}

(using addition mod K).

S-SARP rules out the existence of certain cycles in revealed preference, but it is careful as to where it infers a revealed preference from choice. The subtlety in the definition is the second part that requires either $\tau(s_{k+1}) = \tau(s_k)$ or that $\tau(s_{k+1})$ is saturated at S_{k+1} . In the first case, when s_{k+1} and s_k have the same type, it is revealed that s_{k+1} has a higher priority than s_k . However, when they have different types, we require that $\tau(s_{k+1})$ is saturated at S_{k+1} . When this happens, even though the school could admit fewer type $\tau(s_{k+1})$ students, it accepts more. Thus, in the revealed preference, s_{k+1} is preferred to s_k even if they have different types. This axiom allows us to construct a priority order over students. It is easy to see that S-SARP implies t-WARP.

Theorem D.2. *A choice rule is generated by reserves for some priority if and only if it satisfies gross substitutes, saturated strong axiom of revealed preference, and acceptance.*

Proof. For any $x \leq \xi(\mathcal{S})$, let $F(x) \equiv \{\xi(C(S)) : \xi(S) = x\}$ and

$$\hat{f}(x) = \bigwedge_{f(x) \in F(x)} f(x).$$

Suppose that C satisfies the axioms. Construct the vector r of minimum quotas as in the proof of Theorem 2Flexible Diversitytheorem.2. Then $|C(S)^t| \geq \min\{r_t, |S^t|\}$ like before.

Consider the following binary relation. Let $s \succ^* s'$ if there is S , at which $\{s\} = \{s, s'\} \cap C(S)$ and $\{s, s'\} \subseteq S$, and either $\tau(s) = \tau(s')$ or $\tau(s)$ is saturated at S . By S-SARP, \succ^* has a linear extension \succ to \mathcal{S} .

Third we prove that C is consistent with \succeq , as stated in the definition. Let $s \in C(S)$ and $s' \in S \setminus C(S)$. If $\tau(s) = \tau(s')$ then $s \succ^* s'$ by definition of \succ^* ; hence $s \succ s'$. If $\tau(s) \neq \tau(s')$ then we need to consider the case when $|S^t| > r_t$ where $t = \tau(s)$. The construction of r_t implies that $r_t = \hat{f}(|S^t|, \bar{x}_{-t}) < |S^t|$. Therefore, there exists $S' \subseteq \mathcal{S}$ such that if

$$S' = S^t \cup \left(\bigcup_{\bar{i} \neq t} S^{\bar{i}} \right)$$

then $S^t \setminus C(S^t) \neq \emptyset$. Thus t is saturated at S . Since $s \in C(S)$ and $s' \in S \setminus C(S)$, we get $s \succ s'$, as \succ extends \succ^* .

It remains to show that if C is generated by reserves for some priority, then it satisfies the axioms. It is immediate that it satisfies Acceptance and S-SARP. That C satisfies GS is in the proof of Theorem 2Flexible Diversitytheorem.2.

□

Demanded strong axiom of revealed preference (D-SARP): *There are no sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, such that, for all k*

- (1) $s_{k+1} \in C(S_{k+1})$ and $s_k \in S_{k+1} \setminus C(S_{k+1})$;
- (2) $\tau(s_{k+1}) = \tau(s_k)$ or $\tau(s_k)$ is demanded in S_{k+1} .

(using addition mod K).

D-SARP rules out cycles in the revealed preference of the choice rule, where again we are careful as to when we infer the existence of a revealed preference. It is stronger than t-WARP. The difference between D-SARP and S-SARP is the second component of the definition, s_{k+1} is revealed preferred to s_k only when $\tau(s_k)$ is demanded in S_{k+1} implying that the school could choose more type $\tau(s_k)$ students out of S_{k+1} .

Theorem D.3. *A choice rule is generated by quotas for some priority if and only if it satisfies gross substitutes, rejection maximality, and demanded strong axiom of revealed preference.*

Proof. Suppose that C satisfies the axioms. Define r_t as in the proof of Theorem 3Flexible Diversitytheorem.3. In addition, define \succ^* as follows: $s \succ^* s'$ if there exists $S \supseteq \{s, s'\}$ such that $s \in C(S)$, $s' \notin C(S)$ and either $\tau(s) = \tau(s')$ or $\tau(s')$ is demanded in S . By D-SARP, \succ^* has a linear extension \succ to \mathcal{S} .

To show that C is generated by quotas for some priority we need to show three things. First, we need $|C(S)^t| \leq r_t$ for every $S \subseteq \mathcal{S}$, which follows from the construction of r_t .

Second we show that if $s \in C(S)$, $s' \in S \setminus C(S)$ and $s' \succ s$, then it must be the case that $\tau(s) \neq \tau(s')$ and $|C(S)^{\tau(s')}| = r_{\tau(s')}$. If $\tau(s) = \tau(s')$, then $s \succ^* s'$ and $s \succ s'$, which is a contradiction with the fact that \succ is an extension of

\succ^* . So $\tau(s) \neq \tau(s')$. To prove that $|C(S)^{\tau(s')}| = r_{\tau(s')}$ suppose, towards a contradiction, that $|C(S)^{\tau(s')}| \neq r_{\tau(s')}$, so $|C(S)^{\tau(s')}| < r_{\tau(s')}$.

We shall prove that $\tau(s')$ is demanded in S , which will yield the desired contradiction by D-SARP, as \succ is an extension of \succ^* . Let $S' \equiv S^{\tau(s')}$. We consider three cases.

- First, $|C(S')| = q$ then $|C(S)^{\tau(s')}| < |C(S')^{\tau(s')}|$ (as $s \in C(S)$ and $\tau(s) \neq \tau(s')$), so $\tau(s')$ is demanded in S .
- Second, consider the case when $|C(S')| < q$ and $|C(S')| < |S'|$. Then, by Lemma 5Proof of Theorem 3Flexible Diversitytheorem.3lemma.5, $|C(S')| = r_{\tau(s')}$, so $|C(S')| > |C(S)^{\tau(s')}|$. Hence $\tau(s')$ is demanded in S .
- Third, consider the case when $|C(S')| < q$, and $|C(S')| = |S'|$. Then $|C(S')| > |C(S)^{\tau(s')}|$, as $s' \in S^{\tau(s')} \setminus C(S)^{\tau(s')}$. Thus $\tau(s')$ is demanded in S .

In all three cases we conclude that $s \succ^* s'$. Since \succ is a linear extension of \succ^* , we get $s \succ s'$, a contradiction.

Finally, we need to show that if $s \in S \setminus C(S)$, then either $|C(S)| = q$ or $|C(S)^{\tau(s)}| = r_{\tau(s)}$. This is in the proof of Theorem 3Flexible Diversitytheorem.3.

To finish the proof, suppose that C is generated by quotas for some priority. Then it is easy to see that C satisfies RM and D-SARP. The proof that C satisfies GS is in the proof of Theorem 3Flexible Diversitytheorem.3. \square

APPENDIX E. A GENERAL COMPARATIVE STATIC

Definition E.1. *Choice rule C is **path independent** if for every S and S' , $C(S \cup S') = C(S \cup C(S'))$.*

Definition E.2. *A choice rule is an **expansion** of another choice rule if, for any set of students, any student chosen by the latter is also chosen by the former. (C' is an expansion of C is for every set S , $C'(S) \supseteq C(S)$.)*

For matching markets, stability has proved to be a useful solution concept because mechanisms that find stable matchings are successful in practice (Roth, 2008). Moreover, finding stable matchings is relatively easy. In particular, the deferred acceptance algorithm (DA) of Gale and Shapley (1962) finds

a stable matching, and DA has other attractive properties.⁵ Therefore, it also serves as a recipe for market design. For example, it has been adapted by the New York and Boston school districts (see Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) and Abdulkadiroğlu, Pathak, and Roth (2009)). For completeness, we provide a description of the student-proposing deferred acceptance algorithm.

Deferred Acceptance Algorithm (DA)

Step 1: Each student applies to her most preferred school. Suppose that S_c^1 is the set of students who applied to school c . School c tentatively admits students in $C_c(S_c^1)$ and permanently rejects the rest. If there are no rejections, stop.

Step k : Each student who was rejected at Step $k - 1$ applies to her next preferred school. Suppose that S_c^k is the set of new applicants and students tentatively admitted at the end of Step $k - 1$ for school c . School c tentatively admits students in $C_c(S_c^k)$ and permanently rejects the rest. If there are no rejections, stop.

The algorithm ends in finite time since at least one student is rejected at each step. When choice rules are path independent, DA produces the student-optimal stable matching (Roth, 1984; Aygün and Sönmez, 2013; Chambers and Yenmez, 2013). Therefore, the student-optimal stable mechanism (SOSM) coincides with DA.

Theorem E.1. *Suppose that for each school c , C_c is path independent and C'_c is a path-independent expansion of C_c . Then all students weakly prefer the outcome of SOSM with $(C'_c)_{c \in C}$ to the outcome with $(C_c)_{c \in C}$.*

Proof. We start with the following lemma.

Lemma E.1. *If C satisfies GS and (c, S) blocks a matching μ , then for every $s \in S \setminus \mu(c)$, $(c, \{s\})$ blocks μ .*

Proof. Since (c, S) blocks μ , we have $S \subseteq C_c(\mu(c) \cup S)$. Let $s \in S \setminus \mu(c)$, by substitutability $s \in C(\mu(c) \cup S)$ implies $s \in C(\mu(c) \cup \{s\})$. Therefore, $(c, \{s\})$ blocks μ . \square

⁵For a history of the deferred acceptance algorithm, see Roth (2008).

Since we use two different choice rule profiles and stability depends on the choice rules, we prefix the choice rule profile to stability, individual rationality and no blocking to avoid confusion. For example, we use C -stability, C -individual rationality and C -no blocking.

DA produces the student-optimal stable matching (Roth and Sotomayor, 1990). Denote the student-optimal stable matching with C and C' by μ and μ' , respectively. Since $C_c(\mu(c)) = \mu(c)$ by C -individual rationality of μ by every school c , $C'_c(\mu(c)) \supseteq C_c(\mu(c))$ by the assumption, and $C'_c(\mu(c)) \subseteq \mu(c)$ by definition of the choice rule we get $C'(\mu(c)) = \mu(c)$. Therefore, μ is also C' -individually rational for schools. Since student preference profile is fixed, μ is also C' -individually rational for students. If μ is a C' -stable matching, then μ' Pareto dominates μ since μ' is the student-optimal C' -stable matching. Otherwise, if μ is not a C' -stable matching, then there exists a C' -blocking pair. Whenever there exists such a blocking pair, there also exists a blocking pair consisting a school and a student by Lemma E.1. In such a situation, we apply the following improvement algorithm. Let $\mu^0 \equiv \mu$.

Step k : Consider blocking pairs involving school c_k and students who would like to switch to c_k , say $S_{c_k}^k \equiv \{s : c_k \succ_s \mu^{k-1}(s)\}$. School c_k accepts $C'_{c_k}(\mu^{k-1}(c_k) \cup S_{c_k}^k)$ and rejects the rest of the students. Let $\mu^k(c_k) \equiv C'_{c_k}(\mu^{k-1}(c_k) \cup S_{c_k}^k)$ and $\mu^k(c) \equiv \mu^{k-1}(c) \setminus C'_{c_k}(\mu^{k-1}(c_k) \cup S_{c_k}^k)$ for $c \neq c_k$. If there are no more blocking pairs, then stop and return μ^k , otherwise go to Step $k + 1$.

We first prove by induction that no previously admitted student is ever rejected in the improvement algorithm. For the base case when $k = 1$ note that $C'_{c_1}(\mu(c_1) \cup S_{c_1}^1) \supseteq C_{c_1}(\mu(c_1) \cup S_{c_1}^1)$ by assumption and $C_{c_1}(\mu(c_1) \cup S_{c_1}^1) = \mu(c_1)$ since μ is C -stable. Therefore, $C'(\mu(c_1) \cup S_{c_1}^1) \supseteq \mu(c_1)$, which implies that no students are rejected at the first stage of the algorithm. Assume, by mathematical induction hypothesis, that no students are rejected during Steps 1 through $k - 1$ of the improvement algorithm. We prove that no student is rejected at Step k . There are two cases to consider.

First, consider the case when $c_n \neq c_k$ for all $n \leq k - 1$. Since μ is C -stable, we have $C_{c_k}(\mu(c_k) \cup S_{c_k}^k) = \mu(c_k)$ (as students in $S_{c_k}^k$ prefer c_k to their schools in μ). By assumption, $C'_{c_k}(\mu(c_k) \cup S_{c_k}^k) \supseteq C_{c_k}(\mu(c_k) \cup S_{c_k}^k)$ which implies

$C'_{c_k}(\mu(c_k) \cup S_{c_k}^k) \supseteq \mu(c_k)$. Since $\mu(c_k) \supseteq \mu^{k-1}(c_k)$ we have $C'_{c_k}(\mu^{k-1}(c_k) \cup S_{c_k}^k) \supseteq \mu^{k-1}(c_k)$ by substitutability. In this case no student is rejected at Step k .

Second, consider the case when $c_k = c_n$ for some $n \leq k-1$. Let n^* be the last step smaller than k in which school c_k was considered. Since each student's match is either the same or improved at Steps 1 through $k-1$, we have $\mu^{n^*-1}(c_k) \cup S_{c_k}^{n^*} \supseteq \mu^{k-1}(c_k) \cup S_{c_k}^k$. By construction, $\mu^{n^*}(c_k) = C'_{c_k}(\mu^{n^*-1}(c_k) \cup S_{c_k}^{n^*})$ which implies $\mu^{k-1}(c_k) \subseteq C'_{c_k}(\mu^{k-1}(c_k) \cup S_{c_k}^k)$ by substitutability and the fact that $\mu^{n^*}(c_k) \supseteq \mu^{k-1}(c_k)$ (since n^* is the last step before k in which school c_k is considered). Therefore, no student is rejected at Step k .

Since no student is ever rejected by the improvement algorithm, it ends in a finite number of steps. Moreover, the resulting matching does not have any C' -blocking pair. By construction, it is also C' -individually rational. This shows that there exists a C' -stable matching that Pareto dominates μ . Since μ' is the student-optimal C' -stable matching, we have that μ' Pareto dominates μ for students. \square

APPENDIX F. INDEPENDENCE OF AXIOMS

Here, we check the independence of axioms that are used in Theorems 1Diversity Firsttheorem.1-D.3. The following axiom is useful in our examples below.

Axiom F.1. *Choice rule C satisfies the **strong axiom of revealed preference (SARP)** if there are no sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, such that, for all k*

$$(1) \ s_{k+1} \in C(S_{k+1}) \text{ and } s_k \in S_{k+1} \setminus C(S_{k+1}).$$

(using addition mod K).

SARP is stronger than both D-SARP and S-SARP.

Axioms in Theorem D.1.

Example 1 (GS, t-WARP but not Mon). Let $\mathcal{S} = \{s_1, s_2, s_3\}$, $q = 2$, and $\tau(s_1) = \tau(s_2) = \tau(s_3) = t$. Consider the following choice function: $C(s_1, s_2, s_3) = C(s_1, s_2) = C(s_1, s_3) = C(s_1) = \{s_1\}$, $C(s_2, s_3) = \{s_2, s_3\}$,

$C(s_2) = \{s_2\}$, and $C(s_3) = \{s_3\}$.⁶ Clearly, C satisfies both GS and t-WARP. But it fails Mon since $|\{s_1, s_2, s_3\}^t| \geq |\{s_2, s_3\}^t|$ but $|C(s_1, s_2, s_3)^t| < |C(s_2, s_3)^t|$.

Example 2 (t-WARP, Mon but not GS). Let $\mathcal{S} = \{s_1, s_2, s_3\}$, $\tau(s_1) = \tau(s_2) = t_1$, $\tau(s_3) = t_2$. Consider the following choice function: $C(s_1, s_2) = \{s_1\}$ and $C(S) = S$ for the remaining S . C satisfies t-WARP and Mon. But it fails GS because $s_2 \in C(s_1, s_2, s_3)$ and $s_2 \notin C(s_1, s_2)$.

Example 3 (Mon, GS but not t-WARP). Let $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$, $q = 2$, and $\tau(s_1) = \tau(s_2) = \tau(s_3) = \tau(s_4) = t$. Consider the following choice function: $C(s_1, s_2, s_3, s_4) = C(s_1, s_2, s_3) = C(s_1, s_2, s_4) = \{s_1, s_2\}$, $C(s_1, s_3, s_4) = \{s_1, s_3\}$, $C(s_2, s_3, s_4) = \{s_2, s_4\}$, and $C(S) = S$ for the remaining S . C satisfies Mon and GS. But it fails t-WARP because $s_3 \in C(s_1, s_3, s_4) \setminus C(s_2, s_3, s_4)$ and $s_4 \in C(s_2, s_3, s_4) \setminus C(s_1, s_3, s_4)$.

Axioms in Theorem D.2.

Example 4 (GS, S-SARP but not Acceptance). Consider the choice function in Example 1. C satisfies both GS and SARP (and hence S-SARP). But it fails acceptance since $|C(s_1, s_2, s_3)| = 1 < 2 = q$.

Example 5 (S-SARP, Acceptance but not GS). Let $S = \{s_1, s_2, s_3, s_4\}$ and $T = \{t_1, t_2\}$. Suppose that s_1 and s_2 are of type t_1 and the rest of type t_2 . Let the capacity of the school be 2 and the choice be:

$$C(S) = \begin{cases} S & \text{if } |S| \leq 2 \\ \{s_1, s_2\} & \text{if } \{s_1, s_2\} \subseteq S \\ \{s_3, s_4\} & \text{otherwise.} \end{cases}$$

Note that C violates GS because $s_1 \notin C(s_1, s_3, s_4)$ while $s_1 \in C(s_1, s_2, s_3, s_4)$. However, C satisfies acceptance and S-SARP. Acceptance is obvious. To see that it satisfies S-SARP, let R be the revealed preference relation, where $x R y$ if there is S such that $x \in C(S)$ and $y \in S \setminus C(S)$, and either x and y are of the same type or the type of x is saturated in S .

⁶For ease of notation we write $C(s_i, \dots, s_j)$ for $C(\{s_i, \dots, s_j\})$.

We can only infer $x R y$ when there is S with $S \setminus C(S) \neq \emptyset$. So we can focus in S with $|S| \geq 3$. There are four such sets. When $|S^{t_1}| = 2$ we have $S^{t_1} \setminus C(S)^{t_1} = \emptyset$, so t_1 is never saturated at any S with $|S^{t_1}| = 2$. Therefore we cannot infer any $x R y$ from any S with $\{s_1, s_2\} \subseteq S$. Thus we are only left with the facts that

$$\{s_3, s_4\} = C(s_1, s_3, s_4) = C(s_2, s_3, s_4).$$

That is, $s_3 R s_1$, $s_3 R s_2$, $s_4 R s_1$, and $s_4 R s_2$. Such R is acyclic. So S-SARP is satisfied.

Example 6 (Acceptance, GS but not S-SARP). Consider choice function C introduced in Example 3. We showed that C satisfies GS but fails t-WARP. Since S-SARP is stronger than t-WARP, S-SARP is also violated. It is easy to check that C also satisfies acceptance.

Axioms in Theorem D.3.

Example 7 (GS, D-SARP but not RM). Consider the choice function in Example 1. C satisfies both GS and SARP (and hence D-SARP). But it fails RM since $s_2 \in \{s_1, s_2\} \setminus C(s_1, s_2)$ and $|C(s_1, s_2)| < q = 2$ but $|C(s_1, s_2)^t| < |C(s_2, s_3)^t|$.

Example 8 (D-SARP, RM but not GS). Let $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$, $q = 2$, and $\tau(s_1) = \tau(s_2) = \tau(s_3) = t_1$ and $\tau(s_4) = t_2$. Consider the following choice function: $C(s_1, s_2, s_3, s_4) = C(s_1, s_2, s_3) = \{s_1, s_2\}$, $C(s_1, s_2, s_4) = C(s_1, s_3, s_4) = \{s_1, s_4\}$, $C(s_2, s_3, s_4) = \{s_2, s_4\}$, $C(s_1, s_2) = C(s_1, s_3) = \{s_1\}$, $C(s_2, s_3) = \{s_2\}$, and $C(S) = S$ for the remaining S .

Let \succ be defined as follows: $s \succ s'$ if there exists $S \supseteq \{s, s'\}$ such that $s \in C(S)$, $s' \notin C(S)$ and either $\tau(s) = \tau(s')$ or $\tau(s')$ is demanded in S . We consider every set of students from which a student is rejected and deduce that $s_1 \succ s_2 \succ s_3, s_4$. Since there are no cycles, D-SARP is satisfied. It is easy to see that RM is also satisfied. To see that GS fails, note $s_2 \in C(s_1, s_2, s_3, s_4)$ and $s_2 \notin C(s_1, s_2, s_4)$.

Example 9 (RM, GS but not D-SARP). Consider choice function C introduced in Example 3. C satisfies GS but it fails t-WARP. Since D-SARP

is stronger than t-WARP, D-SARP is also not satisfied. In addition, C also satisfies acceptance, which implies RM.

Axioms in Theorem 1.

Example 10 (GS, within-type \succ -compatibility but not Mon). Consider choice function C introduced in Example 1. Let \succ be as follows: $s_1 \succ s_2 \succ s_3$. Clearly, C satisfies both GS and within-type \succ -compatibility. But it fails Mon.

Example 11 (Within-type \succ -compatibility, Mon but not GS). Consider choice function C introduced in Example 2. Let \succ be as follows: $s_1 \succ s_2 \succ s_3$. Clearly, C satisfies both within-type \succ -compatibility and Mon. But it fails GS.

Example 12 (Mon, GS but not within-type \succ -compatibility). Consider choice function C introduced in Example 3. It satisfies Mon and GS but fails t-WARP. Therefore, it fails within-type \succ -compatibility for any \succ .

Axioms in Theorem 2.

Example 13 (GS, saturated \succ -compatibility but not acceptance). Consider the choice function in Example 1. Let \succ be as follows: $s_1 \succ s_2 \succ s_3$. C satisfies GS and saturated \succ -compatibility. But it fails acceptance since $|C(s_1, s_2, s_3)| = 1 < 2 = q$.

Example 14 (Saturated \succ -compatibility, acceptance but not GS). Consider the choice function in Example 5. Let \succ be as follows: $s_3 \succ s_4 \succ s_1 \succ s_2$. It is clear by the argument in Example 5 that C satisfies saturated \succ -compatibility because \succ agrees with the revealed preference constructed therein. In addition, C also satisfies acceptance. But it fails GS as shown in Example 5.

Example 15 (Acceptance, GS but not saturated \succ -compatibility). Consider the choice function in Example 3 but suppose that all students have different types. C satisfies acceptance and GS. But it fails saturated \succ -compatibility: $\tau(s_3)$ is saturated in $\{s_1, s_3, s_4\}$, $s_3 \in C(s_1, s_3, s_4)$ and $s_4 \notin C(s_1, s_3, s_4)$ imply $s_3 \succ s_4$. On the other hand, $\tau(s_4)$ is saturated in $\{s_2, s_3, s_4\}$, $s_4 \in C(s_2, s_3, s_4)$ and $s_3 \notin C(s_2, s_3, s_4)$ imply $s_4 \succ s_3$. Therefore, C cannot satisfy saturated \succ -compatibility for any \succ .

Axioms in Theorem 3.

Example 16 (GS, within-type \succ -compatibility, demanded \succ -compatibility but not RM). Consider the choice function in Example 1. Let \succ be as follows: $s_1 \succ s_2 \succ s_3$. As argued in Example 6, C satisfies GS but fails RM. Moreover, it satisfies within-type \succ -compatibility and demanded \succ -compatibility.

Example 17 (Within-type \succ -compatibility, demanded \succ -compatibility, RM but not GS). Consider the choice function in Example 8. Let \succ be as follows: $s_1 \succ s_2 \succ s_3 \succ s_4$. As argued in Example 8, C satisfies RM but fails GS. Clearly it satisfies within-type \succ -compatibility. Let us now check saturated \succ -compatibility: the only sets in which a lower priority student is chosen over a higher priority student are $\{s_1, s_2, s_4\}$, $\{s_1, s_3, s_4\}$, and $\{s_2, s_3, s_4\}$. But s_2 is not demanded for $\{s_1, s_2, s_4\}$, s_3 is not demanded for $\{s_1, s_3, s_4\}$ and s_3 is not demanded for $\{s_2, s_3, s_4\}$. Therefore, demanded \succ -compatibility is also satisfied.

Example 18 (Demanded \succ -compatibility, RM, GS but not within-type \succ -compatibility). Consider the choice function in Example 3. C satisfies GS as argued in Example 3 and it satisfies RM as argued in Example 9. Since there is only one type and this type is never demanded in a set, C also satisfies demanded \succ -compatibility for any \succ . However, it fails within-type \succ -compatibility because it fails t-WARP as shown in Example 3.

Example 19 (RM, GS, within-type \succ -compatibility but not demanded \succ -compatibility). Consider the choice function in Example 3 but suppose that all students have different types. C satisfies GS as argued in Example 3 and it satisfies RM as argued in Example 9. It trivially satisfies within-type \succ -compatibility because all students have different types. But it fails demanded \succ -compatibility for any \succ because $\tau(s_4)$ is demanded in $\{s_1, s_3, s_4\}$, so we need $s_3 \succ s_4$. On the other hand, $\tau(s_3)$ is demanded in $\{s_2, s_3, s_4\}$, so we need $s_4 \succ s_3$.

REFERENCES

- ABDULKADIROĞLU, A., P. PATHAK, A. ROTH, AND T. SÖNMEZ (2005):
 “The Boston Public School Match,” *American Economic Review*, 95(2),

- 368–371.
- ABDULKADIROĞLU, A., P. A. PATHAK, AND A. E. ROTH (2009): “Strategy-Proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match,” *American Economic Review*, 99(5), 1954–78.
- AYGÜN, O., AND T. SÖNMEZ (2013): “Matching with contracts: Comment,” *American Economic Review*, 103(5), 2050–2051.
- BAGDE, S., D. EPPLE, AND L. J. TAYLOR (2011): “Dismantling the Legacy of Caste: Affirmative Action in Indian Higher Education,” Working Paper.
- CHAMBERS, C. P., AND M. B. YENMEZ (2013): “Choice and Matching,” Working Paper.
- DUR, U., S. D. KOMINERS, P. A. PATHAK, AND T. SÖNMEZ (2014): “The Demise of Walk Zones in Boston: Priorities vs. Precedence in School Choice,” MIT Working Paper.
- GALE, D., AND L. S. SHAPLEY (1962): “College Admissions and the Stability of Marriage,” *The American Mathematical Monthly*, 69(1), 9–15.
- KOMINERS, S. D., AND T. SÖNMEZ (2012): “Designing for Diversity: Matching with Slot-Specific Priorities,” Working Paper.
- PATHAK, P. A., AND T. SÖNMEZ (2013): “School Admissions Reform in Chicago and England: Comparing Mechanisms by Their Vulnerability to Manipulation,” *American Economic Review*, 103(1), 80–106.
- ROTH, A., AND M. SOTOMAYOR (1990): *Two-sided Matching: A Study in Game-Theoretic Modelling and Analysis*, vol. 18 of *Econometric Society Monographs*. Cambridge University Press, Cambridge England.
- ROTH, A. E. (1984): “Stability and Polarization of Interests in Job Matching,” *Econometrica*, 52(1), 47–57.
- (2008): “Deferred Acceptance Algorithms: History, Theory, Practice and Open Questions,” *International Journal of Game Theory*, 36(3), 537–569.
- SOWELL, T. (2004): *Affirmative Action Around the World: An Empirical Study*. New Haven: Yale University Press.