

# Financial Entanglement: A Theory of Incomplete Integration, Leverage, Crashes, and Contagion Online Appendix

Nicolae Gârleanu, Stavros Panageas, and Jianfeng Yu

## A. Extensions

### A.1. An infinite horizon version of the baseline model

In this section we develop an intertemporal version of the model. The intertemporal version allows us to extend the intuitions of our comparative statics exercises to a framework where the shocks to the participation technology are recurrent.

Specifically, we keep the key assumptions of the baseline model (Section II.B.), and in particular the assumption that investors only participate in an arc surrounding their home location. However, we assume that investors maximize expected discounted utility from consumption

$$-\sum_{t=0}^{\infty} \beta^t E_0 [e^{-\gamma c_{t,i}}],$$

where  $t$  denotes (discrete) calendar time. We assume that dividends at location  $i \in [0, 1)$  are given by

$$(1) \quad D_{t,i} = (1 - \rho) \sum_{k=-\infty}^t \rho^{t-k} \varepsilon_{k,i},$$

where  $\rho < 1$ ,

$$(2) \quad \varepsilon_{t,i} \equiv 1 + \sigma \left( B_i^{(t)} - \int_0^1 B_j^{(t)} dj \right),$$

and  $B_i^{(t)}$  denotes a family of Brownian Bridges on  $[0, 1]$  drawn independently across times  $t \in \{-\infty, \dots, 1, 2, \dots, \infty\}$ .

We make a few observations about this dividend structure. First, we note that equation (2) coincides with equation (3). Accordingly,  $\int_0^1 \varepsilon_{t,i} di = 1$ , and therefore, equation (1) implies that  $\int_0^1 D_{t,i} = 1$  for all  $t$ , so that the aggregate dividend is always equal to one. Second, dividends at individual locations follow AR(1) processes, since equation (1) implies

$$(3) \quad D_{t,i} = (1 - \rho) \varepsilon_{t,i} + \rho D_{t-1,i}.$$

Moreover, since (2) coincides with (3), the increments of two dividend processes at two locations  $i$  and  $j$  have the covariance structure of equation (4).

In terms of participation decisions, we keep the same cost assumption as in the baseline model and further assume that investors participate in a single interval of length  $\Delta_t^{(i)}$  centered at their “home” location. Participation costs are paid period by period in advance of trading. Specifically, an investor’s intertemporal budget constraint is given by

$$(4) \quad c_{t,i} + F_t \left( \Delta_t^{(i)} \right) + \int_0^1 P_{t,j} dX_{t,j}^{(i)} + P_{B,t} X_{B,t}^{(i)} = \int_0^1 (P_{t,j} + D_{t,j}) dX_{t-1,j}^{(i)} + X_{B,t-1}^{(i)}.$$

We note that in equation (4) we allow the entire cost function  $F_t(\Delta) = \kappa_t g_t(\Delta)$  to be different across different periods in order to capture the effect of repeated shocks to the participation technology.

This intertemporal version of the model presents a challenge that is absent in a static framework: If the interest rate varies over time, then the value function of an agent is not exponential in wealth. In fact, a closed-form expression for the value function most likely does not exist. Furthermore, once the value function is no longer exponential, portfolios are no longer independent of wealth, and hence the entire wealth distribution matters — an infinite-dimensional state variable.

In order to maintain the simple structure of the solution, therefore, we make necessary assumptions to achieve a constant interest rate in the presence of random participation costs, while safeguarding market clearing for bond markets, risky-asset markets, and consumption markets.<sup>1</sup> The following proposition states that it is possible to achieve this outcome by judiciously specifying the distribution of the participation costs. The main thrust of the proposition, however, concerns the expression for the risky-asset prices.

**Proposition 1** *There exist an interval  $[\Delta_l, \Delta_u]$ , a (non-trivial) distribution function  $\Psi(\cdot)$  on  $[\Delta_l, \Delta_u]$ , and a cost function  $F(\cdot; \Delta_t) : [\Delta_l, \Delta_u] \rightarrow \mathbb{R}^+$  such that, if  $\Delta_t$  is drawn in an i.i.d. fashion from  $\Psi$ , then*

- (i) *investors optimally choose  $\Delta = \Delta_t$ , thus incurring cost  $F(\Delta_t; \Delta_t)$ ;*
- (ii) *the risk-free rate is constant over time and given as the unique positive solution to*

$$(5) \quad 1 = \beta(1+r) E \left[ e^{\frac{\gamma^2}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta)} \right],$$

*where the expectation is taken over the distribution of  $\Delta$ ;*

- (iii) *the risky-asset prices equal*

$$(6) \quad P_{t,j}(\Delta_t, D_{t,j}) = \phi(D_{t,j} - 1) + \frac{1}{r} - \Phi_1 \omega(\Delta_t) - \Phi_0$$

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<sup>1</sup>Alternatively, in the interest of simplicity, we could fix the interest exogenously to the model and let aggregate lending adjust accordingly.

with  $\phi \equiv \frac{\rho}{1+r-\rho}$  and

$$(7) \quad \Phi_1 = \gamma \frac{r(1-\rho)^2}{(1+r-\rho)^2}$$

$$(8) \quad \Phi_0 = \frac{\Phi_1}{r} \frac{E \left[ e^{\frac{\gamma^2}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta)} \omega(\Delta) \right]}{E \left[ e^{\frac{\gamma^2}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta)} \right]} > 0.$$

Furthermore, investors' optimal portfolios of risky assets are given by (12).

Equation (6) decomposes the price of a security into three components. As in all CARA models, one of these components equals the expected discounted value of future dividends,  $\phi(D_{t,j} - 1) + r^{-1}$ . The other two capture the risk premium. The term  $\Phi_1 \omega(\Delta_t)$  is the risk premium associated with the realization of time- $t+1$  dividend uncertainty, to which each investor is exposed according to the breadth  $\Delta_t$  of her time- $t$  portfolio. Finally,  $\Phi_0$  equals the sum of the expected discounted value of risk premia due to future realizations of dividend innovations and  $\Delta_t$ .

We emphasize that the risk premium decreases with  $\Delta_t$  and is common for all securities. Alternatively phrased, increases in capital movements across locations are correlated with higher prices for all risky securities (and hence lower expected excess returns). Importantly, these movements in the prices of risky securities are uncorrelated with movements in aggregate output or the interest rate, which are both constant by construction.

A further immediate implication of equation (6) is that the presence of repeated shocks to participation costs introduces correlation in security prices that exceeds that of their dividends. Indeed, taking two securities  $j$  and  $k$ , and noting that  $Var(D_{t,j}) = Var(D_{t,k})$ , we can use equation (6) to compute

$$corr(P_{t,j}, P_{t,i}) = \frac{Var(\Phi_1 \omega(\Delta_t)) + \phi^2 cov(D_{t,j}, D_{t,k})}{Var(\Phi_1 \omega(\Delta_t)) + \phi^2 Var(D_{t,j})} > \frac{cov(D_{t,j}, D_{t,k})}{Var(D_{t,j})} = corr(D_{t,j}, D_{t,i}).$$

The intuition is that movements in market integration cause common movements in the pricing of risk which make prices more correlated (and volatile) than the underlying dividends.

We collect some basic properties of the price due to the randomness in  $\Delta_t$  in the following proposition.

- Proposition 2** (i)  $P_{t,i}$  increases with  $\Delta_t$ , and therefore  $corr(P_{t,i}, \Delta_t) > 0$ ;  
(ii)  $E_t[P_{i,t+1} - (1+r)P_{i,t}]$  decreases with  $\Delta_t$ ;  
(iii)  $corr(P_{t,i}, P_{t,j}) > corr(D_{t,i}, D_{t,j})$ ;  
(iv)  $\Phi_0$  is higher (and hence the unconditional expected price is lower) than the one obtaining for  $\Delta_t$  constant and equal to  $E[\Delta]$ .

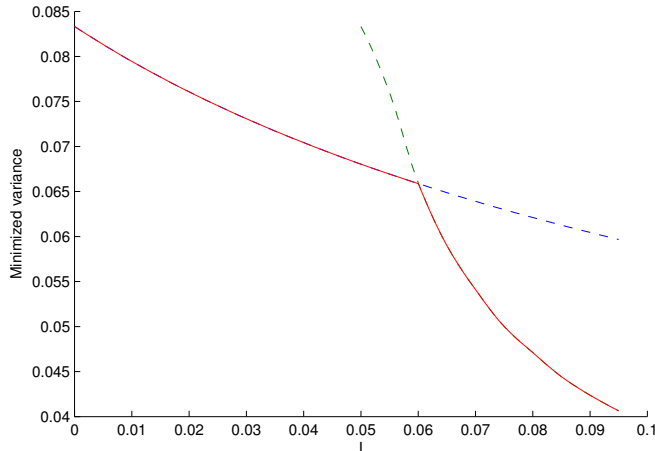


Figure 1: Numerical example to illustrate that  $v(I)$  is non-convex. The figure depicts two (dotted) lines and the minimum of the two lines (solid line). The first dotted line starts at  $I = 0$  and depicts the minimal variance that can be attained when participation costs are equal to  $I$  and the investor chooses to participate only on a single arc centered at her home location. The second dotted line starts at  $I = 0.05$ , i.e. at the minimum expenditure required to invest in two distinct arcs. This second dotted line depicts the minimal variance that can be attained when participation costs are equal to  $I$  and the investor can participate on two separate arcs with locations and lengths chosen so as to minimize variance. The function  $v(I)$  (the minimum of the two dotted lines) is given by the solid line. For this example we chose  $\sigma = 1$ ,  $g(x) = 0.1 \times ((1 - x)^{-6} - 1)$ ,  $f(y) = 0.05 + 0.005 \times ((\frac{1}{2} - y)^{-2} - \frac{1}{0.25})$ . (For  $I > 0.1$  the function  $v(I)$  would in general exhibit further kinks at the critical values  $I_N$ ,  $N \geq 2$ , where the investor is indifferent between  $N$  and  $N + 1$  distinct arcs.)

## A.2. Multiple arcs on the circle

The baseline model assumes that investors participate in markets spanning a single arc of length  $\Delta$  around their “home” location. Extending the results to the general case where investors can choose to participate on multiple, disconnected arcs (as illustrated on the right-most graph of Figure 4) is straightforward and involves essentially no new insights. In this section we briefly sketch how to extend the results of the baseline model to this case and we show that allowing for this extra generality introduces an additional source of non-concavity into an investor’s optimization problem.

To start, we introduce the function

$$v(I) = \min_{N_i, \vec{a}_i, \vec{\Delta}_i, G_j^{(i)}} \text{Var} \left( \int_0^1 D_j dG_j^{(i)} \right)$$

$$s.t. \quad I \geq F \left( \vec{a}_i, \sum_{n=1}^{N_i} \Delta_{i,n} \right).$$

In words, the function  $v(I)$  is the minimal variance, per share purchased, of the portfolio payoff that can be obtained by an investor who is willing to spend an amount  $I$  on participation costs. Proceeding similarly to Section B. under the assumption that  $P_j = P$  for all  $j$ , the facts that  $U$  is exponential and all  $D_j$  are normally distributed imply that maximizing utility over the choice of  $N_i$ ,  $\{a_{i,1}; \dots; a_{i,N_i}\}$ ,  $\{\Delta_{i,1}; \dots; \Delta_{i,N_i}\}$ ,  $G_j^{(i)}$ , and  $w_i^f$  is equivalent to solving<sup>2</sup>

$$(9) \quad \max_{w_i^f, I} Pw_i^f + (1 - w_i^f) \int_0^1 E[D_j] dG_j^{(i)} - \frac{\gamma}{2} (1 - w_i^f)^2 v(I) - I.$$

Given that  $E[D_j] = 1$ , equation (9) can be rewritten as

$$(10) \quad V = \max_{I, w_i^f} Pw_i^f + (1 - w_i^f) - \frac{\gamma}{2} (1 - w_i^f)^2 v(I) - I.$$

In the baseline version of the model (Section B.), where the investor chooses to invest in a single arc around her home location,  $v$  is a convex function of the total cost  $I$ .<sup>3</sup> In the general case where investors' portfolios are invested on disconnected arcs, the function  $v(I)$  is in general non-convex with kinks at the expenditure levels  $I_n$  where it becomes optimal to invest in  $n + 1$  rather than  $n$  distinct arcs. Figure 1 provides an illustration. This non-convexity of  $v(I)$ , which may arise when (and only when) investors participate in markets located on multiple distinct arcs, constitutes an additional reason for the maximization problem (10) to be non-concave. This reason is distinct from the non-concavity arising from the interaction between leverage and participation decisions that we identify in Section C., and strengthens the conclusion that a symmetric equilibrium may not exist.

<sup>2</sup>To ensure that the optimization problem (9) has a solution it is convenient either to impose a collateral constraint such as (23) or to introduce some (potentially small but positive) aggregate risk in dividends. Either of these assumptions coupled with the additional assumption  $\lim_{\Delta \rightarrow 1} g(\Delta) = \infty$  suffices to ensure the existence of a solution to (9). Alternatively, one can ensure that (9) has an interior solution by requiring that, upon plugging in the optimal value of  $w_i^f$ , the maximand in (10) tends to negative infinity as  $I$  goes to infinity. Given the lower bound on  $P$  provided by the autarky equilibrium, it suffices that  $\lim_{I \rightarrow \infty} \left( \frac{\sigma^2}{12} \right)^2 \frac{\gamma}{v(I)} - I = -\infty$ . This condition may be harder to verify than Assumption 1, since it is not readily expressed in terms of primitive parameters.

<sup>3</sup>To see this, note that  $v'(I) = \frac{\omega'(\Delta)}{\kappa g'(\Delta)}$ , where  $\Delta(I) = g^{-1}\left(\frac{I}{\kappa}\right)$ . Differentiating again gives  $v''(I) = \frac{1}{\kappa} \frac{\omega''(\Delta)g'(\Delta) - \omega'(\Delta)g''(\Delta)}{(g'(\Delta))^2} \Delta'(I) > 0$ .

## B. Proofs

**Proof of Lemma 1.** Property 2 follows immediately from integrating (3). To show property 3, note that, for any  $i \in (0, 1)$ ,  $\lim_{d(i,j) \rightarrow 0} D_j = \lim_{j \rightarrow i} D_j = D_i$  a.s. by the continuity of the Brownian motion. Continuity at 0 follows from the fact that  $B_0 = B_1$ .

We turn now to property 1. Since  $E(B_i) = 0$  for all  $i \in [0, 1]$ ,  $E(D_j) = 1$ . To compute  $\text{cov}(D_i, D_j)$  we start by noting that  $\text{cov}(B_s, B_t) = E(B_s B_t) = s(1-t)$  for  $s \leq t$ . Therefore, for any  $t \in [0, 1]$ ,

$$(11) \quad \int_0^1 E(B_t B_u) du = \int_0^t u(1-t) du + \int_t^1 t(1-u) du \\ = \frac{1}{2}(1-t)t^2 + \frac{1}{2}(1-t)^2 t = \frac{t(1-t)}{2}.$$

Accordingly,

$$(12) \quad \text{Var} \left( \int_0^1 B_u du \right) = E \left[ \left( \int_0^1 B_u du \right)^2 \right] = E \left[ \left( \int_0^1 B_u du \right) \left( \int_0^1 B_t dt \right) \right] \\ = \int_0^1 \left( \int_0^1 E(B_u B_t) du \right) dt = \int_0^1 \frac{t(1-t)}{2} dt = \frac{1}{12},$$

where the second line of (12) follows from Fubini's Theorem and (11). Combining (12) and (11) gives

$$(13) \quad \frac{1}{\sigma^2} \text{Var}(D_t) = \text{Var}(B_t) + \text{Var} \left( \int_0^1 B_u du \right) - 2 \text{cov} \left( B_t, \int_0^1 B_u du \right) \\ = t(1-t) + \frac{1}{12} - 2 \int_0^1 E(B_t B_u) du = \frac{1}{12}.$$

This calculation finishes the proof of property 1. For property 4, take any  $s \leq t$  and use (11) and (12) to obtain

$$(14) \quad \frac{\text{cov}(D_s, D_t)}{\sigma^2} = \text{cov} \left( B_s - \int_0^1 B_u du, B_t - \int_0^1 B_u du \right) \\ = E(B_s B_t) - E \left( B_s \int_0^1 B_u \right) du - E \left( B_t \int_0^1 B_u \right) du + \frac{1}{12} \\ = s(1-t) - \frac{s(1-s)}{2} - \frac{t(1-t)}{2} + \frac{1}{12} \\ = \frac{(s-t)(1+s-t)}{2} + \frac{1}{12}.$$

This establishes property 4. ■

**Proof of Proposition 1.** We start by establishing the following lemma.

**Lemma 1** *The (bounded-variation) function  $L$  with  $L_{-\frac{\Delta}{2}-} = 0$  and  $L_{\frac{\Delta}{2}} = 1$  that minimizes  $\text{Var}\left(\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} D_j dL_j\right)$  is given by (12). Moreover, the minimal variance is equal to  $\omega(\Delta)$ .*

**Proof of Lemma 1.** To simplify notation, we prove a “shifted” version of the lemma, namely finding the minimal-variance portfolio on  $[0, \Delta]$  rather than  $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$ . The two versions are clearly equivalent, since covariances depend only on the distances between locations, rather than the locations themselves.

We start by defining  $q(d) = \frac{1}{12} - \frac{d(1-d)}{2}$  and therefore  $q'(d) = -\frac{1}{2} + d$ . In light of (4),  $q(d) = \frac{1}{\sigma^2} \text{cov}(D_i, D_j)$  whenever  $d(i, j) = d$ . If  $L_u = \int_{0-}^u dL_u$  is a variance-minimizing portfolio of risky assets, it must be the case that the covariance between any gross return  $R_s = \frac{D_s}{P}$  for  $s \in [0, \Delta]$  and the portfolio  $\int_{0-}^{\Delta} R_u dL_u = \int_{0-}^{\Delta} \frac{D_u}{P} dL_u$  is independent of  $s$ . Thus, the quantity

$$(15) \quad \frac{1}{\sigma^2} \text{cov}\left(D_s, \int_{0-}^{\Delta} D_u dL_u\right) = \frac{1}{\sigma^2} \left[ \int_{0-}^s \text{cov}(D_s, D_u) dL_u + \int_s^{\Delta} \text{cov}(D_s, D_u) dL_u \right] = \\ = \int_{0-}^s q(s-u) dL_u + \int_s^{\Delta} q(u-s) dL_u$$

is independent of  $s$ . Letting  $\tilde{L}(s) = 1 - L(s)$  and integrating by parts we obtain

$$(16) \quad \int_{0-}^s q(s-u) dL_u = L(s)q(0) - L(0^-)q(s) + \int_{0-}^s L_u q'(s-u) du$$

$$(17) \quad \int_s^{\Delta} q(u-s) dL_u = \tilde{L}(s)q(0) - \tilde{L}(\Delta)q(\Delta-s) + \int_s^{\Delta} \tilde{L}_u q'(u-s) du.$$

Using (16) and (17) inside (15) and recognizing that  $q(0) = \frac{1}{12}$ ,  $L(0^-) = 0$ , and  $\tilde{L}(\Delta) = 0$ , we obtain that (15) equals

$$(18) \quad Q(s) \equiv \frac{1}{12} + \int_{0-}^s L_u q'(s-u) du + \int_s^{\Delta} \tilde{L}_u q'(u-s) du.$$

This expression is independent of  $s \in [0, \Delta]$  if and only if  $Q'(s) = 0$ . Differentiating (18) and setting the resulting expression to zero yields

$$(19) \quad Q'(s) = \int_{0-}^s L_u q''(s-u) du - \int_s^{\Delta} \tilde{L}_u q''(u-s) du + L_s q'(0) - \tilde{L}_s q'(0) \\ = \int_{0-}^{\Delta} L_u du - \Delta + s - L_s + \frac{1}{2} = 0,$$

where we used  $q'(0) = -\frac{1}{2}$ ,  $q'' = 1$ , and  $\tilde{L}(s) = 1 - L(s)$ . Since (19) needs to hold for all  $s \in [0, \Delta]$ , it must be the case that  $L_s = A + s$  for an appropriate constant  $A$ . To determine  $A$ , we substitute  $L_s = A + s$  into (19) and solve for  $A$  to obtain

$$A = \frac{1 - \Delta}{2}.$$

It is immediate that the standardized portfolio corresponding to the solution  $L$  we computed is  $L^*$  of (12).

Using the variance-minimizing portfolio inside (18), implies after several simplifications, that  $Q = \frac{1}{12}(1 - \Delta)^3$  and hence  $\text{cov}\left(D_s, \int_{0^-}^{\Delta} D_u dL_u\right) = Q\sigma^2 = \omega(\Delta)$ . Accordingly,

$$\begin{aligned} \text{Var}\left(\int_{0^-}^{\Delta} D_u dL_u\right) &= \text{cov}\left(\int_{0^-}^{\Delta} D_s dL_s, \int_{0^-}^{\Delta} D_u dL_u\right) = \int_{0^-}^{\Delta} \text{cov}\left(D_s, \int_{0^-}^{\Delta} D_u dL_u\right) dL_s \\ &= \omega(\Delta) \int_{0^-}^{\Delta} dL_s = \omega(\Delta). \end{aligned}$$

■

With Lemma 1 in hand it is possible to confirm that the allocations and prices of Proposition 1 constitute a symmetric equilibrium — assuming that one exists. We already argued that all agents choose the same standardized portfolio (as agent  $\frac{\Delta}{2}$ ). Furthermore, since in a symmetric equilibrium all agents must hold the same allocation of bonds, clearing of the bond market requires  $w_i^f = 0$  for all  $i$ . By equation (18),  $w_i^f = 0$  is supported as an optimal choice for an investor only if  $P_i = P$  is given by (15). Similarly, in light of (19), equation (14) is a necessary optimality condition for the interval  $\Delta^*$ . Since the values of  $P$  and  $\Delta^*$  implied by (15) and (14) are unique, they are necessarily the equilibrium values of  $P$  and  $\Delta^*$  that characterize a symmetric equilibrium. Hence, when a symmetric equilibrium exists, it is unique in the class of symmetric equilibria.

Existence of a symmetric equilibrium implies that  $w_i^f = 0$  is optimal, and so are the choices  $\Delta^*$  and  $G_{i+j}^{(i)} = L_j^*$  given prices  $P_i = P$ . It remains to show that markets clear. We already addressed bond-market clearing. To see that the stock markets clear, we start by noting that, since  $P_i = W_{0,i} = P$  for all  $i$ , the market clearing condition amounts to  $\int_{i \in [0,1)} dG_j^{(i)} = 1$ . We have  $\int_{i \in [0,1)} dG_j^{(i)} = \int_{i \in [0,1)} dL_{j-i}^* = \int_{j \in [0,1)} dL_j^* = 1$ . ■

**Proof of Proposition 2.** Let  $w^*(P)$  denote the set of optimal  $w_i^f$  solving the maximization problem (17) when the price in all markets is  $P$ . We first note that the assumption that no symmetric equilibrium exists implies that there exists no  $P$  such that  $0 \in w^*(P)$ . (If such a  $P$  existed, then we could simply repeat the arguments of Proposition 1 to establish the existence of a symmetric equilibrium with price  $P_i = P$ , and interval choice  $\Delta_i = \Delta^*(P)$ ).

We next show that since there exists no  $P$  such that  $0 \in w^*(P)$ , it follows that  $w^*(P)$  cannot be single-valued for all  $P$ . We argue by contradiction. Suppose to the contrary that  $w^*(P)$  is single-valued. Since the theorem of the maximum implies that  $w^*(P)$  is an upper-hemicontinuous correspondence, it follows that  $w^*(P)$  is actually a continuous function. Inspection of (17) shows that  $w^*(1) = 1$ . Moreover, as  $P \rightarrow -\infty$ , the optimal solution to (17) becomes negative:  $w^*(-\infty) < 0$ . Then an



application of the intermediate value theorem gives the existence of  $P$  such that  $w^*(P) = 0$ , a contradiction.

Combining the facts that a) there exists no  $P$  such that  $0 \in w^*(P)$ , b)  $w^*(P)$  is multi-valued for at least one value of  $P$ , and c)  $w^*(P)$  is upper-hemicontinuous, implies that there exists at least one  $P$  such that  $\{w_1, w_2\} \in w^*(P)$  with  $w_1 > 0$  and  $w_2 < 0$ . An implication of the necessary first-order condition for the optimality of the interval choice  $\Delta^*(P)$  is that  $\Delta^*(P)$  is also multi-valued with  $\Delta_1 < \Delta_2$ . Furthermore, since prices in all locations are equal, the (standardized) optimal portfolio of an agent choosing  $\Delta_k$  is the variance-minimizing portfolio of Proposition 1, denoted  $L^{*,k}$ .

From this point onwards, an equilibrium can be constructed as follows. By definition, the tuples  $\{\Delta_1, w_1, dL^{*,1}\}$  and  $\{\Delta_2, w_2, dL^{*,2}\}$  are optimal. Hence it only remains to confirm that asset markets clear. Define  $\pi \equiv -\frac{w_2}{w_1 - w_2} \in (0, 1)$ . By construction,  $\pi w_1 + (1 - \pi) w_2 = 0$  and, therefore, if in every location  $\pi$  agents choose  $\{\Delta_1, w_1, dL^{*,1}\}$  and the remaining fraction  $(1 - \pi)$  choose  $\{\Delta_2, w_2, dL^{*,2}\}$ , then the bond market clears by construction. To see that the stock markets clear, we start by noting that, since  $P_i = W_{0,i} = P$ , the market clearing condition for stock  $i$  amounts to

$$\pi \int_{[0,1]} dL_{j-i}^{*,1} + (1 - \pi) \int_{[0,1]} dL_{j-i}^{*,2} = 1,$$

which holds because  $L_j^{*,k}$  for  $k \in \{1, 2\}$  is a measure on the circle.

We prove next that symmetric and asymmetric equilibria (with different prices) cannot co-exist. Inspection of (17) shows that the optimal  $w_i^f$  is increasing in  $P$  in the sense that if  $P_1 < P_2$  then  $w_1 < w_2$  for any  $w_1 \in w^*(P_1)$  and  $w_2 \in w^*(P_2)$ .<sup>4</sup> Accordingly, if a symmetric equilibrium exists, i.e., if there is a  $\hat{P}$  such that  $0 \in w^*(\hat{P})$  then there cannot exist  $P \neq \hat{P}$  with the property that  $\{w_1, w_2\} \in w(P)$  and yet  $w_1 > 0$  and  $w_2 < 0$ , which is a requirement for the existence of an asymmetric equilibrium. Hence symmetric and asymmetric equilibria cannot co-exist. The fact that  $w^*(P)$  is an increasing correspondence also implies that asymmetric equilibria are essentially unique, in the sense that asymmetric equilibria associated with different equilibrium prices cannot co-exist. ■

**Remark 1** *The existence proof of an asymmetric equilibrium (when a symmetric equilibrium fails to exist) obtains also in the presence of the leverage constraint (23) that we introduce in Section IV..*

**Proof of Proposition 3.** Conjecture first that in equilibrium  $P_j = P$  for all  $j$  and let  $\pi \in [0, 1]$  denote the fraction of funds invested in the local market.

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<sup>4</sup>To see that this statement is correct, consider the maximum  $\bar{V}(P, w_i^f)$  of the maximand in (17) over  $\Delta$  and note that its cross-partial derivative with respect to  $P$  and  $w_i^f$  is positive:  $\partial_P \partial_{w_i^f} \bar{V}(P, w_i^f) > 0$ .

Assuming that a given investor chooses  $N = 2$  (that is, chooses to invest in her own location and another location at distance  $d$ ), equation (4) allows the computation of the minimal portfolio variance:

$$(20) \quad \widehat{\omega}(d) = \sigma^2 \min_{\pi} \left\{ \left( \pi^2 + (1 - \pi)^2 \right) \frac{1}{12} + 2\pi(1 - \pi) \left( \frac{1}{12} - \frac{d(1 - d)}{2} \right) \right\} \\ = \sigma^2 \left( \frac{1}{12} - \frac{1}{4}d(1 - d) \right).$$

The optimal distance  $d$  for an investor choosing  $N = 2$  satisfies a first-order condition similar to equation (19), namely

$$(21) \quad -\frac{\gamma}{2} (1 - w^f)^2 \frac{\widehat{\omega}'(d)}{\widehat{\omega}^2(d)} = \kappa f'(d).$$

Since  $\widehat{\omega}'(d) = 0$  when and only when  $d = \frac{1}{2}$ , and  $f'(d) = 0$ , it follows that  $d = \frac{1}{2}$  is optimal for an investor choosing  $N = 2$ . Hence the minimal portfolio variance of an investor choosing  $N = 1$  is equal to  $\widehat{\omega}(0) = \frac{\sigma^2}{12}$ , while the minimal portfolio variance for an investor choosing  $N = 2$  is  $\widehat{\omega}(\frac{1}{2}) = \frac{\sigma^2}{48}$ . Assuming that the equilibrium is of the asymmetric, location-invariant type, we can use equation (17) to express the indifference between the choices  $N = 1$ , respectively  $N = 2$  and  $d = \frac{1}{2}$ , as

$$(22) \quad Pw_1^f + (1 - w_1^f) - \frac{\gamma}{2} (1 - w_1^f)^2 \widehat{\omega}(0) = Pw_2^f + (1 - w_2^f) - \frac{\gamma}{2} (1 - w_2^f)^2 \widehat{\omega}\left(\frac{1}{2}\right) - \kappa f_0.$$

Using the first-order conditions for leverage

$$(23) \quad 1 - P = \gamma (1 - w_2^f) \widehat{\omega}\left(\frac{1}{2}\right) = \gamma (1 - w_1^f) \widehat{\omega}(0)$$

inside (22) yields — after some simplifications — the equilibrium price (24).

To verify that the postulated equilibrium is indeed an equilibrium, we proceed as in the proof of Proposition 2. For  $P(\kappa)$  to be an equilibrium price in all locations, it must also be case that  $1 - w_1^f \leq 1 \leq 1 - w_2^f$ , so that setting  $\pi = \frac{w_2^f}{w_2^f - w_1^f} > 0$  ensures market clearing (of bond markets and all risky asset markets). In light of (23), the requirement  $1 - w_1^f \leq 1 \leq 1 - w_2^f$  is equivalent to  $P \in [1 - \gamma \widehat{\omega}(0), 1 - \gamma \widehat{\omega}(\frac{1}{2})]$ . This requirement is satisfied as long as  $\kappa \in (\kappa_1, \kappa_2)$ . ■

**Lemma 2** *Consider an investor located at  $i \notin [-\frac{k}{2}, \frac{k}{2}]$ , and therefore investing in markets  $[i - \frac{\bar{\Delta}}{2}, i + \frac{\bar{\Delta}}{2}]$ . Suppose that  $P(x)$  is continuously differentiable everywhere*

on  $[i - \frac{\bar{\Delta}}{2}, i + \frac{\bar{\Delta}}{2}]$ . With  $dX_l^{(i)}$  the number of shares purchased on the account of an investor at  $i$  in market  $l$  and  $j \equiv i - \frac{\bar{\Delta}}{2}$ ,

$$(24) \quad X_{j+\bar{\Delta}}^{(i)} = \frac{1}{\gamma\omega(\bar{\Delta})} \left[ 1 - \frac{1-\bar{\Delta}}{2} (P_j + P_{j+\bar{\Delta}}) - \int_j^{j+\bar{\Delta}} P_u du \right].$$

Furthermore, the function  $X$  is given by

$$(25) \quad X_{j+l}^{(i)} = \frac{P'_{j+l}}{\gamma\sigma^2} + X_{j+\bar{\Delta}}^{(i)} \frac{1-\bar{\Delta}+2l}{2} + \frac{1}{1-\bar{\Delta}} \frac{P_{j+\bar{\Delta}} - P_j}{\gamma\sigma^2}.$$

If an investor is located at  $i \in [-\frac{k}{2}, \frac{k}{2}]$  and only invests in market  $i$  then the respective demand for risky asset  $i$  is given by

$$(26) \quad \hat{X}_i^{(i)} = \frac{1}{\gamma\omega(0)} (1 - P_i).$$

**Proof of Lemma 2.** Notice that optimization problem of agent  $i$  is equivalent to

$$(27) \quad \max_X P_i + \int_{j^-}^{j+\bar{\Delta}} (1 - P_u) dX_u - \frac{\gamma}{2} \text{Var} \left( \int_{j^-}^{j+\bar{\Delta}} D_u dX_u \right)$$

Thus, the first-order condition requires that

$$(28) \quad \gamma \text{cov} \left( D_s, \int_{j^-}^{j+\bar{\Delta}} D_u dX_u \right) = 1 - P_s$$

for all  $s \in [j, j + \bar{\Delta}]$ . Letting  $q(d)$  be defined as in Lemma 1 we can rewrite (28) as

$$(29) \quad \int_{j^-}^s q(s-u) dX_u + \int_s^{j+\bar{\Delta}} q(u-s) dX_u = \frac{1 - P_s}{\gamma\sigma^2}.$$

Let  $\tilde{X}(s) = X(j + \bar{\Delta}) - X(s)$  and integrating by parts we obtain

$$(30) \quad \int_{j^-}^s q(s-u) dX_u = X(s)q(0) - X(j^-)q(s) + \int_j^s X_u q'(s-u) du$$

$$(31) \quad \int_s^{j+\bar{\Delta}} q(u-s) dX_u = \tilde{X}(s)q(0) - \tilde{X}(\bar{\Delta})q(\Delta-s) + \int_s^{j+\bar{\Delta}} \tilde{X}_u q'(u-s) du$$

Substituting (30) and (31) into (29), recognizing that  $q(0) = \frac{1}{12}$ ,  $X(j^-) = 0$ , and  $\tilde{X}(j + \bar{\Delta}) = 0$ , we obtain

$$(32) \quad \frac{1}{12} X(j + \bar{\Delta}) + \int_j^s X_u q'(s-u) du + \int_s^{j+\bar{\Delta}} \tilde{X}_u q'(u-s) du = \frac{1 - P_s}{\gamma\sigma^2}.$$

Since this relation must hold for all  $s$ , we may differentiate both sides of (32) to obtain

$$(33) \quad \int_j^s X_u q''(s-u) du - \int_s^{j+\bar{\Delta}} \tilde{X}_u q''(u-s) du + X_s q'(0) - \tilde{X}_s q'(0) = -\frac{P'_s}{\gamma\sigma^2}.$$

This equation holds for all  $s \in (j, j + \bar{\Delta})$ . Noting that  $q'' = 1$ ,  $q'(0) = -\frac{1}{2}$ ,  $\tilde{X}(s) = X(j + \bar{\Delta}) - X(s)$ , and using (33) to solve for  $X_s$  yields

$$(34) \quad X_s = \int_j^{j+\bar{\Delta}} X_u du + \left(s - j + \frac{1}{2} - \bar{\Delta}\right) X(j + \bar{\Delta}) + \frac{P'_s}{\gamma\sigma^2}.$$

Integrating (34) from  $j$  to  $j + \bar{\Delta}$  and solving for  $\int_j^{j+\bar{\Delta}} X_u du$  leads to

$$(35) \quad \int_j^{j+\bar{\Delta}} X_u du = \frac{1}{1-\bar{\Delta}} \left[ X(j + \bar{\Delta}) \bar{\Delta} \left(\frac{1-\bar{\Delta}}{2}\right) + \frac{P(j + \bar{\Delta}) - P(j)}{\gamma\sigma^2} \right],$$

so that

$$(36) \quad X_s = \frac{1}{1-\bar{\Delta}} \left[ X(j + \bar{\Delta}) \bar{\Delta} \left(\frac{1-\bar{\Delta}}{2}\right) + \frac{P(j + \bar{\Delta}) - P(j)}{\gamma\sigma^2} \right] + \left(s - j + \frac{1}{2} - \bar{\Delta}\right) X(j + \bar{\Delta}) + \frac{P'_s}{\gamma\sigma^2}.$$

Evaluating (32) at  $s = j + \bar{\Delta}$ , and noting that  $q'(s) = -\frac{1}{2} + s$  leads to

$$(37) \quad \frac{1}{12} X(j + \bar{\Delta}) + \int_j^{j+\bar{\Delta}} X_u \left[-\frac{1}{2} + (\bar{\Delta} - u)\right] du = \frac{1 - P_{j+\bar{\Delta}}}{\gamma\sigma^2}.$$

An implication of (34) is that  $X_u = X_j + \frac{P'_u - P'_j}{\gamma\sigma^2} + X(j + \bar{\Delta})u$ . Using this expression for  $X_u$  inside (37), carrying out the requisite integrations and using integration by parts to express  $\int_j^{j+\bar{\Delta}} \left(\frac{P'_u}{\gamma\sigma^2}\right) u du = \frac{P_{j+\bar{\Delta}}}{\gamma\sigma^2}(j + \bar{\Delta}) - \frac{P_j}{\gamma\sigma^2}j - \int_j^{j+\bar{\Delta}} \frac{P_u}{\gamma\sigma^2} du$ , leads (after some simplifications) to

$$(38) \quad X(j + \bar{\Delta}) \left(\frac{1}{12} + \frac{\bar{\Delta}^3}{6} - \frac{\bar{\Delta}^2}{4}\right) - \frac{\bar{\Delta}(1-\bar{\Delta})}{2} \left(X_j - \frac{P'_j}{\gamma\sigma^2}\right) - \frac{P_{j+\bar{\Delta}} - P_j}{2\gamma\sigma^2} + \int_{j-}^{j+\bar{\Delta}} \frac{P_u - P_j}{\gamma\sigma^2} du = \frac{1 - P_{\bar{\Delta}}}{\gamma\sigma^2}.$$

Finally, evaluating (34) at  $j$  gives

$$(39) \quad \left(X_j - \frac{P'_j}{\gamma\sigma^2}\right) = \int_j^{j+\bar{\Delta}} X_u du + \left(\frac{1}{2} - \bar{\Delta}\right) X(j + \bar{\Delta}).$$

Equations (35), (38), and (39) are three linear equations in three unknowns. Solving for  $X(j + \bar{\Delta})$  and using the definition of  $\omega(\bar{\Delta})$  leads to (24). Equation (36) simplifies to (25). Finally, (26) is a direct consequence of (28) when  $\bar{\Delta} = 0$ . ■

**Proof of Proposition 4.** For any  $j \in (\frac{k}{2}, \frac{1}{2}]$  and  $l \in (-\frac{\bar{\Delta}}{2}, \frac{\bar{\Delta}}{2})$ , we have from Lemma 2:

$$\begin{aligned} X_{j-\frac{\bar{\Delta}}{2}}^{(j)} &= \frac{P'_{j-\frac{\bar{\Delta}}{2}}}{\gamma\sigma^2} + \frac{P_{j-\frac{\bar{\Delta}}{2}} - P_{j+\frac{\bar{\Delta}}{2}}}{\gamma\sigma^2(1-\Delta)} + \frac{1-\Delta}{2} X_{j+\frac{\bar{\Delta}}{2}}^{(j)} \\ dX_{j+l}^{(j)} &= \left( \frac{P''_{j+l}}{\gamma\sigma^2} + X_{j+\frac{\bar{\Delta}}{2}}^{(j)} \right) dl \\ X_{j+\frac{\bar{\Delta}}{2}}^{(j)} - X_{(j+\frac{\bar{\Delta}}{2})^-}^{(j)} &= -\frac{P'_{j+\frac{\bar{\Delta}}{2}}}{\gamma\sigma^2} - \frac{P_{j-\frac{\bar{\Delta}}{2}} - P_{j+\frac{\bar{\Delta}}{2}}}{\gamma\sigma^2(1-\Delta)} + \frac{1-\Delta}{2} X_{j+\frac{\bar{\Delta}}{2}}^{(j)}. \end{aligned}$$

Specialize the first equation to  $j = \frac{1}{2} + \frac{\bar{\Delta}}{2}$ , the second to  $j = \frac{1}{2} - l$  for all  $l \in (-\frac{\bar{\Delta}}{2}, \frac{\bar{\Delta}}{2})$ , and the third to  $j = \frac{1}{2} - \frac{\bar{\Delta}}{2}$  and aggregate to obtain the total demand for asset  $\frac{1}{2}$ :

$$1 = \frac{1-\Delta}{2} X_{\frac{1}{2}+\bar{\Delta}}^{(\frac{1}{2}+\frac{\bar{\Delta}}{2})} + \frac{1-\Delta}{2} X_{\frac{1}{2}}^{(\frac{1}{2}-\frac{\bar{\Delta}}{2})} + \int_{-\frac{\bar{\Delta}}{2}}^{\frac{\bar{\Delta}}{2}} X_{\frac{1}{2}-l+\frac{\bar{\Delta}}{2}}^{(\frac{1}{2}-l)} dl + \frac{P''_{\frac{1}{2}}}{\gamma\sigma^2} \Delta + \frac{P_{\frac{1}{2}-\bar{\Delta}} + P_{\frac{1}{2}+\bar{\Delta}} - 2P_{\frac{1}{2}}}{2\gamma\sigma^2(1-\Delta)}.$$

Suppose now that  $P_j \geq 1 - \gamma\omega(\bar{\Delta})$  on  $[\frac{1}{2} - \Delta, \frac{1}{2} + \Delta]$ , with strict inequality on a positive measure set. It then follows from equation (24) that  $X_{j+\frac{\bar{\Delta}}{2}}^{(j)} \leq 1$ , so that

$$0 < \frac{P''_{\frac{1}{2}}}{\gamma\sigma^2} \Delta + \frac{P_{\frac{1}{2}-\bar{\Delta}} + P_{\frac{1}{2}+\bar{\Delta}} - 2P_{\frac{1}{2}}}{2\gamma\sigma^2(1-\Delta)}.$$

This inequality contradicts the assumption that  $P$  is maximized at  $\frac{1}{2}$ . ■

**Proof of Proposition 1.** We adopt a guess-and-verify approach. We start by noting that the beginning-of-period wealth of investor  $i$  at time  $t+1$  is  $W_{t+1,i} \equiv \int_0^1 (P_{t+1,j} + D_{t+1,j}) dX_{t,j}^{(i)} + X_{B,t}^{(i)}$ . We then conjecture that, as long as

$$(40) \quad F(\Delta) = \frac{M}{\gamma} + \frac{\gamma}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta_t)$$

for some  $M > -\frac{\gamma^2}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta_u)$ , (ii) and (iii) obtain. We show at the end that the function  $F$  can be chosen to ensure (i).

We also conjecture and verify that investors' holdings of risky assets  $X_{t,j}^{(i)}$  coincide with  $G_{t,j}^{(i)}$  of Proposition 1, and that their bond holdings equal

$$(41) \quad X_{B,t}^{(i)} = W_{t,i} - (1+r)\bar{P}_{t,i} - r\Phi_t,$$

where  $\bar{P}_{t,i} \equiv \int_0^1 P_{t,j} dX_{t,j}^{(i)}$  is the average price that investor  $i$  pays for her portfolio. Here, to simplify notation, we defined  $\Phi_t \equiv \Phi_1 \omega(\Delta_t) + \Phi_0$ .

We first ensure that with these postulates markets clear. Clearly, all risky markets clear, since the holdings of risky assets are the same as in Proposition 1. To show that bond markets clear, we proceed inductively. First we note that investors are endowed with no bonds at time zero. Hence  $\int_0^1 X_{B,-1}^{(i)} di = 0$  and therefore  $\int_0^1 W_{0,i} di = \int_0^1 (P_{0,i} + D_{0,i}) di$ . Next we postulate that  $\int_0^1 X_{B,t-1}^{(i)} di = 0$ , so that  $\int_0^1 W_{t,i} di = \int_0^1 P_{t,i} di + \int_0^1 D_{t,j} dj$ . Integrating our postulate (41) for  $X_{B,t}^{(i)}$  across all investors, we obtain

$$(42) \quad \int_0^1 X_{B,t}^{(i)} di = \int_0^1 W_{t,i} di - (1+r) \int_0^1 \bar{P}_{t,i} di - r\Phi_t.$$

We next note that that (a)  $\int_0^1 D_{t,j} dj = 1$ , by construction of the dividend process; (b)  $\int_0^1 W_{t,i} di = \int_0^1 P_{t,i} di + \int_0^1 D_{t,j} dj = r^{-1} - \Phi_t + 1$ , using the induction hypothesis, (6), and (a); and (c)  $\int_0^1 \bar{P}_{t,i} di = \int_0^1 P_{t,i} di = r^{-1} - \Phi_t$ . Using these three facts, it follows immediately that the right-hand side of (42) is zero, so that the bond market clears.

If investors set their bond holdings according to (41), then their budget constraint implies a consumption of

$$(43) \quad c_{t,i} = W_{t,i} - \frac{1}{1+r} X_{B,t}^{(i)} - \bar{P}_{t,i} - F_t.$$

Using the definition of  $W_{t,i}$  and market clearing condition for bond holdings inside (43), and integrating across  $i$  implies that the market for consumption goods clears:  $\int_0^1 c_{t,i} di = 1 - F_t$ .

Having established market clearing given the postulated policies and prices, we next turn to optimality. Equation (43) implies

$$(44) \quad \begin{aligned} c_{t+1,i} - c_{t,i} &= W_{t+1,i} - W_{t,i} - \frac{1}{1+r} \left( X_{B,t+1}^{(i)} - X_{B,t}^{(i)} \right) - (\bar{P}_{t+1,i} - \bar{P}_{t,i}) - (F_{t+1} - F_t) \\ &= \left( \frac{r}{1+r} \right) (W_{t+1,i} - W_{t,i}) + \frac{r}{1+r} (\Phi_{t+1} - \Phi_t) - (F_{t+1} - F_t), \end{aligned}$$

where the second line follows from (41). We next use the definition of  $W_{t,i}$  and (41) to obtain

$$(45) \quad \begin{aligned} W_{t+1,i} - W_{t,i} &= \int_0^1 (P_{t+1,j} + D_{t+1,j}) dX_{t,j}^{(i)} + X_{B,t}^{(i)} - W_{t,i} \\ &= \int_0^1 (P_{t+1,j} + D_{t+1,j}) dX_{t,j}^{(i)} - (1+r) \bar{P}_{t,i} - r\Phi_t. \end{aligned}$$

Substituting (45) into (44) and using (6) and (41) leads to

$$(46) \quad c_{t+1,i} - c_{t,i} = \left( \frac{r}{1+r} \right) \left[ (1+\phi) \int_0^1 D_{t+1,j} dX_{t,j}^{(i)} - (1+r)\phi \int_0^1 D_{t,j} dX_{t,j}^{(i)} - (1-r\phi) \int_0^1 dX_{t,j}^{(i)} \right] - (F_{t+1} - F_t).$$

Next use the fact  $\bar{D}_{t+1,j} = \rho \bar{D}_{t,j} + (1-\rho) \bar{\varepsilon}_{t+1,j}$  along with  $\phi = \frac{\rho}{1+r-\rho}$ ,  $(1+\phi)\rho = (1+r)\phi$ , and  $(1+\phi)(1-\rho) = (1-r\phi)$  inside (46) to arrive at

$$(47) \quad c_{t+1,i} - c_{t,i} = \left( \frac{r}{1+r-\rho} \right) (1-\rho) \int_0^1 (\varepsilon_{t+1,j} - 1) dX_{t,j}^{(i)} - (F_{t+1} - F_t).$$

Having established (47), the dynamics of agent  $i$ 's consumption under our postulate, we next turn attention to the Euler equations, starting with the bond Euler equation

$$(48) \quad 1 = \beta(1+r) E_t e^{-\gamma(c_{t+1,i} - c_{t,i})}.$$

Substituting (47) into (48) and noting that  $\int_0^1 (\varepsilon_{t+1,j} - 1) dX_{t,j}^{(i)}$  is normally distributed with mean zero and variance  $\omega(\Delta_t)$  gives

$$(49) \quad 1 = \beta(1+r) e^{\frac{\gamma^2}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta_t) - \gamma F_t} E_t (e^{\gamma F_{t+1}}).$$

Now suppose that for any  $r$  and a given desired distribution  $\Psi(\Delta)$  we set

$$(50) \quad F_t(\Delta_t; r) = \frac{M}{\gamma} + \frac{\gamma}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta_t).$$

Then equation (49) can be written as (5). Since  $(1+r) E e^{\frac{\gamma^2}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta)}$  is equal to 1 when  $r = 0$  and increases monotonically to infinity as  $r$  increases, it follows that there exists a unique positive  $r$  such that equation (5) holds. For that value of  $r$ , all investors' bond Euler equations are satisfied.

Finally, we need to determine  $\Phi_t$  so as to ensure that the Euler equations for risky assets hold, i.e., that

$$(51) \quad P_{t,j} = \beta E_t \left[ e^{-\gamma(c_{t+1,i} - c_{t,i})} (P_{t+1,j} + D_{t+1,j}) \right].$$

To that end, we use (6) and (3) to express (51) as

$$(52) \quad \frac{1}{r} - \Phi_t + \phi(D_{t,j} - 1) = \beta E_t \left[ e^{-\gamma(c_{t+1,i} - c_{t,i})} \left( \frac{1}{r} - \Phi_{t+1} + (1+\phi)(\rho D_{t,j} + (1-\rho)\varepsilon_{t+1,j}) - \phi \right) \right].$$

We next note that

$$(53) \quad \beta E_t \left[ e^{-\gamma(c_{t+1,i} - c_{t,i})} \right] (1 + \phi) \rho D_{t,j} = \frac{(1 + \phi) \rho}{1 + r} D_{t,j} = \phi D_{t,j}$$

using (48). Equation (53) simplifies (52) to

$$(54) \quad \begin{aligned} \frac{1}{r} - \Phi_t - \phi &= \beta E_t \left[ e^{-\gamma(c_{t+1,i} - c_{t,i})} \left( \frac{1}{r} - \Phi_{t+1} + (1 + \phi)(1 - \rho) \varepsilon_{t+1,j} - \phi \right) \right] \\ &= \frac{1}{r(1 + r)} - \beta E_t \left[ e^{-\gamma(c_{t+1,i} - c_{t,i})} \Phi_{t+1} \right] + \frac{1}{1 + r} (-\phi + (1 + \phi)(1 - \rho)) \\ &\quad + (1 + \phi)(1 - \rho) \beta E_t \left[ e^{-\gamma(c_{t+1,i} - c_{t,i})} (\varepsilon_{t+1,j} - 1) \right]. \end{aligned}$$

Using (47), Stein's Lemma, the fact that  $\text{cov} \left( \int_0^1 (\varepsilon_{t+1,j} - 1) dX_{t,j}^{(i)}, \varepsilon_{t+1,j} \right) = \omega(\Delta)$  (see Proposition 1), and (48) implies

$$(55) \quad \beta E_t \left[ e^{-\gamma(c_{t+1,i} - c_{t,i})} \varepsilon_{t+1,j} \right] = \frac{1 - \gamma \frac{r}{1+r-\rho} (1 - \rho) \omega(\Delta_t)}{1 + r}.$$

Substituting (55) into (54) gives linear equations in  $\Phi_0$  and  $\Phi_1$ , solved by (7), respectively (8).

To complete the proof of the claim that  $\Delta_t$  is chosen optimally, we provide an explicit example of a family of functions for  $F_t(\Delta)$  that has the desired properties. To start, we compute the value function of an investor adopting the policies of Proposition 1. Equation (48) along with (47) imply that

$$\begin{aligned} V(W_{t,i}, \Delta_t) &= -\frac{1}{\gamma} \sum_{t \geq 0} \beta^t E_t \left[ e^{-\gamma c_{t,i}} \right] \\ &= -\frac{1}{\gamma} e^{-\gamma c_{0,i}} \left( 1 + \sum_{t \geq 1} \beta^t E_t \left[ e^{-\gamma \sum_{m=0}^{t-1} (c_{m+1,i} - c_{m,i})} \right] \right) \\ &= -\frac{1}{\gamma} e^{-\gamma c_{0,i}} \sum_{t \geq 0} (1 + r)^{-t} = -\frac{1 + r}{\gamma r} e^{-\gamma c_{0,i}}. \end{aligned}$$

In turn, equations (41), (43), and (50) imply

$$(56) \quad V(W_{t,i}, \Delta_t) = -\frac{1 + r}{\gamma r} e^{-\frac{\gamma r}{1+r} W_{t,i} + z(\Delta_t)},$$

where  $z_t(\Delta_t) \equiv -\frac{r}{1+r} \gamma \Phi_t + M + \frac{\gamma^2}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1 - \rho)^2 \omega(\Delta_t)$ .

Next we suppose that we no longer impose that the investor choose  $\Delta = \Delta_t$ , (where  $\Delta_t$  is the time- $t$  random draw of  $\Delta$  that we imposed in Proposition 1). Instead  $\Delta$  is chosen optimally. However, prices are still given by  $P_{t,j}(\Delta_t, D_{t,j})$  from



equation (6). We will construct a function  $\kappa_t g_t(\Delta)$  that renders the choice  $\Delta = \Delta_t$  optimal at the total cost specified in (50).

Throughout we let  $\bar{X}_t^{(i)}(\Delta; \Delta_t)$  denote the optimal number of total risky assets chosen by investor  $i$ , and assuming that that investor chooses  $\Delta$  and prices are given by  $P_{t,j}(\Delta_t, D_{t,j})$ . For future reference, we note that by construction of the price function  $P_{t,j}(\Delta_t, D_{t,j})$  it follows that  $\bar{X}_t^{(i)}(\Delta_t; \Delta_t) = 1$ . Using (56) the first order condition characterizing an optimal  $\Delta$  is

$$F'_t(\Delta) = h(\Delta; \Delta_t),$$

where

$$(57) \quad h(\Delta; \Delta_t) = -\frac{1}{1+r} \frac{\gamma}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \left( \bar{X}_t^{(i)}(\Delta; \Delta_t) \right)^2 \omega'(\Delta).$$

Next we fix a value of  $\Delta_t$  and we simplify notation by writing  $h(\Delta)$  rather than  $h(\Delta; \Delta_t)$ . We also let  $q(x)$  denote some continuous function with  $q(0) = 1$  and  $q(x) > 1$  for  $x > 0$ . Let  $\eta \in [0, 1]$ , take some positive (small)  $\varepsilon < \frac{\Delta_t}{2}$ , and consider the function

$$(58) \quad F'_t(\Delta) = \begin{cases} \frac{\Delta}{\varepsilon} \eta h(\varepsilon) & \text{for } \Delta \leq \varepsilon \\ \eta h(\Delta) & \text{for } \Delta \in (\varepsilon, \Delta_t - \varepsilon] \\ \eta h(\Delta_t - \varepsilon) \frac{\Delta_t - \Delta}{\varepsilon} + h(\Delta_t) \frac{\Delta - \Delta_t + \varepsilon}{\varepsilon} & \text{for } \Delta \in (\Delta_t - \varepsilon, \Delta_t] \\ h(\Delta) q(\Delta) & \text{for } \Delta > \Delta_t + \varepsilon. \end{cases}$$

By construction,  $F'_t(0) = 0$  and  $F'_t(\Delta)$  is continuous and increasing in  $\Delta$ . More importantly,  $F'_t(\Delta_t) = h(\Delta_t)$ , and hence  $\Delta = \Delta_t$  satisfies the necessary first order condition (57). Moreover, since  $F'_t(\Delta) < (>) h(\Delta)$  for  $\Delta < (>) \Delta_t$ , it follows that  $\Delta = \Delta_t$  is optimal for any  $\varepsilon > 0$  and  $\eta \in [0, 1]$ . Finally,

$$(59) \quad \lim_{\varepsilon \rightarrow 0} \int_0^{\Delta_t} F'_t(\Delta) = \eta \int_0^{\Delta_t} h(x) dx > 0.$$

Now suppose that nature draws  $\Delta_t = \Delta_u > 0$ . By choosing  $M$  that is sufficiently close to  $-\frac{\gamma}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\bar{\Delta})$  it follows that

$$(60) \quad 0 < \frac{M}{\gamma} + \frac{\gamma}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta_u) < \int_0^{\Delta_u} h(x) dx.$$

Combining equations (59) and (60) it follows that for sufficiently small  $\varepsilon > 0$  there exists some  $\eta \in [0, 1]$  so that

$$(61) \quad \int_0^{\Delta_u} F'_t(x) dx = \frac{M}{\gamma} + \frac{\gamma}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta_u) > 0.$$

Hence, when  $\Delta_t = \Delta_u$  the cost function  $\kappa_t g_t(\Delta)$  renders  $\Delta = \Delta_u$ , while also satisfying (50). The same argument implies that for any value of  $\Delta_t$  that satisfies

$$(62) \quad 0 < \frac{M}{\gamma} + \frac{\gamma}{2} \left( \frac{r}{1+r-\rho} \right)^2 (1-\rho)^2 \omega(\Delta_t) < \int_0^{\Delta_t} h(x) dx,$$

there exists  $\eta \in [0, 1]$  and sufficiently small  $\varepsilon > 0$  such that the optimal  $\Delta$  coincides with  $\Delta_t$ , and (50) holds. Continuity of  $\omega(\Delta_t)$  and of  $\int_0^{\Delta_t} h(x) dx$  in  $\Delta_t$  implies that as long as  $\underline{\Delta}$  is sufficiently close to  $\Delta_u$ , there always exists  $\eta \in [0, 1]$  and  $\varepsilon > 0$  (both depending on the random draw  $\Delta_t$ ) such that  $\Delta = \Delta_t$  is optimal and (50) holds. ■

**Proof of Proposition 2.** Parts (i)–(iii) are proved in the main body of the text. Part (iv) comes down to noticing that

$$(63) \quad \text{cov}(e^z, z) > 0$$

for any random variable  $z$  — in particular, for  $z = \omega(\Delta)$ . The second statement of (iv) follows from the first and Jensen’s inequality applied to the convex function  $\omega$ . ■

## C. An interpretation of participation costs

Throughout the paper we maintain the assumption that participation in “distant” markets incurs participation costs. In this appendix<sup>5</sup> we discuss how these costs could arise as information-acquisition costs that permit an investor to avoid the lower net returns earned by an investor unfamiliar with the asset class. Indeed, we wish to re-emphasize that we construe the notion of distance broadly, as a stand-in for the level of familiarity of investors in one location with all aspects of the financial environment in another. We also wish to emphasize that our notion of locations is meant to be very broad, in particular encompassing asset classes that may be especially opaque to uninformed investors (e.g., mortgage pools and small stocks in distant countries).

### C.1. Regular firms and common investors

Timing and the set of locations are the same as in Section II.. In each location there are measure-one continua of investors and firms, but both are of two types: Investors are either “common investors” or “swindlers”, while firms are either “regular” or “fraudulent”.

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<sup>5</sup>This appendix borrows from Gârleanu et al. (2013). We present a self-contained version of the model to keep the effort required of the interested reader to a minimum. Another related paper that derives portfolio concentration as a result of endogenous information acquisition is Van Nieuwerburgh and Veldkamp (2010).

Common investors in each location constitute a fraction  $\nu \in (0, 1)$  of the population in that location. They are identically endowed with an equal-weighted portfolio of all regular firms in that location  $i$ . The total measure of regular firms in each location is also  $\nu$ . All regular firms in location  $i$  produce the same random output:  $\hat{D}_{ik} = \hat{D}_i$ , where  $k$  identifies the firm. The dividend is given by equation (3), as in the paper:

$$(64) \quad \hat{D}_i \equiv 1 + \left( B_i - \int_0^1 B_j dj \right).$$

Swindlers are a fraction  $1 - \nu$  of the population in each location. Each swindler is endowed with the entirety of shares (normalized to one) of one fraudulent firm. Fraudulent firms produce zero output ( $\hat{D}_{ik} = 0$ ).

For every firm in every location, there is a market for shares where any investor can submit a demand. As before, there exists a market for the riskless bond, available in zero net supply. Since investors don't consume at time zero, we use the bond as the numeraire, and normalize its price to one ( $r = 0$ ).

## C.2. Budget constraints

Letting  $\hat{B}^{ci}$  denote the amount that a common investor in location  $i$  invests in riskless bonds, and  $\hat{X}_{jk}^{ci}$  denote a bivariate signed measure giving the number shares of firm  $k$  in location  $j$  she invests in, the time-one wealth of a common investor located in  $i$  is given by

$$(65) \quad \hat{W}_1^{ci} \equiv \hat{B}^{ci} + \int_{j \in \mathcal{L}} \int_{k \in [0,1]} D_{jk} d\hat{X}_{jk}^{ci}.$$

The first term on the right hand side of (65) is the amount that the investor receives from her bond position in period 1, while the second term captures the portfolio-weighted dividends of all the firms that the investor holds. The time-zero budget constraint of a common investor in location  $i$  is given by

$$(66) \quad \hat{B}^{ci} + \int_{j \in \mathcal{L}} \int_{k \in [0,1]} \hat{P}_{jk} d\hat{X}_{jk}^{ci} = \frac{1}{\nu} \int_{k \in [0,1]} \hat{P}_{ik} \rho_{ik} dk,$$

where  $\rho_{ik}$  is an indicator function taking the value one if the firm  $k$  in location  $i$  is a regular firm and zero otherwise, and  $\hat{P}_{jk}$  refers to the price of security  $k$  in location  $j$ . The left-hand side of (66) corresponds to the sum of the investor's bond and risky-security spending, while the right-hand side reflects the value of the (equal-weighted) portfolio of regular firms the investor is endowed with.

## C.3. Signals

Each investor may obtain a signal of the type — regular or fraudulent — of every firm in every location. The precision of these signals depends on the locations of the investor and the firm.

Specifically, we assume that each fraudulent firm in every location is assigned in an i.i.d. fashion a uniformly distributed index  $u \in [0, 1 - \nu]$ , which reflects the difficulty with which it can be identified as fraudulent. This index is not observed by anyone.

After the index is drawn, investors in every location  $i$  may obtain a signal about each firm in location  $j$ . (All investors in  $i$  who choose to become informed obtain the same signal about any given firm.) This signal characterizes the firm as either regular or fraudulent. The signal is imperfect. It correctly identifies every regular firm as such. However, it fails to identify all fraudulent firms: it correctly identifies a fraudulent firm that has drawn an index  $u$  with probability  $\pi_u$  and misclassifies it as regular with probability  $1 - \pi_u$ . Note that we take  $\pi_u$  independent of  $i$  and  $j$ .

We introduce the index  $u$  to obtain a heterogeneous distribution of the demand for the shares of fraudulent firms in the same location by investors in other locations. We assume that for some (positive-measure) set of values  $u$   $\pi_u = 1$ , i.e., there exist in every location a positive measure of fraudulent firms that are correctly classified as fraudulent by all informed agents. No one knows, however, the identity of these firms before trading takes place. (Even conditional on prices and each agent's own trades, only the swindler can infer the  $u$  of her own firm in equilibrium).

Given this setup, Bayes' rule implies that the probability that a firm in location  $j$  is regular given that investor  $i$ 's signal identifies it as regular is given by

$$(67) \quad p \equiv \frac{\nu}{\nu + \int_0^{1-\nu} (1 - \pi_u) du}.$$

The law of large numbers implies then that  $p$  can also be interpreted as the fraction of firms in a given location  $j$  that are regular, given that the signal of investor  $i$  has identified them as regular.

The signals are costly. Specifically, the investor in location  $i$  can acquire signals about all the firms in a given location  $j$ , by paying exactly the same "participation costs" that we assume in Section B.. We denote the cost function by  $\hat{F}$ .

## C. 4. Earnings manipulation and swindler's problem

We next introduce an assumption whose sole purpose is to ensure endogenously that agents do not short fraudulent shares. Before proceeding, we note that one can dispense with all the assumptions of this section, by simply imposing a no-shorting constraint.

Specifically, we assume that swindlers have the ability to manipulate the earnings of fraudulent firms. A swindler  $l$  in location  $i$  has the ability to borrow any amount  $\hat{L}^{il} \geq 0$  of her choosing at time 0, divert these funds into the firm, and report earnings equal to  $\hat{L}^{il}$  in period 1. (Equivalently, we could assume that the swindler can take an action to produce earnings  $\hat{L}^{il}$  by incurring a personal non-pecuniary cost of effort, which would have a value  $\hat{L}^{il}$  in monetary terms.)

The budget constraint of a swindler is similar to (66) except that  $\nu^{-1} \int_{k \in [0,1]} \hat{P}_{ik} \rho_{(i,k)} dk$  is replaced by  $\hat{P}_{il}$ :

$$(68) \quad \hat{B}^{sil} + \int_{j \in \mathcal{L}} \int_{k \in [0,1]} \hat{P}_{jk} d\hat{X}_{jk}^{sil} = \hat{P}_{il}.$$

Note that, as before, the notation allows investors' portfolios to have atoms, which is further useful here because, in equilibrium, swindlers optimally hold a non-infinitesimal quantity of shares of their own firms. We denote the post-trade number of shares held by the swindler who owns firm  $l$  in location  $i$  by  $\hat{S}^{il} = d\hat{X}_{il}^{sil}$ .

The time-1 wealth of a swindler is

$$(69) \quad \hat{W}_1^{sil} \equiv \hat{B}^{sil} + \int_{j \in \mathcal{L}} \int_{k \in [0,1]} \hat{D}_{jk} d\hat{X}_{jk}^{sil} + \hat{L}^{il} (\hat{S}^{il} - 1).$$

The difference with (65) is the term  $\hat{L}^{il} (\hat{S}^{il} - 1)$ , which represents the swindler's consumption gains when performing a diversion  $\hat{L}^{il}$ , while owning a post-trade number of shares in her company equal to  $\hat{S}^{il}$ . This increase is intuitive. If  $\hat{S}^{il} - 1 < 0$ , i.e., if the swindler reduces her ownership of shares by being a net seller, then she has no incentive to perform earnings diversion since she will recover only a fraction of her personal funds that she diverts into the company; thus  $\hat{L}^{il} = 0$ . If, however, the swindler is a net buyer of her own security ( $\hat{S}^{il} - 1 > 0$ ), then the ability to manipulate earnings becomes infinitely valuable, since  $\hat{L}^{il}$  can be chosen arbitrarily large. Intuitively, the swindler can report arbitrarily large profits at the expense of outside investors who hold negative positions (short sellers) in the fraudulent firm. As a consequence, in equilibrium all other investors optimally refrain from shorting, even when they know for sure that the firm is fraudulent. The reasoning is as follows: With positive probability  $\pi_u = 1$ ; accordingly all investors' signals identify the fraudulent firm as such, and investors do not find it optimal to submit a positive demand for that firm in equilibrium. Therefore, any prospective shorter understands that *any* short position that she establishes implies  $\hat{S}^{il} > 1$ , and is (unboundedly) loss-making. Since the quality of the signal ( $u$ ) is not observed by anyone<sup>6</sup>, a prospective shorter must assign positive probability to such an occurrence, and hence avoids short-selling.

Before proceeding, we reiterate that our earnings-manipulation assumption serves only as a deterrent to shorting, and is intentionally stylized so as to expedite the presentation of the results that follow. In reality there are many other reasons why investors are deterred from short selling firms with tightly controlled float, such as "short squeezes", which we do not model here.<sup>7</sup> The exact nature of the shorting

<sup>6</sup>In equilibrium only the swindler can infer the  $u$  of her own firm.

<sup>7</sup>A short squeeze refers to the possibility of cornering the shorting market by restricting the amount of securities that are available for lending and forcing short sellers to close

deterrent is inconsequential for our results. (See, e.g., Lamont (2012) for an empirical study documenting various short-selling deterrence mechanisms employed by firms.)

## C.5. Optimization problem

All investors are maximizing a CARA utility with parameter  $\hat{\gamma}$  over time-1 wealth.

Common investors are price takers. Taking the interest rate, the prices for risky assets, and the actions of the swindlers as given for all firms in all locations, a common investor maximizes her expected utility subject to (66). The investor conditions on her own information set  $\mathcal{F}_i$  (i.e., on her signals about every security), as well as on the prices of all securities in all markets.

The problem of the swindler is similar to the one of the common investor with two exceptions: a) she takes into account the impact of her trading on the price of her stock, and b) she needs to decide whether to manipulate the earnings of her company. Similarly to a common investor, the swindler who owns firm  $l$  in location  $i$  maximizes her utility over  $\hat{B}^{sil}$  and  $d\hat{X}_{jk}^{sil}$  subject to the budget constraint (68).

## C.6. Equilibrium

An equilibrium is a collection of prices  $\hat{P}_i$  for all risky assets, asset demands, and bond holdings expressed by all investors in all locations, such that: 1) Markets for all securities clear:  $\nu \int_{i \in \mathcal{L}} d\hat{X}_{jk}^{ci} + \int_{i \in \mathcal{L}} \int_{l=\nu}^1 d\hat{X}_{jk}^{sil} = 1$  for all  $j, k$ ; 2) Risky-asset and bond holdings,  $\{\hat{X}_{jk}^{ci}, \hat{B}^{ci}\}$ , are optimal for regular investors in all locations given prices and the investors' expectations; 3) Optimal bond holdings  $\hat{B}^{sil}$ , diversion amounts  $\hat{L}^{sil}$ , and asset holdings for all securities  $\hat{X}_{jk}^{sil}$  (including a swindler's own holdings of her own firm  $\hat{S}^{il}$ ) are optimal for swindlers given their expectations; 4) All investors update their beliefs about the type of stock  $k$  in location  $j$  by using all available information to them — prices, interest rate, and private signals — and Bayes' rule, whenever possible.

Our equilibrium concept contains elements of both a rational expectations equilibrium and a Bayes-Nash equilibrium. All investors make rational inferences about the type of each security based on their signals, the equilibrium prices, and the interest rate, by using Bayes' rule and taking the optimal actions of all other investors (regular and swindlers) in all locations as given. The assumption of a continuum of regular investors implies that they are price takers in all markets. In forming dividend anticipations about a given security, they take prices and the optimal diversion strategies of swindlers as given.

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out their positions. Short squeezes can be detrimental to short sellers. We do not model these deterrence mechanisms here since they would require the introduction of more trading periods in the model.

Swindlers, on the other hand, are endowed with the shares of a fraudulent company and can manipulate its earnings. Thus they take into account the impact of their trades on the security they are trading. In formulating a demand for their security, swindlers have to consider how different prices might affect the perceptions of other investors about the type of their security. As is standard, Bayes' rule disciplines investors' beliefs only for demand realizations that are observed in equilibrium. As is usual in a Bayes-Nash equilibrium, there is freedom in specifying how out-of-equilibrium prices affect investor posterior distributions of security types.

We note that the distinction between regular investors, who are rational price-takers and swindlers who are strategic about the impact of their actions on the price of their firm is helpful for expediting the presentation of results, but not crucial. We can show<sup>8</sup> that our equilibrium concept is the limit (as the number of traders approaches infinity) of a sequence of economies with finite numbers of traders — both regular and swindlers — who are all strategic about their price impact and rational about their inferences, as in Kyle (1989).

## C.7. Non-participation

We are finally ready to state the result of this appendix, which states that non-participation can arise as the result of a choice not to obtain signals about given markets. As in Section II., we focus for simplicity on location invariance.

**Proposition 3** *Consider an equilibrium to the economy in Section II., defined by the price  $P$  and the share holdings  $X_j^{(i)}$ . Consider also the class of asymmetric-information economies, defined in this appendix, that are indexed by  $\nu$  and obey the restrictions  $\hat{\gamma}(\nu) = \gamma/\nu$ ,  $\hat{F} = \nu F$ , and  $p(\nu)/\nu$  decreases in  $\nu$  with  $\lim_{\nu \rightarrow 0} p(\nu)/\nu = \infty$ .*

*Then there exists a value  $\bar{\nu} > 0$  such that, as long as  $\nu \leq \bar{\nu}$ , an equilibrium exists in the asymmetric-information economy that exhibits the following properties.*

- (i) *All prices in location  $i$  are equal,  $\hat{P}_{ik} = \hat{P}_i = \hat{P} = pP$ .*
- (ii) *There is neither shorting nor dividend manipulation in equilibrium— in particular, of fraudulent firms.*
- (iii) *If an investor  $i$  in the participation-cost economy participates in a set  $\mathcal{S}_i$  of locations, then the same investor acquires signals for locations  $\mathcal{S}_i$  and no others, and only trades in these locations; in fact, the share holdings are proportional (excluding swindler's holdings of own firm share):*

$$(70) \quad \int_{k \in \mathcal{L}} d\hat{X}_{jk}^{ci} = \int_{k \in \mathcal{L}} d\hat{X}_{jk}^{sil} = \frac{\nu}{p} dX_j^{(i)}.$$

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<sup>8</sup>See Gârleanu et al. (2013) for details.

Proposition 3 provides an information-theoretic interpretation of participation costs. The investors in the main body of the paper can be thought of as acting in an environment in which they have the following choice. They can either pay a “participation” cost — exactly the same as specified in Section B. — to learn informative signals of the types of securities in a given location, or simply limit their investments in that location to uninformed (index) portfolios. According to Proposition 3, in case their information advantage in markets where they are informed is sufficiently large compared to markets where they are uninformed, then it is optimal to allocate their limited capital exclusively in locations where they are not subject to an informational disadvantage, thus foregoing some diversification benefits. Put differently, they only invest in a location when they have paid the cost to become informed about it.

### C. 8. Proof of Proposition 3

**Proof.** We structure the proof in two steps. We start by assuming that no shorting or dividend manipulation is possible, and verify that the prescribed prices and portfolios constitute an equilibrium. We then check that property (ii) obtains as a result of optimal behavior.

*Step 1.* Suppose first that investors in the asymmetric-information economy (AIE) obtain signals and invest in the same locations  $\mathcal{S}_i$  that the investors in the base-case, participation-cost economy (PCE) choose to participate in. We note that a bad signal identifies a firm as fraudulent for sure, and therefore the investor does not purchase any share in such a firm. Since the signals on any market  $j$  contain noise (so that every firm has the same probability of being misclassified), it is optimal to allocate the funds in location  $j$  in an equal way across the firms that are identified as regular. In particular, letting  $K_j^i$  be the set of firms in location  $j$  for which agent  $i$ 's signal is positive, the quantity

$$(71) \quad d\hat{Z}_j^i \equiv \int_{k \in K_j^i} \rho_{jk} d\hat{X}_{jk}^{ci}$$

is deterministic and equal to

$$(72) \quad d\hat{Z}_j^i = p \int_{k \in K_j^i} d\hat{X}_{jk}^{ci} \equiv p d\hat{X}_j^{ci}.$$



Given the prices, all competitive investors maximize

$$\begin{aligned}
(73) \quad & E \left[ \int_{j \in \mathcal{S}_i} \int_{k \in K_j^i} \left( \hat{D}_{jk} - \hat{P} \right) d\hat{X}_{jk}^{ci} \right] - \frac{\hat{\gamma}}{2} \text{Var} \left( \int_{j \in \mathcal{S}_i} \int_{k \in K_j^i} \hat{D}_{jk} d\hat{X}_{jk}^{ci} \right) \\
&= E \left[ \int_{j \in \mathcal{S}_i} \int_{k \in K_j^i} \left( \hat{D}_j - \hat{P} \frac{1}{p} \right) \rho_{jk} d\hat{X}_{jk}^{ci} \right] - \frac{\hat{\gamma}}{2} \text{Var} \left( \int_{j \in \mathcal{S}_i} \int_{k \in K_j^i} \hat{D}_j \rho_{jk} d\hat{X}_{jk}^{ci} \right) \\
&= \int_{j \in \mathcal{S}_i} \left( E[p\hat{D}_j] - \hat{P} \right) d\hat{X}_j^{ci} - \frac{\hat{\gamma}}{2} \text{Var} \left( \int_{j \in \mathcal{S}_i} p\hat{D}_j d\hat{X}_j^{ci} \right) \\
(74) \quad &= \int_{j \in \mathcal{S}_i} (E[D_j] - P) d\hat{Z}_j^i - \frac{\gamma}{2\nu} \text{Var} \left( \int_{j \in \mathcal{S}_i} D_j d\hat{Z}_j^i \right).
\end{aligned}$$

It consequently follows that, if  $dX_j^{(i)}$  satisfies the first-order conditions of the investor in the PCE,  $d\hat{Z}_j^i = \nu dX_j^{(i)}$  satisfies the first-order conditions of a common investor in the AIE. We therefore conclude that (70) holds on  $\mathcal{S}_i$ .

To verify that the markets for all regular firms clear, it suffices to note that common investors in  $i$  purchase  $d\hat{Z}_j^{ci}$  shares of regular firms in location  $j$ , thus  $\nu dX_j^{(i)}$ . The swindlers in  $i$  solve exactly the same problem, in addition to that of trading in their own firm. Consequently the total demand for regular firms in location  $j$  is  $\nu \int_i dX_j^{(i)} = \nu$ , the same as the supply. We verify later that markets clear for the fraudulent firms, as well.

We next check that for  $\nu$  sufficiently low no agent wishes to invest outside  $\mathcal{S}_i$ . The agent's concave objective implies that, if a better portfolio existed than the one optimal on  $\mathcal{S}_i$ , then trading towards that portfolio would be beneficial. Thus, if  $\hat{W}_1^{ci}$  is the optimal wealth achievable on  $\mathcal{S}_i$ , then, for  $t \in (0, 1)$ ,

$$\begin{aligned}
& E[\hat{W}_1^{ci}] - \frac{\hat{\gamma}}{2} \text{Var}(\hat{W}_1^{ci}) \\
& \leq E \left[ \hat{W}_1^{ci} + t \int_{j,k} \left( \hat{D}_{jk} - \hat{P} \right) d\hat{X}_{jk} \right] - \frac{\hat{\gamma}}{2} \text{Var} \left( \hat{W}_1^{ci} + t \int_{j,k} \hat{D}_{jk} d\hat{X}_{jk} \right),
\end{aligned}$$

which gives, by letting  $t$  go to zero,

$$(75) \quad E \left[ \int_{j,k} \left( \hat{D}_{jk} - \hat{P} \right) d\hat{X}_{jk} \right] - \hat{\gamma} \text{Cov} \left( \hat{W}_1^{ci}, \int_{j,k} \hat{D}_{jk} d\hat{X}_{jk} \right) \geq 0.$$

Since both sides are linear in  $d\hat{X}_{jk}$ , and the inequality holds with equality for  $j \in \mathcal{S}_i$ , it must hold for at least one location  $j \notin \mathcal{S}_i$ . (Remember that the agent

cannot distinguish between firms in location  $j$ .) Thus,

$$\begin{aligned}
0 &\leq E \left[ \int_k \left( \hat{D}_{jk} - \hat{P} \right) d\hat{X}_{jk} \right] - \hat{\gamma} Cov \left( \hat{W}_1^{ci}, \int_k \hat{D}_{jk} d\hat{X}_{jk} \right) \\
&= (\nu - \hat{P}) \int_k d\hat{X}_{jk} - \hat{\gamma} Cov \left( \hat{W}_1^{ci}, \nu \hat{D}_j \right) \int_k d\hat{X}_{jk} \\
&= (\nu - pP - \gamma Cov(W_1^i, \nu D_j)) \int_k d\hat{X}_{jk}, \\
(76) \quad &= \nu \left( 1 - \gamma Cov(W_1^i, D_j) - \frac{p}{\nu} P \right) \int_k d\hat{X}_{jk},
\end{aligned}$$

where we used the fact that agent  $i$  in the AIE takes the same risky positions (when restricted to  $\mathcal{S}_i$ ) as agent  $i$  in the PCE, multiplied by  $\nu$ .

Expression (76), however, is clearly negative as long as  $P > 0$  and  $p/\nu$  large enough. This conclusion represents a contradiction, thus implying that the investor cannot achieve a higher utility by purchasing (a non-zero measure of) shares located outside  $\mathcal{S}_i$ .

We can write down the investor's certainty-equivalent as a function of the choice  $\mathcal{S}_i$  by combining (74) with (70) and adding the information costs:

$$\begin{aligned}
&\nu \left( \int_{j \in \mathcal{S}_i} (E[D_j] - P) dX_j^{(i)} - \frac{\gamma}{2} Var \left( \int_{j \in \mathcal{S}_i} D_j dX_j^{(i)} \right) \right) - \hat{F} \\
(77) \quad &= \nu \left( \int_{j \in \mathcal{S}_i} (E[D_j] - P) dX_j^{(i)} - \frac{\gamma}{2} Var \left( \int_{j \in \mathcal{S}_i} D_j dX_j^{(i)} \right) - F \right).
\end{aligned}$$

The tradeoff between diversification and costs is therefore the same in the AIE as in the PCE, implying that any choice of  $\mathcal{S}_i$  optimal in the AIE is optimal in the PCE.

*Step 2.* Consider now the problem of a swindler. Her investment in her own (fraudulent) firm is independent of the choices she makes with respect to information acquisition and investment in the other firms — in particular, in these respects she behaves just like a common investor.

In her own firm, the swindler submits a demand that may affect prices. We assume the following off-the-equilibrium-path beliefs: a firm whose price is not equal to  $P$  is inferred to be fraudulent with probability one. Since no one buys a firm believed fraudulent for sure, the swindler's only chance of making a profit off their fraudulent firm is to submit a demand that is perfectly elastic at price  $P$ . Indeed, it is immediate to see that, if the demand by investors other than the swindler at  $P$  is positive, then the swindler makes a profit, equal to  $P$  times this demand.

Furthermore, if no investor shorts this firm, then the aggregate demand (excluding the swindler) is positive, and therefore the swindler does not manipulate dividends. We discussed this decision in detail above, following equation (69).

Conversely, if the aggregate demand by investors other than the swindler were negative, then the swindler would manipulate in unlimited amounts —  $\hat{L}^{il} = \infty$  —

and the agents who shorted would make an infinite loss. The assumptions made on  $\pi$  (i.e., on the correlation structure of the firm-specific shocks  $u$ ) imply that, with strictly positive probability, all investors' signals identify firm  $l$  as fraudulent. (However, investors do not know that other investors have also identified such a firm as fraudulent). This means that, with positive probability, no investor beside the swindler submits a positive demand for this firm. A shorting agent, therefore, would make an infinite loss for any short position, since in that case market clearing would imply  $\hat{S}^{il} > 1$ .<sup>9</sup> ■

## D. An alternative formulation of the leverage constraint

In this section we elaborate further on the interaction between borrowing constraints and high price sensitivity to participation costs. Specifically, we introduce a “limited-liability” constraint that places an endogenous bound on borrowing, and show that it plays a similar role to constraint (23) in the text. In particular, we show a qualitatively similar amplification result to Section IV., as depicted in Figure 8: An increase in the participation cost parameter implies an amplified – and possibly discontinuous – reaction of the equilibrium price as compared to the case where the limited-liability constraint is not imposed.

Before formalizing and analyzing the constraint, we provide a new dividend structure. An important novel feature of this structure is that all dividends are non-negative, so that the notion of limited liability is economically meaningful. Specifically, let  $\Gamma_j$  be a Gamma process on  $[0, 1)$ , so that for  $u > s$  we have

$$\Gamma_u - \Gamma_s \sim \Gamma(k(u - s); \nu).$$

Extending  $d\Gamma$  to the entire real line as before — that is, via  $d\Gamma_s = d\Gamma_{s \bmod 1}$  — we define

$$(78) \quad D_j = \mu + \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} w_{s-j} d\Gamma_s$$

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<sup>9</sup>To make this argument precise for an economy with a continuum of agents, in Gârleanu et al. (2013) we consider a sequence of finite economies with increasing numbers of strategic traders and show that investors do not find it optimal to short stocks in any equilibrium along the sequence. However, their price impact declines monotonically as the number of traders increases. (Intuitively, the reason is that the danger of trading against a swindler receiving zero outside demand for her share is present no matter how small is the size of a short position.) Since we want to construe our economy with a continuum of agents as a limit of finite economies, we must assume that any short position against a swindler receiving zero outside demand would give rise to earnings manipulation.

for some  $\mu \geq 0$  and weights  $w_i \geq 0$  periodic with period 1 and symmetric around 0. In the interest of concreteness, in our numerical illustration below we define  $w_i = 1$  if  $i \in [-\frac{1}{4}, \frac{1}{4}]$  and  $w_i = 0$  otherwise. Conveniently, for this choice of  $w$ ,  $D_j$  and  $D_{j+\frac{1}{2}}$  are independent.

More important, specification (78) generally implies that dividends are positive and the joint distribution of the dividends in any  $n$  locations depends exclusively on the distances on the circle between the locations.

An agent located in location  $i$  maximizes utility over end-of-period wealth  $W_{1,i}$  net of participation costs, that is, she maximizes

$$-\frac{1}{\gamma} E_0 e^{-\gamma(W_{1,i}-F_i)},$$

where  $F_i$  refers to the participation costs incurred by the agent, depending on her participation choices. For the participation costs we adopt the same structure as in Section IV.. Specifically we assume that by paying a cost  $\kappa$ , an investor can participate not only in her location but also in the location diametrically “opposite” hers on the circle. Otherwise the investor can only invest in the risky asset in her own location. Proceeding as in Section IV., the indifference of agent  $i$  between investing exclusively in location  $i$  and incurring the cost  $\kappa$  to participate also in location  $i + \frac{1}{2}$  means

$$(79) \quad \max_{1-w_2^f} -E e^{-\gamma\left(P+\frac{1}{2}(1-w_2^f)\sum_{j=i,i+\frac{1}{2}}(D_j-P)-\kappa\right)} = \max_{1-w_1^f} -E e^{-\gamma\left(P+(1-w_1^f)(D_i-P)\right)},$$

where we have used the definition of an agent’s objective and her budget constraint. Note that  $1 - w_2^f$ , respectively  $1 - w_1^f$ , is the leverage choice of an agent who decides to invest across the two locations, respectively only in her own location.

For future reference, we provide an analytic expression for the dependence of  $P$  on  $\kappa$  in the absence of any constraint on leverage. The implicit function theorem applied to equation (79) yields<sup>10</sup>

$$(80) \quad \frac{dP}{d\kappa} = -\frac{1}{w_1^f - w_2^f} < 0,$$

Now suppose that, due to the no-recourse nature of lending contracts, borrowing is restricted so as to ensure that there is no default in equilibrium.<sup>11</sup> Thus, borrowing is subject to the constraint

$$(81) \quad X_2^S D^{\min} + (X_2^B - \kappa) \geq 0,$$

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<sup>10</sup>It is possible to show that there exist values of  $\kappa$  for which only asymmetric equilibria exist.

<sup>11</sup>Richer contracts, through which the borrower and lender share some risk, can be envisaged. Note, however, that such a contract would allow an agent (partial) diversification across locations at zero cost, and thus run counter to the central friction of the paper.

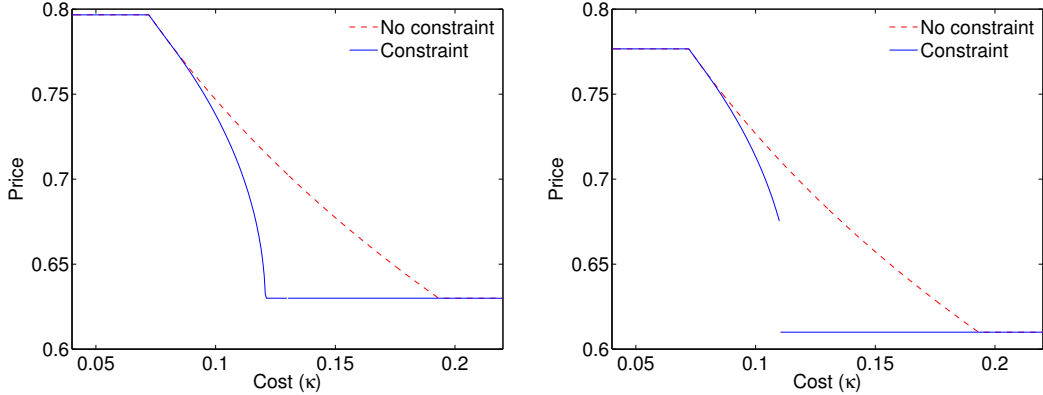


Figure 2: The figure illustrates the higher sensitivity of the price to the diversification cost in the presence of the constraint. In the left panel, the price decreases continuously to the value obtaining with no diversification. In the right panel, the price jumps to this value when  $\kappa = \mu$ . In either case, the slope of the solid line (the price in the presence of a leverage constraint) exhibits a steeper decline than the dotted line (the price in the absence of a leverage constraint). The common parameters used here are  $k = 20$ ,  $v = 10$ , and  $\gamma = 10$ ; in the left panel  $\mu = 0.13$ , while in the right one  $\mu = 0.11$ .

where  $D^{\min}$  is the smallest possible dividend in period 1,  $X_2^S$  is the number of shares chosen by investor  $a$ , and  $-X_2^B$  is the amount borrowed by the investor. Using the time-zero budget constraint and noting that  $D^{\min} = \mu$  and  $X_2^S - 1 = -w_2^f$ , equation (81) becomes

$$(82) \quad -w_2^f(P - \mu) \leq \mu - \kappa.$$

Attaching a Lagrange  $\lambda$  to the constraint (82) we obtain

$$(83) \quad \frac{dP}{d\kappa} = -\frac{1 + \lambda}{w_2^f - (1 + \lambda)w_1^f} < -\frac{1}{w_2^f - w_1^f} < 0.$$

Equation (83) shows an amplification effect: In the presence of the constraint small changes in  $\kappa$  translate into larger drops in the price than would obtain in its absence. This stronger reaction of the price in the presence of the constraint is illustrated in Figure 2, and is analogous to Figure 8, where the constraint is modeled as a collateral constraint. The intuition for the increased price sensitivity, captured by (83), is quite immediate: An increase in  $\kappa$  not only requires the price to decrease in order to avoid a decrease in  $V_2$  relative to  $V_1$  (the effect behind equation (80)); it also pushes the price down to counteract the direct effect of tightening of the constraint.

Depending on the parameters, the price may even drop discontinuously to the value obtaining in the no-diversification equilibrium. The point is made most starkly in the case  $\kappa = 0$  and  $\mu = 0$ . At these values there is diversification, and no leverage. Any increase in  $\kappa$ , on the other hand, drives the price discontinuously down to the no-diversification value. More generally, it can happen that, as  $\kappa$  approaches  $\mu$ , enough agents continue to diversify — even if their leverage is virtually zero — for the price to be above the no-diversification value that obtains when  $\kappa > \mu$ . However, once  $\kappa$  exceeds  $\mu$  the price drops discontinuously, as the right panel of Figure 2 illustrates. Thus, as in Section IV., a small change in  $\kappa$  can cause the nature of the equilibrium to change, which induces a discontinuous change in the price.

To conclude, even if one modeled borrowing limitations as resulting from a no-default requirement, the price function is steeper in  $\kappa$  than when the constraint is absent, and can even be discontinuous, similar to section IV., where the constraint is modeled as a simple leverage constraint.

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