Online Appendix of Banking, Liquidity and Bank Runs in an Infinite Horizon Economy

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This online appendix describes our computation procedure.

Here we describe how we compute impulse responses to shocks to Z_t, when bank runs may occur. We assume the shock comes in the first period and then Z_t obeys a deterministic path back to steady state, following the first order process given by equation (A2). Accordingly, our computational procedures boils down to computing nonlinear perfect foresight paths that allow for sunspot equilibria to arise.

We describe our procedure for the case where a single bank run occurs before the economy returns to steady state, though it is straightforward to generalize to the case of multiple bank runs. In particular, suppose the economy starts in a no-bank run equilibrium in the steady state and then is hit with a negative shock to Z_t at $t = 1$. It stays in the no-bank run equilibrium until t^* when a bank run occurs, assuming the condition (28) for a bank run equilibrium is met. After the bank run it then returns to the no-bank run equilibrium until it converges back to the steady state. Suppose further that after T periods from the initial shock (either productivity shock or sunspot shock) the economy is back to steady state. Let $\{Z_t\}_{t=1}^{T}$ be the exogenous path of Z_t over this period and let $\{X_t\}_{t=1}^{t^*+T}$ be the path of the vector of endogenous variables X_t.

Then there are three steps to computing the response of the economy to the run experiment, which involve working backwards: First, one needs to calculate the saddle path of the economy from the period after the run happens back to its steady state, i.e. $\{X_t\}_{t=t^*+1}^{t^*+T}$. Second, given $\{X_t\}_{t=t^*+1}^{t^*+T}$, and given that a run occurs at t^*, one can then compute X_{t^*}, the values of the endogenous variables at the time of the run. Third, one needs to compute the saddle path of the economy starting from the initial shock in $t = 1$ back to the steady state and then select the first $t^* - 1$ elements to obtain $\{X_t\}_{t=1}^{t^*-1}$. (The elements of this saddle path from from t^* to T can be ignored since the run happens at t^*.)

What aids in the computation of the three pieces of the impulse is that we know the initial value of the endogenous state for each piece. For the initial piece, the endogenous state begins at its steady state value. For the second piece, the bank run at date t, $N_t = 0$. For the final piece, which begins the period after the run, N_t depends on the endowment of entering bankers and $K_{t-1}^h = 1$ and $D_{t-1} = 0$.

The details of the algorithm will be different depending on whether the run is anticipated or not. Below we explain the implementation of the algorithm in these two cases.

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UNANTICIPATED RUN

1) Compute $\{X_t\}_{t=t^*+1}^{t^*+T}$

Let T be a time after which the system is assumed to be back in steady state. Let $\{X_t\}_{t=t^*+1}^{t^*+T+1}$ be the solution of the system given by the equilibrium equations at each $t = t^* + 1, ..., t^* + T$

$$C_t^h + \frac{(1-\sigma)}{\sigma}(N_t - W) + \frac{\alpha}{2}\left(K_t^h\right)^2 = Z_t + Z_t W^h + W$$

$$Q_t + \alpha K_t^h = \beta\left[\frac{C_t^h}{C_{t+1}^h}(Z_{t+1} + Q_{t+1})\right]$$

$$1 = \beta\left[\frac{C_t^h}{C_{t+1}^h} R_{t+1}\right]$$

$$Q_t\left(1 - K_t^h\right) = \phi_t N_t$$

$$\theta\phi_t = \beta\left(1 - \sigma + \sigma\theta\phi_{t+1}\right)\left[\phi_t\left(\frac{(Z_{t+1} + Q_{t+1})}{Q_t} - R_{t+1}\right) + R_{t+1}\right]$$

$$Q_t\left(1 - K_t^h\right) = N_t + D_t$$

$$N_t = \sigma\left[(Z_t + Q_t)\left(1 - K_{t-1}^h\right) - D_{t-1}R_t\right] + W + \varepsilon_t$$

where $\varepsilon_t = 0$ for $t \neq t^*$ and $\varepsilon_{t^*+1} = \sigma W$ (which ensures that at $N_{t^*+1} = (1+\sigma)W$).

In addition we have the terminal condition $X_{t^*+T+1} = X^{SS}$ as well as the initial conditions for the state given by $K_{t^*}^h = 1$ and $D_{t^*} = 0$.

2) Compute X_{t^*} from

$$C_{t^*}^h + \frac{\alpha}{2} = Z_{t^*} + Z_{t^*} W^h$$

$$Q_{t^*} + \alpha = \beta\left[\frac{C_{t^*}^h}{C_{t^*+1}^h}(Z_{t^*+1} + Q_{t^*+1})\right]$$

$$1 = \beta\left[\frac{C_{t^*}^h}{C_{t^*+1}^h} R_{t^*+1}\right]$$

$$K_{t^*}^h = 1$$

$$N_{t^*} = 0$$

$$D_{t^*} = 0.$$

3) Compute $\{X_t\}_{t=1}^{t^*-1}$

Let $\left\{X'_t\right\}_{t=1}^{T+1}$ be the solution of

$$C_t^h + \frac{(1-\sigma)}{\sigma}(N_t - W) + \frac{\alpha}{2}\left(K_t^h\right)^2 = Z_t + Z_t W^h + W$$

$$Q_t + \alpha K_t^h = \beta\left[\frac{C_t^h}{C_{t+1}^h}(Z_{t+1} + Q_{t+1})\right]$$

$$1 = \beta\left[\frac{C_t^h}{C_{t+1}^h} R_{t+1}\right]$$

$$Q_t\left(1 - K_t^h\right) = \phi_t N_t$$

$$\theta\phi_t = \beta\left(1 - \sigma + \sigma\theta\phi_{t+1}\right)\left[\phi_t\left(\frac{(Z_{t+1} + Q_{t+1})}{Q_t} - R_{t+1}\right) + R_{t+1}\right]$$

$$Q_t\left(1 - K_t^h\right) = N_t + D_t$$

$$N_t = \sigma\left[(Z_t + Q_t)\left(1 - K_{t-1}^h\right) - D_{t-1}R_t\right] + W$$

$$X'_{T+1} = X^{SS}$$

given initial conditions for the state given by $K_0^h = K^{h,SS}$, $D_0 = D^{SS}$ and $R_1 = R^{SS}$. Then $\{X_t\}_{t=1}^{t^*-1} = \left\{X'_t\right\}_{t=1}^{t^*-1}$. Note that the run never occurs in the neighborhood of the steady state and by assumption the run occurs at most once in our example. Thus we restrict the attention to $t^* + T \leq 2T$.

Anticipated Run

We now allow for an endogenously determined probability of a run p_t, as described in the text. This means allowing for an additional equation for p_t. In addition, in order to perform steps 1 and 3 in this case, we need to compute the values households' consumption and asset prices that would materialize if the run happened at each time t. This is because when there is a probability of a run, consumption and asset prices depend on what is expected to happen if a run actually occurs in the subsequent period.

First note that the endogenous state variables at $t+1$ are reset fresh when a run occurs at date t as

$$K_t^h = 1,\ D_t = 0,\ \text{and } N_{t+1} = (1+\sigma)W^b.$$

The only other state variable is the exogenous aggregate productivity Z_{t+1}. (Our economy has endogenous "amnesia" after the run.) Hence, we can always compute the saddle path of the economy back to steady state after a run occurs at t by initializing the endogenous state as above and picking the appropriate path for the exogenous state, i.e. $(Z_s)_{s=t+1}^{t+T}$. We denote the endogenous variable vector at date s when a run occurs at t along such path as $\left\{{}_tX_s^{**}\right\}_{s=t+1}^{t+T}$.

Secondly, given the saddle path after the date-t bank run, we can the compute asset price and household consumption when a run occurs at date t as a function of Z_t as

$$Q_t^* = Q^*(Z_t) \text{ and } C_t^{h*} = C^{h*}(Z_t).$$

Let the endogenous variable vector starting at that the time of run t until $t + T$ be $\left\{{}_t X_s^{**}\right\}_{s=t}^{t+T}$. Note that all the subsequent endogenous variable vectors are only a function of Z_s, the date of the last run t and the present date s.

Basic Step We assume that at $T + 1$ the productivity is back in steady state and $Z_t = Z^{SS}$ for $t \geq T + 1$. Then we learn that the asset price and household consumption when the run occurs at date $t \geq T + 1$ are given as

$$Q_t^* = Q^*(Z^{SS}),\ C_t^{h*} = C^{h*}\left(Z^{SS}\right), \text{ for all } t \geq T + 1.$$

We also learn all the subsequent endogenous variable vectors only depend upon the time since the last run:

$$\left\{{}_J X_t^{**}\right\}_{t=J}^{J+T} = \left\{{}_{J+1} X_t^{**}\right\}_{t=J+1}^{J+1+T} \equiv \{\chi^{**}(t-J)\}_{t-J=0}^{T}$$

for $J \geq T + 1$.

We can now compute $Q^*\left(Z^{SS}\right)$, $C^{h*}\left(Z^{SS}\right)$ and $\left\{{}_J X_t^{**}\right\}_{t=J}^{J+T} = \{\chi^{**}(t-J)\}_{t-J=0}^{T}$ for $J \geq T + 1$. We focus on parametrizations such that a run is not possible in steady state, although the technique, with minor modifications, is easily applicable to cases in which a run can occur also in steady state.

$Q^*\left(Z^{SS}\right)$, $C_h^*\left(Z^{SS}\right)$ and $\left\{{}_J X_t^{**}\right\}_{t=J+1}^{J+T}$ are the solution of the following system of equations at each $t = J + 1, ..., J + T$.

$$p_t = 1 - \min\left\{\frac{\left(Z^{SS} + Q^*\left(Z^{SS}\right)\right)\left(1 - K_t^h\right)}{\bar{R}_{t+1} D_t}; 1\right\}$$

$$C_t^h + \frac{(1-\sigma)}{\sigma}\left(N_t - W\right) + \frac{\alpha}{2}\left(K_t^h\right)^2 = Z^{SS} + Z^{SS} W^h + W$$

$$Q_t + \alpha K_t^h = \beta\left[(1-p_t)\frac{C_t^h}{C_{t+1}^h}\left(Z^{SS} + Q_{t+1}\right) + p_t \frac{C_t^h}{C^{h*}\left(Z^{SS}\right)}\left(Z^{SS} + Q^*\left(Z^{SS}\right)\right)\right]$$

$$1 = \beta \bar{R}_{t+1}\left[(1-p_t)\frac{C_t^h}{C_{t+1}^h} + p_t \frac{C_t^h}{C^{h*}\left(Z^{SS}\right)}\min\left\{\frac{\left(Z_{t+1} + Q^*\left(Z^{SS}\right)\right)\left(1 - K_t^h\right)}{\bar{R}_{t+1} D_t}; 1\right\}\right]$$

$$Q_t\left(1 - K_t^h\right) = \phi_t N_t$$

$$\theta\phi_t = (1-p_t)\beta\left(1 - \sigma + \sigma\theta\phi_{t+1}\right)\left[\phi_t\left(\frac{\left(Z_{t+1} + Q_{t+1}\right)}{Q_t} - \bar{R}_{t+1}\right) + \bar{R}_{t+1}\right]$$

$$Q_t\left(1 - K_t^h\right) = N_t + D_t$$

$$N_t = \sigma\left[\left(Z_t + Q_t\right)\left(1 - K_t^h\right) - D_{t-1}\bar{R}_t\right] + W + \varepsilon_t$$

$$C^{h*}\left(Z^{SS}\right)+\frac{\alpha}{2}=Z^{SS}+Z^{SS}W^{h}$$
$$Q^{*}\left(Z^{SS}\right)+\alpha=\beta\left[\frac{C^{h*}\left(Z^{SS}\right)}{C_{J+1}^{h}(Z^{SS})}\left(Z^{SS}+Q_{J+1}(Z^{SS})\right)\right],$$

and the terminal condition

$${}_{J}X_{J+T+1}^{**}=X^{SS},$$

given initial conditions for the state given by $K_J^h=1$ and $D_J=0$. The variable $\varepsilon_t=0$ for $t\neq J+1$ and $\varepsilon_{J+1}=\sigma W$.

Inductive Step: From the Basic Step, we have $Q^{*}\left(Z^{SS}\right)$, $C^{h*}\left(Z^{SS}\right)$ and $\left\{{}_{J}X_t^{**}\right\}_{t=J}^{J+T}$ for $J\geq T+1$. We find the endogenous variables after a bank run by solving inductively for $J=T,T-1,...,1$. Given $\left\{Q^{*}\left(Z_t\right)\right\}_{t=J+1}^{J+T+1}$, $\left\{C^{h*}\left(Z_t\right)\right\}_{t=J+1}^{J+T+1}$ and $\left\{{}_{J+1}X_t^{**}\right\}_{t=J+1}^{J+T}$, we find $\left\{Q^{*}\left(Z_t\right)\right\}_{t=J}^{J+T}$,$\left\{C^{h*}\left(Z_t\right)\right\}_{t=J}^{J+T}$ and $\left\{{}_{J}X_t^{**}\right\}_{t=J}^{J+T-1}$, for $J=1,2,...,T$.

Let $\left\{{}_{J}X_t^{**}\right\}_{t=J+1}^{J+T+1}$ be the solution of the system given by the equilibrium equations at each $t=J+1,...,J+T+1$

$$p_t=1-\min\left\{\frac{\left(Z_{t+1}+Q^{*}\left(Z_{t+1}\right)\right)\left(1-K_t^h\right)}{\bar{R}_{t+1}D_t};1\right\}$$

$$C_t^h+\frac{(1-\sigma)}{\sigma}\left(N_t-W\right)+\frac{\alpha}{2}\left(K_t^h\right)^2=Z_t+Z_tW^h+W$$
$$Q_t+\alpha K_t^h=\beta\left[(1-p_t)\frac{C_t^h}{C_{t+1}^h}\left(Z_{t+1}+Q_{t+1}\right)+p_t\frac{C_t^h}{C^{*}\left(Z_{t+1}\right)}\left(Z_{t+1}+Q^{*}\left(Z_{t+1}\right)\right)\right]$$
$$1=\beta\bar{R}_{t+1}\left[(1-p_t)\frac{C_t^h}{C_{t+1}^h}+p_t\frac{C_t^h}{C_h^{*}\left(Z_{t+1}\right)}\min\left\{\frac{\left(Z_{t+1}+Q^{*}\left(Z_{t+1}\right)\right)\left(1-K_t^h\right)}{\bar{R}_{t+1}D_t};1\right\}\right]$$
$$Q_t\left(1-K_t^h\right)=\phi_tN_t$$
$$\theta\phi_t=(1-p_t)\beta\left(1-\sigma+\sigma\theta\phi_{t+1}\right)\left[\phi_t\left(\frac{\left(Z_{t+1}+Q_{t+1}\right)}{Q_t}-\bar{R}_{t+1}\right)+\bar{R}_{t+1}\right]$$
$$Q_t\left(1-K_t^h\right)=N_t+D_t$$
$$N_t=\sigma\left[\left(Z_t+Q_t\right)\left(1-K_t^h\right)-D_{t-1}\bar{R}_t\right]+W+\varepsilon_t$$

and the terminal condition

$${}_{J}X_{J+T+1}=X^{SS}$$

given initial conditions for the state given by $K_J^h=1$ and $D_J=0$. The variable $\varepsilon_t=0$ for $t\neq J+1$ and $\varepsilon_{J+1}=\sigma W$. Here $C^{h*}\left(Z_{t+1}\right)$ and $Q^{*}\left(Z_{t+1}\right)$ are the elements of ${}_{J+1}X_{J+1}^{**}$ in the previous iteration.

Find $Q^{*}\left(Z_J\right)$ and $C^{h*}\left(Z_J\right)$ from

$$C^{h*}\left(Z_J\right)+\frac{\alpha}{2}=Z_J+Z_JW^h$$

$$Q^{*}\left(Z_{J}\right)+\alpha=\beta\left[\frac{C^{h*}\left(Z_{J}\right)}{C^{h**}\left(Z_{J+1}\right)}\left(Z_{J+1}+Q^{**}\left(Z_{J+1}\right)\right)\right]$$

where $C^{h**}\left(Z_{J+1}\right)$ and $Q^{**}\left(Z_{J+1}\right)$ are elements of ${}_{J}X_{J+1}^{**}$. Use $Q^{*}\left(Z_{J}\right)$ and $C^{h*}\left(Z_{J}\right)$ and the other obvious values to form ${}_{J}X_{J}^{**}$.

This procedure yields $\left\{Q^{*}\left(Z_{t}\right)\right\}_{t=1}^{T}$ $\left\{C^{h*}\left(Z_{t}\right)\right\}_{t=1}^{T}$ and $\left\{\left\{{}_{J}X_{t}^{**}\right\}_{t=J}^{J+T}\right\}_{J=1}^{T}$. Given these we have $\left\{X_{t}\right\}_{t=t^{*}}^{T}=\left\{{}_{t^{*}}X_{t}^{**}\right\}_{t=t^{*}}^{T}$ when a run actually occurs at t^{*}, and step 3, appropriately modified in order to account for the endogenous probability of a run, yields $\left\{X_{t}\right\}_{t=1}^{t^{*}}$.