For Online Publication: Appendix for Government Policy with Time Inconsistent Voters

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April 30, 2015

Abstract

In this supplementary appendix, we study various extensions of the model in the paper. More specifically, in Section A we allow for election in the first period; in Section B we study the setting with arbitrary number of periods and convex distortions discussed in the body of the paper; in Section C we analyze a model analogous to the one presented in the paper in which agents are characterized by Gul and Pesendorfer (2004) preferences; in Section D we allow for consumption in period 1, in addition to periods 2 and 3.

A. Period 1 Elections

The model studied in the paper allowed for government actions and elections in periods 2 and 3. We now extend the model to consider elections in period 1 as well.

The objective of this extension is to evaluate whether collective action in period 1 could effectively satisfy the demand for commitment agents display in period 1 or at least limit the distortions associated with debt accumulation in period 2. It turns out, however, that the equilibrium consumption sequence and the total amount of resources destroyed is completely unaffected by period 1 elections. The only thing that changes is that there is a multiplicity of equilibria determining the timing of distortions.

The economic environment is the same as the one assumed in the previous sections. There are two candidates running for office, both in period 1 and in period 2. The candidates are office motivated. The policy space is extended to allow candidates to offer a transfer y_1 and a lump-sum tax t_1 in period 1, as well as a transfer y_2 and tax t_2 in period 2 (elections in period 3 are redundant as before). Debt financing is allowed. Tax collection in any period carries distortions of a unit loss $\eta > 0$ for every unit collected. For this robustness check, we focus on the case of high debt limit, $\overline{d} \geq d^{**}$.

By taxing themselves in period 1 and investing the proceeds in the liquid asset agents can effectively commit resources for consumption in period 2 and hence reduce debt accumulation. On the other hand, if the proceeds of taxes carried to period 2 are smaller than d^{**}, in per-capita terms, a strict majority of agents in period 2 will support a positive debt level so as to increase consumption in period 2. Let $t = t_1 - y_1$ denote per-capita taxes in period 1 and $d = y_2 - t_2$ denote debt in period 2. It turns out that even though by taxing themselves in period 1 agents can indeed limit debt accumulation in period 2, this strategy simply shifts some of the repayment of debt from period 3 to period 1, but does not alter total distortions and has no ultimate effects on consumption profiles.

Proposition A. 1 *In the economy with consumption and elections in every period and with high debt limit, $\overline{d} \geq d^{**}$, the set of equilibria is characterized by pairs of period 1 taxes and period 2 debt of the form (t, d) such that $t \in [0, d^{**}]$ and $d = d^{**} - t$; total agents' distortions and consumption profiles are unchanged relative to the case in which elections take place only in period 2.*[1]

It is easy to show that debt limits would be the only way to reduce distortions even in the model with period 1 elections. The reason is that any limit on period 1 surpluses would

[1] Only if the distortion on taxes at $t = 1$ were smaller than the distortion on taxes at $t = 3$ election in period 1 would help reducing debt in period 2 and hence distortions in period 3.

just shift the financing to higher debt in the second period. It is also easy to show that in the case of convex distortions the indeterminacy would disappear and period 1 elections would have no effect whatsoever.

Of course, the debt limit may be endogenously determined via voting. Suppose, for instance, that in period 1 agents vote on the *debt limit* that would affect the debt imposed in period 2 as in the model studied thus far. In such a setting, all agents would favor low debt limits in period 1. In fact, since illiquid assets allow agents to commit without experiencing the loss of wealth that results from distortionary debt, equilibrium would entail a debt limit fixed at zero. Of course, if agents could vote again on the debt limit in period 2 (prior to determining the debt level itself, as in the model studied thus far), they would collectively choose a positive debt limit and consumption would be distorted (relative to the commitment paths). This suggests the importance of timing in constitutional reform. Since most amendments take a substantial amount of time to pass, changes in debt limits are likely to occur a significant time prior to the 'temptation' of consumption. Even if multiple elections occurred over such amendments, it would be difficult to achieve a super-majority to agree *over time* on an increase on the debt limit itself (as pointed above, early in the process, one would expect voters to reject debt limit increases).

B. Arbitrary Number of Periods and Convex Distortions

We now study an economy that lasts for an arbitrary number of period T. We later consider the limit case as $T \to \infty$. In this section, we allow for consumption in period 1 as this in fact simplifies the notation in this case. For simplicity, in this section we assume that agents' preferences also satisfy certain Inada conditions, namely we assume that $\lim_{c \to 0} u'(c) = \infty$. The analysis illustrates the robustness of the main messages of the paper when there is repeated feedback from voters, be it through elections per-se, or via other channels such as electoral polls.

As in the analysis of the previous sections, we assume that agents can choose to invest in liquid or illiquid assets and that these assets have equal zero interest rate. An illiquid asset with maturity m, acquired in period t, pays off in period $t + m$ and cannot be sold before then. To isolate the effects of the interaction of time-inconsistency and fiscal policy on debt accumulation as in the previous sections, we make the strong assumption that in any period $t = 1, ..., T - 2$ illiquid assets are available with any maturity m between 2 and $T - t$. A liquid asset has maturity 1. We assume that liquid assets are available in any period $t = 1, ..., T - 1$. Absent government intervention, by appropriate choice of the mix of liquid and illiquid assets with different maturities, an agent can commit to any desired consumption stream.

Elections occur in any period $t \geq 2$. Period T elections are vacuous and period 1 elections can be shown to be irrelevant (when distortions are convex). Let D_t denote accumulated debt at t, while d_t denotes the deficit at time t.

Consider first the economy with linear distortions η such that $\beta(1+\eta) < 1$. It is easy to extend the analysis of Section 4.2 to construct an equilibrium in which debt is accumulated until period $T-1$ and is repaid completely only in the last period, time T. Furthermore, at each time $2 \leq t \leq T-1$ agents consume exclusively off of debt; whereas at time T, agents consume off of time 1 savings:

$$c_t = d_t, \quad \text{for any } 2 \leq t < T; \;\; c_T = s_{1T} - (1+\eta)D_{T-1}.$$

With linear distortions agents have no incentive to smooth debt repayment and the repayment is thus concentrated at time T.[2] Debt then explodes as the number of periods increases. It is clear, however, that the linearity of distortions plays a fundamental role in this construction. We show next that even when distortions are strictly convex, so that there is a motive to smooth repayments over time, debt accumulation can be large when voters are time inconsistent and the political system does not impose debt limits.

To clarify notation, though somewhat redundantly, we make explicit the distinction between deficit and repayment and let $d_t \geq 0$ and $q_t \geq 0$ denote, respectively deficits and repayments at time t. Recall that D_t denotes debt accumulated up to (and including) time t: $D_t = \sum_{\tau=2}^{t} d_t - q_t$. We assume that tax distortions $\eta(q)$ are smooth, non-negative, strictly increasing and strictly convex in q:

$\eta(q)$ is a twice continuously differentiable function which satisfies:

$$\eta(q) > 0, \eta'(q) > 0, \eta''(q) > 0, \text{ for } q > 0\text{; and } \eta(0) = \eta'(0) = 0 \tag{1}$$

The total cost of repayment q, defined as $A(q) = q\,(1+\eta(q))$, is then also increasing and convex, strictly for any $q > 0$, with $A(0) = 0$ and $A'(0) = 1$.

As in the economy studied in the previous sections, at equilibrium deficits and repayments are the outcome of the electoral process, while investments in the liquid and illiquid assets available in financial markets are derived from individual choices, taking as given election outcome.

[2]Note that in our formulation distortions are incurred only when debt is repaid, independently of how far in the future it is in fact repayed (recall returns are zero). This is intended to represent an environment in which debt is repaid by means of distortionary taxation.

At time $t = 1$ agents invest in liquid and illiquid assets, determining a sequence of savings in illiquid assets s_{1t} for any time $t > 1$. Agents can in principle rebalance their asset portfolio at any time $t > 1$. But since a full set of illiquid assets are available in financial markets, agents can effectively implement commitment strategies and hence the option to rebalance investment portfolios in the future has no effect on the equilibrium consumption sequence nor on the equilibrium of the electoral process.[3] Therefore, for any given deficit and repayment sequence $\{d_t, q_t\}_{t=2}^{T}$, the investment problem of any agent at time $t = 1$ involves the choice of the sequence of period 1 savings in illiquid assets $\{s_{1t} \geq 0\}_{t=1}^{T}$ corresponding to the maximization problem:

$$\begin{array}{l} \max\ u(s_{11}) + \beta \sum_{t=2}^{T}\ u(s_{1t} + d_t - A\left(q_t\right)) \\ s.t.\ \ \sum_{t=1}^{T}\ s_{1t} = k \end{array} \tag{2}$$

The political economy problem at any election at time $t \geq 2$ involves two candidates running for office choosing electoral platforms to maximize the probability of elections. As in Section 3, however, the strategic interaction between the candidates is reduced at equilibrium to the solution of a single choice problem at any election time t. In the economy with an arbitrary but finite number of periods T, by backward induction, this problem can be reduced to the choice of maps $d_t(D_{t-1}),\ q_t(D_{t-1}) \geq 0$, for given time 1 transfers $\{s_{1\tau}\}_{\tau=t}^{T}$ and given expected future maps $d_\tau(D_{\tau-1}), q_\tau(D_{\tau-1})$ for all $t+1 \leq \tau \leq T$ to

$$\begin{array}{l} \max\ u\left(s_{1t} + d_t(D_{t-1}) - A\left(q_\tau(D_{\tau-1})\right)\right) + \beta \sum_{\tau=t+1}^{T}\ u\left(s_{1\tau} + d_\tau(D_{\tau-1}) - A\left(q_\tau(D_{\tau-1})\right)\right) \\ s.t.\ \ \sum_{t=2}^{T}\ d_t - q_t = 0. \end{array} \tag{3}$$

Of course, deficits and repayments will not both be positive at the same time t: $d_t \cdot q_t = 0$.

We are now ready to characterize equilibrium debt accumulation and repayment. To better illustrate the structure of equilibrium it is convenient to re-consider first the case in which $T = 3$ and construct the equilibrium for the case of convex distortions. The $T = 3$ economy is special in that debt accumulation necessarily occurs in period 2 and repayment is concentrated in the last period $T = 3$. The first order condition of the political economy problem (the commitment constraint from our analysis of linear distortions) is given by:

$$u'\left(d_2 + s_{12}\right) = \beta A'(q_3) u'\left(s_{13} - A(q_3)\right).$$

In contrast with the model with linear distortions, however, the equilibrium level of debt need not be determined by a corner solution for consumption. At an interior solution, positive

[3]Note however, that the same consumption pattern could in principle be obtained with different transfer sequences, if portfolio rebalancing after period 1 were allowed.

savings s_{12} and s_{13} are chosen to smooth consumption so that $u'(d_2 + s_{12}) = u'(s_{13} - A(q_3))$ and hence

$$\beta A'(q_3) = 1 \tag{4}$$

and $c_2 = c_3$. When, on the other hand, a corner solution obtains with $s_{12} = 0$ (s_{13} is always positive by Inada conditions), $u'(d_2) < u'(s_{13} - A(q_3))$ and $c_2 = d_2 > c_3$. In this case, the political economy conditions imply that $\beta A'(q) < 1$. Corner solutions obtain when the $A'(q)$ does not grow sufficiently quickly, given the size of the debt, to guarantee that $\beta A'(q) = 1$. Alternatively, these corners arise when the size of the economy, measured by the total endowment k, is not large enough to ensure that debt and q are sufficiently large to guarantee that $\beta A'(q) = 1$. This is also the case in general, for $T > 3$: when k is relatively small with respect to q,the economy behaves effectively like the one with linear distortions and $\beta(1 + \eta) < 1$: it displays corners of debt accumulation until $t = T - 1$, with repayment concentrated in the last period, at T.[4]

From now on we restrict ourselves to the more interesting case in which, fixing the function $A(q)$, the total endowment k is sufficiently "large." In this case, when $T > 3$, we show that the dynamics of fiscal policy is characterized by two distinct phases: debt is accumulated first and then repaid. It still turns out that in the debt accumulation phase agents are at a corner in the sense that they consume exclusively off of government spending, as in the economy with linear distortions. However, debt repayment is smoothed over time and the equilibrium is interior during the repayment phase.

Proposition B. 1 *In the economy with $T > 3$ the equilibrium consumption sequence has the following properties: there exists a $\tilde{t} \geq 2$ such that: for $t \leq \tilde{t}$ the government accumulates debt; for $t > \tilde{t}$ the government gradually repays the debt. Furthermore:*

1. *In the repayment phase, for $t > \tilde{t}$, the equilibrium is interior and $c_t = s_{1t} - A(q_t)$.*

2. *Up to the last period of the debt accumulation phase, for $2 \leq t \leq \tilde{t} - 1$, agents consume exclusively off of deficit-financed spending: $s_{1t} = 0, c_t = d_t$;[5] in contrast, in the last period accumulation period ($t = \tilde{t}$) savings and debt are both positive $c_{\tilde{t}} = s_{1\tilde{t}} + d_{\tilde{t}}$, with $s_{1t} > 0$.*

Proof. The proof proceeds as follows. First of all we derive first order conditions of the two maximization problems discussed in the text:

[4]This is formally shown as a by-product of the proof of Proposition 8 in the Online Appendix.

[5]This statement is empty if $\tilde{t} = 2$. However, for T and/or k sufficiently large we must have $\tilde{t} > 2$.

- The investment problem of any agent at time $t = 1$ choosing the sequence of period 1 savings in illiquid assets $\{s_{1t} \geq 0\}_{t=1}^{T}$, for any given deficit and repayment sequence $\{d_t, q_t\}_{t=2}^{T}$:

$$\begin{array}{ll} \max & u(s_{11}) + \beta \sum_{t=2}^{T} \ u(s_{1t} + d_t - A\left(q_t\right)) \\ s.t. & \sum_{t=1}^{T} \ s_{1t} = k; \end{array} \tag{I}$$

- The political economy problem at any election at time $t \geq 2$, reduced to the choice of maps $d_t(D_{t-1}),\ q_t(D_{t-1}) \geq 0$, for given time 1 transfers $\{s_{1\tau}\}_{\tau=t}^{T}$ and given expected future maps $d_\tau(D_{\tau-1}), q_\tau(D_{\tau-1})$ for all $t+1 \leq \tau \leq T$:

$$\begin{array}{ll} \max & u\left(s_{1t} + d_t(D_{t-1}) - A\left(q_\tau(D_{\tau-1})\right)\right) + \beta \sum_{\tau=t+1}^{T} \ u\left(s_{1\tau} + d_\tau(D_{\tau-1}) - A\left(q_\tau(D_{\tau-1})\right)\right) \\ s.t. & \sum_{t=2}^{T} \ d_t - q_t = 0. \end{array} \tag{PE}$$

We then derive several implications, notably regarding the structure of the debt accumulation and repayment phases at equilibrium. To this end we exploit the condition that k is large enough, but we obtain as a by-product a characterization of the structure of equilibria when the condition is not imposed. Finally, we derive properties of the consumption sequence at equilibrium.

Recall the government's budget balance, the constraint in Problem PE is:

$$\sum_{t=2}^{T} \ d_t - q_t = 0. \tag{5}$$

By definition, $D_t = \sum_{\tau=2}^{t} \ d_\tau - q_\tau$. It then follows that (5) can also be written as $D_t + \sum_{\tau=t+1}^{T} \ d_\tau - q_\tau = 0$, for any $t \geq 2$; which in turn implies, for $t = T$, $D_T = 0$. Furthermore, using again the definition of D_t and taking derivatives, $dD_t = dd_t$ and $dD_t = -dq_t$. Let $J_q(\tau)$ (respectively $J_d(\tau)$) denote the subset of periods $j > \tau$ such that $q_j > 0$ (respectively. $d_j \geq 0$ with $q_t = 0$). Therefore, government's budget balance, (5), implies

$$\sum_{j \in J_q(\tau)} \frac{\partial q_j}{\partial D_\tau} - \sum_{j \in J_d(\tau)} \frac{\partial d_j}{\partial D_\tau} = 1. \tag{6}$$

Notice that, at equilibrium, $d_t > 0$ for some $2 \leq t \leq T$. This can be shown by contradiction and along the lines of Proposition 2. Government budget balance, equation (5), implies that $q_\tau > 0$ for some $2 \leq \tau \leq T$. Consider a period $2 \leq \tau < T$ such that $q_\tau > 0$. The first order condition of Problem PE at τ is:

$$0 = A'(q_\tau)u'(s_{1\tau} - q_\tau) - \beta \left[\sum_{j \in J_q(\tau)} A'(q_j)u'(c_j)\frac{\partial q_j}{\partial D_\tau} - \sum_{j \in J_d(\tau)} u'(c_j)\frac{\partial d_j}{\partial D_\tau} \right] \tag{7}$$

Consider instead a period $2 \leq t < T$ such that $d_t \geq 0$, with $q_t = 0$. The first order condition of Problem PE at t is:

$$0 = u'(d_t + s_{1t}) - \beta \left[\sum_{j \in J_q(t)} A'(q_j) u'(c_j) \frac{\partial q_j}{\partial D_\tau} - \sum_{j \in J_d(t)} u'(c_j) \frac{\partial d_j}{\partial D_\tau} \right] \tag{8}$$

Recall that, by the implications of (5) derived above, $\sum_{j \in J_q(t)} \frac{\partial q_j}{\partial D_\tau} - \sum_{j \in J_d(t)} \frac{\partial d_j}{\partial D_\tau} = 1$. Furthermore, it can be shown that the first order conditions of problem PE imply that

$$\frac{\partial q_j}{\partial D_\tau} > 0 \text{ and } \frac{\partial d_j}{\partial D_\tau} < 0, \quad \text{for all } j > \tau. \tag{9}$$

This is a consequence of consumption smoothing and can be formally shown by deriving envelope conditions from (8).

Notice that the solution of Problem I requires

$$u'(d_j + s_{1j}) \leq u'(s_{1j'} - q_{j'}) \text{ for all } j \in J_d(1),\ j' \in J_q(1), \tag{10}$$

with equality for all j, j' such that $s_{1j}, s_{1,j'} > 0$ (that is, when the solution of Problem I is interior). As a consequence, in particular, the solution of Problem I requires that $u'(c_j)$ be constant for all j such that $q_j > 0$, that is for $j \in J_q(1)$. This is so because, by Inada conditions, $q_j > 0$ implies $s_{1j} > 0$.

Conditions (7) and (8) allow us to characterize the structure of the debt accumulation and repayment phases at equilibrium. We show that i) $q_T > 0$ and that ii) $q_\tau > 0$ implies that $q_j > 0$ for all $j > \tau$. To prove *i)* we proceed by contradiction, postulating that $d_T \geq 0$ with $q_T = 0$. Consider first the case in which $q_{T-1} > 0$. Then 7) implies

$$A'(q_{T-1}) u'(c_{T-1}) = \beta u'(d_T + s_{1T})$$

But $q_{T-1} > 0$ implies that $A'(q_{T-1}) > 1$, while $\beta < 1$. As a consequence, the condition cannot be satisfied as it requires $u'(c_{T-1} < u'(d_T + s_{1T})$, which is in contradiction with (10) and hence with the solution of Problem I. The same logic applies to any candidate equilibrium characterized by an uninterrupted sequence of $d_j \geq 0$, from some t up to T and $q_{t-1} > 0$. We conclude $q_T > 0$. The proof of ii) also runs by contradiction, postulating that $q_\tau > 0$ and $d_j \geq 0$ with $q_j = 0$, for some $j > \tau$ (recall that $d_t q_t = 0$ and hence $d_t > 0$ implies $q_t = 0$). Consider first the case that $q_{T-2} > 0$, and $d_{T-1} \geq 0$ with $q_{T-1} = 0$. Recall we have just shown that $q_T > 0$. Then (7) implies

$$\begin{aligned} A'(q_{T-2}) u'(c_{T-2}) &= \beta \left[A'(q_T) u'(c_T) \frac{\partial q_T}{\partial D_{T-2}} - u'(d_{T-1} + s_{1T-1}) \frac{\partial d_{T-1}}{\partial D_{T-2}} \right] \\ u'(d_{T-1} + s_{1T-1}) &= \beta A'(q_T) u'(c_T) \end{aligned}$$

But $q_{T-2} > 0$, $q_T > 0$ imply $A'(q_{T-2})$, $A'(q_T) > 0$ and $u'(c_{T-2}) = u'(c_T)$ as an implication of (10). Furthermore, Using (9), the first equation can then be written as $A'(q_{T-2}) = \beta \left[A'(q_T) \frac{\partial q_T}{\partial D_{T-2}} + \frac{u'(c_{T-1})}{u'(c_T)} \mid \frac{\partial d_{T-1}}{\partial D_{T-2}} \mid \right]$ and by government's budget balance, equation (5), $\frac{\partial q_T}{\partial D_{T-2}} + \mid \frac{\partial d_{T-1}}{\partial D_{T-2}} \mid = 1$. Also, $d_{T-1} \geq 0$ with $q_{T-1}) = 0$ implies $\frac{u'(c_{T-1})}{u'(c_T)} \leq 1$ by (10). As a consequence, the first equation implies $\beta A'(q_T) > 1$, which when substituted into the second requires $\frac{u'(c_{T-1})}{u'(c_T)} > 1$, a contradiction with (10). The same logic applies to any candidate equilibrium such that $q_\tau > 0$ is followed at some $t > \tau$ by $q_t = 0$ with $d_t \geq 0$. We conclude that $q_\tau > 0$ implies that $q_j > 0$ for all $j > \tau$.

We now show that, for k large enough, $\beta A'(q_T) > 1$ (recall that $q_T > 0$). Suppose on the contrary that $\beta A'(q_T) \leq 1$. Conditions (7) and (8) then imply that $d_{T-1} > 0$ and:

$$u'(d_{T-1}) = \beta A'(q_T) u'(c_T).^6 \tag{11}$$

It is straightforward to show that Problem I implies that, at equilibrium, c_T must increase without bound with the total size of the economy, k. Then, keeping $\beta A'(q_T)$ bounded above by 1, (11) implies that d_{T-1} also increases unboundedly with k. But $q_T \geq d_{T-1}$ by (5) and hence, for k large enough, it must be that $\beta A'(q_T) > 1$, the desired contradiction.

Thus, we only need to consider the case in which $\beta A'(q_T) > 1$. In this case, conditions (7) and (8) imply that $q_{T-1} > 0$. Indeed the solution of Problem PE involves then repayments $q_\tau > 0$ for any $\tau \leq T$ greater than some $\tilde{t} \geq 2$. In this case, condition (7), the first order condition of Problem PE, takes the form:

$$0 = A'(q_\tau) u'(s_{1\tau} - q_\tau) - \beta \left[\sum_{j=\tau+1}^{T} A'(q_j) u'(c_j) \frac{\partial q_j}{\partial D_\tau} \right]$$

Furthermore, (6) reduces to $\sum_{j=\tau+1}^{T} \frac{\partial q_j}{\partial D_\tau} = 1$. But, if $q_\tau > 0$ for any τ greater than some $\tilde{t} \geq 2$, the first order conditions corresponding to the agent's optimization at time $t = 1$ are interior and $s_{1j} > 0$, for any $\tau \leq j \leq T$. The implication of (10) that we derived above then implies that $u'(c_j)$ is constant for any $j \geq \tau$. As a consequence c_j as well as $c_j + \frac{\partial q_j}{\partial D_\tau}$ are constant in j and so is $\frac{\partial q_j}{\partial D_\tau}$. In particular, then, $\frac{\partial q_\tau}{\partial D_\tau} = \frac{1}{T-\tau}$. Summing up, the first order conditions of Problem PE are reduced to

$$A'(q_\tau) = \frac{\beta}{T-\tau} \left[\sum_{j=\tau+1}^{T} A'(q_j) \right]. \tag{12}$$

[6] Indeed, this argument implies that, at equilibrium, $q_t = 0$ and $d_t > 0$ for any $2 \leq t \leq T-1$: debt is accumulated until period $T-1$ and repayed at time T.

This ends our proof of part 1 in the statement of Proposition 8.

Having shown that, for k large enough, the dynamics of debt has an accumulation phase followed by a re-payment phase, and having characterized the equilibrium conditions of the repayment phase, we now study the debt accumulation phase. Let $\tilde{t}$ denote the last time τ such that deficit is strictly positive. We now show that the equilibrium condition at $\tilde{t}$ is interior and hence $u'(c_{\tilde{t}}) = u'(c_\tau)$, for any $\tilde{t} < \tau \leq T$. We proceed by contradiction. We have shown that a repayment $q_\tau > 0$ occurs when expected future marginal distortions $\frac{\beta}{T-\tau}\left[\sum_{j=\tau+1}^{T} A'(q_j)\right] > 1$. At $\tilde{t}$ then:

$$\frac{\beta}{T-\tilde{t}}\left[\sum_{j=\tilde{t}+1}^{T} A'(q_j)\right] \leq 1.$$

Assume by way of contradiction that $\frac{\beta}{T-\tilde{t}}\left[\sum_{j=\tilde{t}+1}^{T} A'(q_j)\right] < 1$. This implies $\frac{\beta}{T-\tilde{t}}A'(q_{\tilde{t}+1}) < 1 - \frac{\beta}{T-\tilde{t}}\left[\sum_{j=\tilde{t}+2}^{T} A'(q_j)\right]$. But since $q_{\tilde{t}+1} > 0$ by assumption, the first order conditions at $\tilde{t}+1$ imply $\frac{\beta}{T-\tilde{t}}\left[\sum_{j=\tilde{t}+2}^{T} A'(q_j)\right] > 1$, and hence $\frac{\beta}{T-\tilde{t}}A'(q_{\tilde{t}+1}) < 0$ which is impossible. We can conclude then that

$$\frac{\beta}{T-\tilde{t}}\left[\sum_{j=\tilde{t}+1}^{T} A'(q_j)\right] = 1 \tag{13}$$

which implies $u'(d_{\tilde{t}}) = u'(c_{\tilde{t}+1})$.[7]

We now study the debt accumulation phase up to period $\tilde{t}$. Consider the first order conditions for Problem PE at $\tilde{t}-1$, equation (8). In the accumulation phase these are reduced to:

$$0 = u'(s_{1\tilde{t}-1} + d_{\tilde{t}-1}) + \beta u'(s_{1\tilde{t}} + d_{\tilde{t}})\left[\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-1}} + \frac{\beta}{T-t}\left[\sum_{\tau=t}^{T} A'(q_\tau)\right]\frac{\partial q_\tau}{\partial D_{\tilde{t}-1}}\right].$$

But the (interior) first order conditions of the agent optimization choice at time $t=1$ implies that $c_{\tilde{t}}$ is constant and hence $D_{\tilde{t}}$ is also constant:

$$\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-1}} = -1 \text{ and } \frac{\partial q_\tau}{\partial D_{\tilde{t}-1}} = 0, \text{ for any } \tau > \tilde{t}$$

[7]In other words, if $u'(d_{\tilde{t}}) < u'(c_{\tilde{t}+1})$ then in fact the equilibrium will have an extra period of debt accumulation; that is, the last period of debt accumulation will in fact be $\tilde{t}+1$. As a consequence, note that a corner solution with $s_{1\tilde{t}} = 0$ can in fact occur in the $T=3$ economy, in which debt is necessarily accumulated at $t=2$ and there cannot be an extra period of accumulation as repayment must occur at $t=3$; see the analysis of this case in the text.

It follows then that the first order conditions in Problem I must hold at the corner $s_{\tilde{t}-1} = 0$:

$$u'(d_{\tilde{t}-1}) = \beta u'(s_{1\tilde{t}} + d_{\tilde{t}})$$

The argument can be extended backwards to imply that $c_\tau = d_\tau$, for any $2 \leq \tau \leq \tilde{t} - 1$. Consider period $\tilde{t} - 2$. Using again the fact that the (interior) first order conditions of the agent optimization choice at time $t = 1$ imply that $c_{\tilde{t}}$ is constant and hence $D_{\tilde{t}}$ is also constant, the first order condition of Problem PE are reduced to:

$$0 = u'(s_{1\tilde{t}-2} + d_{\tilde{t}-2}) + \beta \left[u'(d_{\tilde{t}-1}) \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + u'(s_{1\tilde{t}} + d_{\tilde{t}}) \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} \right]$$

Substituting $u'(d_{\tilde{t}-1}) = \beta u'(s_{1\tilde{t}} + d_{\tilde{t}})$ we have

$$0 = u'(s_{1\tilde{t}-2} + d_{\tilde{t}-2}) + \beta u'(s_{1\tilde{t}} + d_{\tilde{t}}) \left[\beta \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} \right]$$

But $\left[\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} \right] = -1$; and hence (9) implies that $\mid \left[\beta \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} \right] \mid < 1$. Then again the first order conditions of Problem I must hold at the corner $s_{\tilde{t}-2} = 0$. Furthermore, using the first order conditions we obtained at $\tilde{t} - 2$ and $\tilde{t} - 1$ it follows directly by concavity, using $\mid \left[\beta \frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} \right] \mid < 1$, that $d_{\tilde{t}-2} > d_{\tilde{t}-1}$. Proceeding recursively back in time, more generally for any $2 \leq \tilde{t} - 1$, the first order conditions will have a similar structure:

$$0 = u'(s_{1t} + d_t) + \beta u'(s_{1\tilde{t}} + d_{\tilde{t}}) \left[\sum_{j=t+1}^{\tilde{t}} \epsilon_j \frac{\partial d_j}{\partial D_t} \right] \tag{14}$$

for some $0 \leq \epsilon_j \leq 1$ such that $|: \sum_{j=t+1}^{\tilde{t}} \epsilon_j \frac{\partial d_j}{\partial D_t} :|:< 1$. As a consequence, $c_\tau = d_\tau$, for any $2 \leq \tau \leq \tilde{t} - 1$. This ends our proof of part 2 in the statement of the proposition. ∎

It may be surprising that, even in the case of convex distortions, in the accumulation phase agents consume exclusively off of deficit (the equilibrium deficit is determined by a corner condition for savings). The intuition for this result is the following: the marginal condition that characterizes the voting equilibrium at $\tilde{t}$ essentially determines the maximal level of debt $D_{\tilde{t}}$. Other things being equal, this condition trades off the marginal cost of future distortions due to an increase in debt and the marginal benefit of an increase in consumption at $\tilde{t}$. At every time $t < \tilde{t}$, therefore, an increase in debt has a positive marginal effect on current consumption without affecting the level of debt at $\tilde{t}$ (since consumption in period $\tilde{t}$ falls by the same amount), and hence without affecting the future cost of distortions at the margin. The smoothing of distortions therefore only plays a role in the repayment phase.

The following corollary characterizes the equilibrium in more detail.

Corollary B. 1 *In the economy with $T > 3$, in equilibrium, the sequences of deficits, repayments, and consumption have the following properties:*

1. *In the repayment phase, for $t > \tilde{t}$, the sequence of repayments $q_t > 0$ is strictly increasing over time and consumption c_t is constant in t.*

2. *Up to the last period of the debt accumulation phase, for $2 \leq t \leq \tilde{t} - 1$, consumption c_t and the deficit d_t are decreasing over time; in contrast, consumption in period $\tilde{t}$ is equalized to subsequent consumption $c_{\tilde{t}} = c_t$ for any $t > \tilde{t}$.*

Proof. In the proof of Proposition B.1 we have shown that the first order condition of Problem PE, in the repayment phase, are reduced to (12). We now show that $A'(q_t)$ is an increasing sequence in t. To this end it is sufficient to write (12) recursively as follows:

$$\begin{aligned}
A'(q_{T-1}) &= \beta A'(q_T), \\
A'(q_{T-2}) &= \frac{\beta}{2}(1+\beta) A'(q_T), \\
A'(q_{T-3}) &= \frac{\beta}{3}(1+\beta)\left(1+\frac{\beta}{2}\right) A'(q_T), \\
&\ldots \\
A'(q_t) &= \frac{\beta}{T-t}\prod_{j=1}^{T-t-1}\left(1+\frac{\beta}{j}\right) A'(q_T).
\end{aligned}$$

It can now be directly checked that the sequence $\frac{\beta}{T-t}\prod_{j=1}^{T-t-1}\left(1+\frac{\beta}{j}\right)$ is increasing in t for given T.

The sequence $A'(q_t)$ is then increasing in t and so is the sequence q_t, as $A(q)$ is strictly convex. Furthermore, as $q_t > 0$ in the repayment phase, s_{1t} must also be, by Inada conditions. The solution of Problem I is then interior, which implies that consumption is equalized across time. Finally, in the proof of Proposition B.1 we have shown that at $\tilde{t}$ distortions must satisfy (13) and that the solution of Problem I is interior at $\tilde{t}$. This implies that $c_{\tilde{t}} = c_{\tilde{t}+1}$. But, by our previous characterization of the repayment phase in this corollary, $c_{\tilde{t}+1} = c_\tau$, for any $\tau > \tilde{t} + 1$.

We now show that d_t is decreasing over time in the debt accumulation phase. To this end we need to show that the absolute value of the expression $\left[\sum_{j=t+1}^{\tilde{t}} \alpha_j \frac{\partial d_j}{\partial D_t}\right]$ in equation (14) is increasing in t.

We first establish that $\frac{\partial d_j}{\partial D_t}$ change by the same factor for any j when t changes:

$$\frac{\frac{\partial d_j}{\partial D_t}}{\frac{\partial d_{j'}}{\partial D_t}} = \frac{\frac{\partial d_j}{\partial D_{t'}}}{\frac{\partial d_{j'}}{\partial D_{t'}}},\ 2 < j, j' \leq \tilde{t},\ 2 \leq k < \min\{j, j'\} \tag{15}$$

Indeed, consider the first order condition at time $\tilde{t}-1$: $u'(c_{\tilde{t}-1}) = \beta u'(c_{\tilde{t}})$. Differentiating, the Envelope Theorem implies,

$$u''(c_{\tilde{t}-1})\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} = \beta u''(c_{\tilde{t}})\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}};$$

but also that

$$u''(c_{\tilde{t}-1})\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-3}} = \beta u''(c_{\tilde{t}})\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-3}}.$$

It follows that $\frac{\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}}}{\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}}} = \frac{\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-3}}}{\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-3}}}$. It is straightforward to see that in fact the argument holds for any $2 \leq k < \tilde{t}-1$. Furthermore, the same logic can be repeated on the first order condition at time $\tilde{t}-2$. In fact, after differentiating and recalling that, for any $\tau > t$, $\frac{\partial d_{tau}}{\partial D_t} < 0$ by (9), we obtain:

$$u''(c_{\tilde{t}-2})\frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-k}} = \beta u''(c_{\tilde{t}}) \mid \beta\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{t\tilde{-}1}}{\partial D_{\tilde{t}-2}} \mid \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-k}};$$

and hence $\frac{\frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-k}}}{\frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-k}}}$ is constant in k. Once again, the same argument holds for $\tilde{t}-3$, $\tilde{t}-4$ and so on backwards until period 2.

Simplify notation by letting $\mid \sum_{j=t+1}^{\tilde{t}} \epsilon_j \frac{\partial d_j}{\partial D_t} \mid$ be denoted Γ_t. Then, developing first order conditions backwards from $\tilde{t}-1$ we have:

$$\begin{aligned}
\Gamma_{\tilde{t}-1} &= \beta \\
\Gamma_{\tilde{t}-2} &= \mid \beta\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} \mid \\
\Gamma_{\tilde{t}-3} &= \mid \beta\Gamma_{\tilde{t}-2}\frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-3}} + \beta\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-3}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-3}} \mid \\
\Gamma_{\tilde{t}-4} &= \mid \beta\Gamma_{\tilde{t}-3}\frac{\partial d_{\tilde{t}-3}}{\partial D_{\tilde{t}-4}} + \Gamma_{\tilde{t}-2}\frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-3}} + \Gamma_{\tilde{t}-2}\frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-4}} + \beta\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-4}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-4}} \mid \\
&\quad .....
\end{aligned}$$

Using (15), however the sequence of first order conditions can be written as follows:

$$\begin{aligned}
\Gamma_{\tilde{t}-1} &= \beta \\
\Gamma_{\tilde{t}-2} &= |\!: \beta\frac{\partial d_{\tilde{t}-1}}{\partial D_{\tilde{t}-2}} + \frac{\partial d_{\tilde{t}}}{\partial D_{\tilde{t}-2}} \mid \\
\Gamma_{\tilde{t}-3} &= \beta\Gamma_{\tilde{t}-2} \mid \frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-3}} + \Gamma_{\tilde{t}-2}\left(1- \mid \frac{\partial d_{\tilde{t}-2}}{\partial D_{\tilde{t}-3}} \mid\right) \\
\Gamma_{\tilde{t}-4} &= \beta\Gamma_{\tilde{t}-3} \mid \frac{\partial d_{\tilde{t}-3}}{\partial D_{\tilde{t}-4}} \mid + \Gamma_{\tilde{t}-3}\left(1- \mid \frac{\partial d_{\tilde{t}-3}}{\partial D_{\tilde{t}-4}} \mid\right) \\
&\quad .....
\end{aligned}$$

and hence Γ_t is increasing in t. ■

We have shown that along the repayment phase, the first order conditions reduce to:

$$A'(q_t) = \frac{\beta}{T-t}\left[\sum_{\tau=t+1}^{T} A'(q_\tau)\right] \tag{12}$$

Equation (12) implies that distortions are indeed smoothed at the margin: it requires in fact that at any time t in the repayment phase, the marginal distortion $A'(q_t)$ be equal to the average future marginal distortion, $\frac{1}{T-t}\left[\sum_{\tau=t+1}^{T} A'(q_\tau)\right]$ discounted by β. It is the discounting by $\beta > 0$ which induces an increasing sequence of marginal distortions $A'(q_t)$ and hence of repayments q_t.

Furthermore, in equilibrium, it has to be the case that, at any time t in the repayment phase, the marginal distortion is > 1. Otherwise the agent would vote for accumulating debt in period t: an increase in consumption at t would have a larger effect at the margin than the induced marginal cost of future distortions. This is guaranteed by the condition that k be large enough. Finally, as $q_t > 0$ in the repayment phase, s_{1t} must also be, by Inada conditions. The solution of the problem of the agent at time 1, Problem (I), is then interior: agents at time 1 will use transfers to equalize consumption and hence c_t will be constant. This is the case for the entire repayment phase.

The last period of the debt accumulation phase is characterized by the fact that at the margin an increase in consumption has the same effect as the induced cost of future distortions. Indeed, we show in the following section that the first order condition of Problem (PE) at $\tilde{t}$ reduces to:

$$\frac{\beta}{T-\tilde{t}}\left[\sum_{\tau=\tilde{t}+1}^{T} A'(q_\tau)\right] = 1. \tag{13}$$

Furthermore, we show that the solution of Problem (I) is interior at $\tilde{t}$. This implies that $c_{\tilde{t}} = c_{\tilde{t}+1}$; and, by our characterization of the repayment phase in this corollary, $c_{\tilde{t}+1} = c_\tau$, for any $\tau > \tilde{t}+1$. Note that this is a T period version of the equation that characterizes the interior equilibrium repayment in the $T = 3$ economy, equation (4).

On the contrary, in the debt accumulation phase, up to period $\tilde{t}-1$, the agent consumes off of deficit spending; that is, transfers from time 1 are zero. As a consequence, consumption is not equalized across periods. In fact, it is declining over time due to the agents' self-control problem ($\beta < 1$).

We discuss now the consequences of extending the horizon T of this economy. We investigate whether the maximal debt $D_{\tilde{t}}$ grows without bound as T goes to infinity. To this end we construct a sequence of replica economies by allowing the aggregate endowment of the economy to grow at the same rate as T, so that the endowment per period remains constant along the sequence, and consumption does not become infinitesimal nor unboundedly large in every period. More precisely, the replica economies are characterized by aggregate endowment ρk and ρT periods, for some $\rho > 1$ (such that ρT is an integer). This construction guarantees that the characterization obtained in Proposition 8 and Corollary 9 hold for all replicas, along the sequence for which $\rho \to \infty$.

Let $\tilde{t}(\rho)$ denote the last accumulation period at the equilibrium of the replica economy corresponding to ρ; $\tilde{t}(1)$ is then the last accumulation period in the original economy with endowment k and T periods.

Corollary B. 2 *Along the sequence of replica economies, the maximal level of debt $D_{\tilde{t}(\rho)}$ increases with ρ and $D_{\tilde{t}(\rho)} \to \infty$ as $\rho \to \infty$.*[8]

Proof. Consider an economy with aggregate endowment k and T periods such that the characterization in Proposition 8 holds. Now consider replicas of this economy characterized by aggregate endowment ρk and ρT periods, for some $\rho > 1$. Let $\tilde{t}(\rho)$ denote the last accumulation period at the equilibrium of the replica economy; $\tilde{t}(1)$ is then the last accumulation period in the original economy with endowment k and T periods. Let $c_t(\rho)$ (respectively. $c_t(1)$) denote the consumption at period t in the replica economy (respectively. in the original economy). We show that the maximal debt of any replica ρ increases with respect to the original economy, $D_{\tilde{t}(\rho)} > D_{\tilde{t}(1)}$. As a consequence, the sequence $D_{\tilde{t}(\rho)}$ increases in ρ. The proof proceeds by contradiction. Assume by way of contradiction that $D_{\tilde{t}(\rho)} \leq D_{\tilde{t}(1)}$. Consider first the case in which $D_{\tilde{t}(\rho)} = D_{\tilde{t}(1)}$ and the sequence of repayments is unchanged, satisfying (12). Note that in this case, as the characterization in Proposition 8 holds, consumption is constant at equilibrium from the last accumulation period up to the last period (hence over the repayment period). Therefore, we must have

$$
\begin{aligned}
\tilde{t}(\rho) &= T + \tilde{t}(1) \\
c_t(1) &= \frac{k - c_1(1)}{\tilde{t}(1) + 1}, \quad \text{for any } \tilde{t}(1) \leq t \leq T \\
c_{T+t}(\rho) &= \frac{2k - c_1(\rho)}{\tilde{t}(1) + 1}, \quad \text{for any } \tilde{t}(1) \leq t \leq T
\end{aligned}
$$

From the first order condition of Problem I it can be shown that, while $c_1(\rho) > c_1(1)$, $c_{T+t}(\rho) > c_t(1)$, for any $t > \tilde{t}(1)$. As a consequence, comparing the first order condition of

[8] It should be noted that $\tilde{t}(\rho)$ also grows without bounds along the sequence of replica economies; see the Online Appendix.

Problem PE in the replica economy at at $T+\tilde{t}(1)-1$ with that of of the original economy at $\tilde{t}(1)-1$ implies that the deficit in the replica economy is higher than in the original economy. Solving backwards the first order condition of Problem PE we have that the maximal debt accumulated in the replica economy must be higher than in the original economy, $D_{\tilde{t}(\rho)} > D_{\tilde{t}(1)}$ yielding the desired contradiction. Note that, as a consequence of equations (12) and (13), if $D_{\tilde{t}(\rho)} = D_{\tilde{t}(1)}$ the sequence of repayments must indeed be unchanged. A similar argument can be applied to the case in which $D_{\tilde{t}(\rho)} < D_{\tilde{t}(1)}$. In this case in fact the repayment phase still needs to satisfy equations (12) and (13). As a consequence, if $D_{\tilde{t}(\rho)} < D_{\tilde{t}(1)}$ the repayment phase is possibly shorter. A fortiori then $c_{T+t}(\rho) > c_t(1)$ for any t such that $T+t$ is in the repayment phase of the replica economy. Comparing the first order condition of Problem PE in the replica economy at at $\tilde{t}(\rho)-1$ with that of of the original economy at $\tilde{t}(1)-1$ and solving backwards the first order condition of Problem PE produces a contradiction, as in the previous case.

We conclude that along the sequence of replica economies the maximal debt accumulated must be increasing, $D_{\tilde{t}(\rho)}$ increases with ρ. In fact, equations (12) and (13) imply that the repayment phase $\rho T-\tilde{t}(\rho)$ must also be increasing. But the sequence of maximal debt cannot have an upper bound. If it did, the sequence $\tilde{t}(\rho)$ would be bounded and the length of the repayment phase would instead grow to infinity, $\rho T-\tilde{t}(\rho)\to\infty$. This is not possible. In fact, in the proof of Corollary 9 we have shown that the repayment phase can be alternatively characterized solving (12) recursively. Proceeding along these lines we obtain

$$\lim_{\rho\to\infty}\frac{\beta}{\rho T-\tilde{t}(\rho)}\prod_{j=1}^{\rho T-\tilde{t}(\rho)-1}\left(1+\frac{\beta}{j}\right)=0 \text{ if } \lim_{\rho\to\infty}\left[\rho T-\tilde{t}(\rho)\right]\to\infty$$

This can be shown by applying the ratio convergence test (after a log transformation). As a consequence, $A'(q_{\tilde{t}(\rho)})\to 0$ as $\rho\to\infty$. In other words, the right-hand-side of equation (13) converges to 0 as $\rho\to\infty$, violating equation (13) itself. ■

The intuition is as follows. Consider a k and a T such that the characterization in Proposition 8 holds. Now double both k and T (that is, consider the replica economy corresponding to $\rho=2$). At equilibrium all transfers must go to support consumption in period 1, in the last accumulation period, and in the repayment phase. If the repayment phase stayed the same in terms of its length and of the size of repayments, consumption along the repayment phase would be larger: the aggregate endowment available to transfer over the same number of periods would have essentially doubled. In this case, however, the political process represented by the solution to Problem (PE) would require smoothing and hence higher deficits and for a longer number of periods implying a higher debt. Indeed, at equilibrium, the repayment phase is longer and the maximal debt it supports is higher. More generally, along the sequence of replica economies in which the aggregate endowment grows at the same rate as the number of periods, the maximal level of debt increases. But the sequence of maximal debt cannot have an upper bound as this would imply that the length of the repayment phase is finite in the limit and consumption along this phase would grow unboundedly, violating the first order conditions of the accumulation phase.

Another possible intuition for the result is that spreading the repayment of any finite amount of debt over a large number of future periods induces smaller and smaller marginal distortions that converge to 0. As a consequence, debt accumulation must also grow without bounds as the number of periods in the repayment phase.

In conclusion, when voters are time inconsistent, while convex distortions induce debt repayments to be smoothed over time, debt accumulation can nonetheless be very large to the point that debt grows without bound as the number of periods increases. This is the case, of course, unless debt limits are imposed. In other words, debt limits are necessary to limit the inefficient distortions which the economy must incur to repay large accumulated debts at equilibrium.

Other mechanisms may limit debt accumulation and hence distortions. Reducing the frequency of elections so that there is voting every $n > 1$ periods may lead to greater political commitment. A formal analysis of the effect of such a restrictions turns out to be complex. One important modeling choice is how the government is expected to behave in non-election periods. If the government chooses policies in non-election periods by attempting to satisfy popular opinion (governing by opinion polls), then the outcome would be equivalent to the one characterized in this Section. However, if the government can commit in election periods to its behavior in non-election periods, then this would presumably enhance political commitment and reduce debt accumulation, thereby producing beneficial effects on the equilibrium outcome.

C. Gul-Pesendorfer Preferences

The paper provides an analysis of voters who are characterized by quasi-hyperbolic preferences. One could also contemplate a setting in which agents experience temptation costs in each period a-la Gul and Pesendorfer (2001, 2004). In this Section, we show that the underlying forces driving our results do not change in such an alternative modeling setup. Indeed, suppose that, as in Gul and Pesendorfer two functions u and v govern an individual's valuations of choices from a set X. We adopt the assumption on temptation in Gul and Pesendorfer (2004), i.e, temptation in period t is given by the option of consuming the maximal feasible amount in period t. To their model we first introduce the possibility of illiquid assets and then add government debt.

As in the baseline model used in the paper, there is a wealth k and three periods. In period 1 the agent does not consume but just saves for subsequent periods. If there is no access to illiquid assets, and therefore, no possibility of commitment, in period 1 the agent can only pass on all the wealth to period 2, and in period 2 the agent chooses how much to consume. Thus, in this case, payoffs are given by

$$\begin{aligned}
&U_3(c_3) = u(c_3), \\
&U_2(c_2, c_3) = u(c_2) + v(c_2) - v(c_2 + c_3) + u(c_3), \\
&U_1(c_2, c_3) = U_2(c_2, c_3)
\end{aligned}$$

Let c_2^U, c_3^U be the solution of this problem when no illiquid assets are available, i.e., the non commitment solution. The first order conditions for this solution are:

$$u'(c_2^U) = u'(c_3^U) - v'(c_2^U), \tag{16}$$

In contrast, when illiquid assets are available, the situation is quite different. In this case the maximal feasible amount of consumption by agent 2 is s_{12}, agent 1's saving choice. Therefore, self 1, by choosing $s_{12} < k$, can reduce the temptation of self 2 with respect to the case of illiquid assets. This will indeed be the case at equilibrium with illiquid assets.[9] Let us begin the characterization of equilibrium with period 3. Given savings s_{13} in illiquid assets in the first period as well as savings in the second period s_{23}, utility in the third period is

$$U_3 = u(s_{13} + s_{23}).$$

In period 2, given savings s_{13} in illiquid assets and s_{12} in assets that are now liquid, utility is given by

$$U_2 = u(s_{12} - s_{23}) + v(s_{12} - s_{23}) - v(s_{12}) + u(s_{13} + s_{23}).$$

As we noticed, if $s_{12} < k$, the fact that assets s_{13} are illiquid reduces the temptation for the agent in period 2. Thus, the optimal solution in period 1 is to choose s_{12}, s_{13} to maximize

$$U_1 = u(s_{12}) + u(s_{13})$$

because, by ensuring that $s_{12} = c_2$, this eliminates temptations in period 2. Let c_2^*, c_3^* be the solution to this maximization problem, i.e., the commitment solution. Note that c_2^*, c_3^*

[9] Assuming that self 1 does not consume and hence experience no instantaneous temptation induces a more clear-cut result, but the same arguments would go through if we were to allow for period 1 consumption.

satisfies:

$$u'(c_2) = u'(c_3). \tag{17}$$

Contrasting equations (17) and (16) highlights the demand for commitment. Indeed, absent commitment, in period 2, the agent would want to shift resources from period 3 to period 2 whenever $v' > 0$ and $u''(c) + v''(c) < 0$.

We now introduce the possibility of government debt. For the purpose of this Web Appendix we assume that there are no distortions in order to make the comparison with the $\beta - \delta$ model used in the paper more direct.

Assume that d^e is the candidate equilibrium level of government debt. From the optimal savings and portfolio choices of the agent we must have:

$$U_3 = u(s_{13} + s_{23} - d^e),$$

$$U_2 = u\left(s_{12} - s_{23} + d^e\right) + v\left(s_{12} - s_{23} + d^e\right) - v\left(s_{12} + d^e\right) + u(s_{13} + s_{23} - d^e).$$

So, if $d^e \leq c_2^*$, the optimal solution in period 1 sets $s_{12} = c_2^* - d^e$, $s_{13} = c_3^* + d^e$ which allows restoring the full commitment utilities in all periods.

However, as long as the debt limit $\overline{d}$ is below the non-commitment level of consumption, c_2^U, the equilibrium debt will be raised up to the debt limit. Consider on the contrary a debt level d such that $d < \overline{d} \leq c_2^U$, in period 2, the **actual** payoff function determining voting over government debt that candidates implicitly maximize is

$$U_2 = u\left(c_2^* + d\right) + v\left(c_2^* + d\right) - v\left(s_{12} + \overline{d}\right) + u(s_{13} + s_{23} - d).$$

Thus, whenever $d < \overline{d}$, the agent has an incentive to vote for higher debt.

This reasoning can easily be extended to show that when $\overline{d} > c_2^U$, then equilibrium debt is equal to c_2^U thus showing the analogue of our Proposition 1 for the case of Gul and Pesendorfer preferences. The case of distortions can also be treated in a similar fashion.

D. Period One Consumption

The paper focused on an environment in which consumption takes place only in periods 2 and 3. In principle, individuals could also make consumption decisions while planning for future consumption. Foreseeing their future behavior, individuals can then adjust their immediate consumption and thereby affect their future budget. We now consider such settings. As in the paper, there is a measure 1 of voters who live for three periods. In period 1 voters have a wealth k from which to finance consumption over three periods. Preferences over consumption sequence c_1, c_2, c_3 are given by

$$
\begin{aligned}
&U_1(c_1, c_2, c_3) = u(c_1) + \beta\delta u(c_2) + \beta\delta^2 u(c_3), \\
&U_2(c_2, c_3) = u(c_2) + \beta\delta u(c_3), \\
&U(c_3) = u(c_3),
\end{aligned} \tag{18}
$$

where u is a continuous and strictly concave utility function. We also assume that the utility function is three times continuously differentiable. As in the paper, we assume that $\delta = 1$ and that agents are sophisticated. We use the notation used in the paper for the commitment and no-commitment consumption choices.

While period-one consumption may affect the budget left for one's period-two self, the demand for commitment is similar to that without period-one consumption. Namely, commitment leads to lower second period consumption: $c_2^* < c_2^U$.

Consider first the benchmark in which debt is non-distortionary.

In period 1 an agent who predicts equilibrium per-capita debt levels of d, chooses savings intended for period 2, denoted by s_{12} and for period 3, denoted by s_{13}, to solve

$$\max_{s_{12}, s_{13}} u(c_1) + \beta u(s_{12} + d - s_{23}) + \beta u(s_{13} + s_{23} - d).$$

In period 2 a voter with preference parameter β chooses savings s_{23} to solve

$$\max_{s_{23}} u(s_{12} + d - s_{23}) + \beta u(s_{13} + s_{23} - d).$$

The political process proceeds as in the paper.

Equilibrium Characterization. The Incomplete Ricardian Equivalence characterized in Proposition 1 in the paper still holds. Namely, we have that:

Proposition D. 1 (Incomplete Ricardian Equivalence)

1. *If $\overline{d} \leq c_2^*$ then both candidates offer platforms with debt $\overline{d}$. Equilibrium consumption is (c_1^*, c_2^*, c_3^*).*

2. *If $c_2^* < \overline{d} < c_2^U$ then both candidates offer platforms with debt $\overline{d}$. In equilibrium, second-period consumption is $c_2 = \overline{d}$.*

3. *If $\overline{d} \geq c_2^U$ then any d* such that *$c_2^U \leq d \leq k$ is part of an equilibrium. Equilibrium consumption is $\left(c_1^U, c_2^U, c_3^U\right)$.*

Proof. 1. Assume by way of contradiction that equilibrium debt is $d^* < \overline{d}$. If this is the case, a voter can implement the commitment sequence of consumption c_1^*, c_2^*, c_3^* by choosing $s_{12} = c_2^* - d^*$, and $s_{13} = c_3^* + d^*$. This is feasible since $d^* < \overline{d} < c_2^*$. Hence, these are the optimal choices for the voter. But, by definition of c_2^*, c_3^*, $u'(c_2^*) > \beta u'(c_3^*)$, and therefore, in period 2 all voters would vote for a candidate who offered a slightly higher debt. Thus, the only debt that can be part of an equilibrium is $\overline{d}$. Given a debt of $\overline{d}$, in period 1, each voter chooses $s_{12} = c_2^* - \overline{d}, s_{13} = c_3^* + \overline{d}$. Given these saving choices, none of the voters would vote for a candidate that offered a lower debt in the second period, proving that debt and this sequence of consumption constitute a unique equilibrium.

2. Assume by way of contradiction that, in equilibrium, a debt $d^* < \overline{d}$ is implemented. As in part (1), voters choose savings to restore commitment as much as possible. Assume that $c_2^* < d^*$ (otherwise, the proof of part (1) applies). Each agent maximizes

$$\begin{aligned} & u(c_1) + \beta u(c_2) + \beta u(k - c_1 - c_2) \\ s.t. \quad c_2 \ \geq \ & d^*. \end{aligned}$$

The first order conditions yield

$$u'(c_1) = \beta u'(k - c_1 - d^*) > u'(c_2) = u'(d^*)$$

because $d^* > c_2^*$ (recall that $u'(c_2^*) = u'(c_3^*)$). This means that the agent sets $s_{12} = 0$ since second-period consumption is already higher than desired by the first-period self. However, since $d^* < c_2^U$, $u'(d) > \beta u'(c_3)$. Thus, in period 2 all voters would vote for higher debts contradicting the assumption that d is an equilibrium debt level. Finally, to conclude that a debt of $\overline{d}$ is indeed part of an equilibrium, observe that, given $\overline{d}$, by similar reasoning, the optimal saving choices of all voters would lead to $u'\left(\overline{d}\right) > \beta u'(c_3)$. Thus, no voter would vote for lower debts.

3. We first show that the claimed outcomes are part of an equilibrium. Given any candidate equilibrium debt $k > d^* \geq c_2^U$ that is expected by voters in period 1, an optimal policy of a voter in period 1 is a choice of $s_{12} = 0$ and $s_{13} = c_3^U - \left(d^* - c_2^U\right)$. In addition, given d^*, in equilibrium, $s_{23} = d^* - c_2^U$ is to be saved in period 2 for period 3. Given this policy, by the definition of c_2^U, c_3^U, we have

$$u'\left(c_2^U\right) = \beta u'\left(c_3^U\right)$$

giving no incentive to any period-2 self to change her savings plan away from s_{23}. Suppose now that the period-1 self were to change (e.g., increase) s_{13}. Then, the period-2 self would make an offsetting change (reduction) in s_{23} to restore period 2 optimality. Any change in

s_{12} would similarly be offset (recall that since $d^* \geq c_2^U$, even if $s_{12} = 0$, the period-2 self can unilaterally choose c_2^U). Thus, the period-1 self has no incentive to deviate.[10]

Given these policies for the voters, consider a deviation to $d < d^*$ in period 2. As long as the deviation is small ($d \geq c_2^U$), all voters are indifferent (they can just make an offsetting reduction in s_{23} to restore the desired consumption sequence). If the deviation is large ($d < c_2^U$), then voters who can no longer make such offsetting reduction in s_{23}. All voters would therefore vote against a candidate offering such a deviation. A deviation to $d > d^*$ would leave all voters indifferent because they could make offsetting changes in s_{23}.

Consider now a candidate equilibrium debt $d^* < c_2^U$. Such an expected debt would constrain period-2 consumption for the voters, leading to victory in period 2 for a candidate offering $d > d^*$. ■

When debt is distortionary, the analysis changes slightly when one accounts for consumption in the first period. The equilibrium characterization is analogous to that corresponding to the case in which consumption occurs only in the second and third periods. Indeed, let $c_1^*(d), c_2^*(d)$, and $c_3^*(d)$ be the commitment sequence of consumption given debt d, namely, the solution to the following problem:

$$\begin{aligned} & \max \left\{u(c_1) + \beta\left(u(c_2) + u(c_3)\right)\right\} \\ s.t. \quad c_1 + c_2 + c_3 &= k - \eta d \end{aligned}$$

Analogously, let $c_1^U(d), c_2^U(d)$, and $c_3^U(d)$ be the corresponding quantities without commitment. We define d^* as the solution of $c_2^*(d^*) = d^*$.[11]

We now introduce an artificial constrained-maximization problem for a voter of preference parameter $\beta(1+\eta) < 1$.

$$\begin{aligned} & \max u(c_1) + \beta\left[u(c_2) + u(c_3)\right] \\ s.t. \;\; u'(c_2) &= \beta(1+\eta)\, u'(c_3), \\ c_1 + c_2 + c_3 &= k - d\eta. \end{aligned} \tag{19}$$

Notice that when there is consumption in the first period, the optimal consumption is not simply prescribed by the second-period constraint, since the resources available to the second-period self are endogenous and determined by consumption in the first period. Denote

[10] There are multiple ways for the period-1 self to implement the uncommitted sequence, involving increasing s_{12} and s_{23} by the same amounts with offsetting reductions to s_{13}. All these are weakly dominated by the proposed sequence.

[11] Notice that $c_2^*(0) \geq 0$, while $c_2^*(k/\eta) = 0 < k/\eta$, and so the Intermediate Value Theorem guarantees the existence of such a d^*.

by $(c_1^\eta(d), c_2^\eta(d), c_3^\eta(d))$ the consumption sequence that solves the problem . We now define d^{**} to be the solution of $d^{**} = c_2^\eta(d^{**})$.[12] It is easy to show that $d^* < d^{**}$.

Proposition D. 2 (Distortionary Equilibrium Debt)

1. *If $\beta(1+\eta) > 1$ then in equilibrium there is no debt and consumption is given by (c_1^*, c_2^*, c_3^*).*

2. *Assume that $\beta(1+\eta) < 1$. If $\overline{d} \leq d^*$, then equilibrium debt is given by $\overline{d}$ and consumption is given by $\left(c_1^*\left(\overline{d}\right), c_2^*\left(\overline{d}\right), c_3^*\left(\overline{d}\right)\right)$. If $d^* < \overline{d} \leq d^{**}$, then equilibrium debt is given by $\overline{d}$ and period 2 consumption is given by $c_2 = \overline{d}$. If $\overline{d} > d^{**}$, then debt is given by d^{**} and period 2 consumption is given by $c_2 = d^{**}$.*

Proof. 1. We first show that there is an equilibrium with zero debt. Given an expected second-period debt of zero, in period 1 voters choose the mix of liquid and illiquid assets $s_{12} = c_2^*$ and $s_{13} = c_3^*$ that implements the commitment consumption sequence (c_1^*, c_2^*, c_3^*). Given this mix of savings, $u'(c_2^*) = u'(c_3^*)$. Thus, if $\beta(1+\eta) > 1$, $u'(c_2^*) < \beta(1+\eta)u'(c_3^*)$ and voters have no incentive to vote for positive debt. Consider now any level of expected debt d. The mix of savings has to be such that $u'(s_{12}+d) \leq u'(s_{13}+s_{23}-d)$. But then $u'(s_{12}+d) < \beta(1+\eta)u'(s_{13}+s_{23}-d)$, inducing voters to vote to reduce debt.

2. Consider now the case in which $\beta(1+\eta) < 1$. Given any $\overline{d} < d^*$ and any expected $d \leq \overline{d}$, optimal savings in period 2 are given by $s_{23} = 0$ and s_{12}, s_{13} are such that $u'(s_{12}+d) = u'(s_{13}-d)$. Thus, $u'(s_{12}+d) > \beta(1+\eta)u'(s_{13}-d)$ and voters would vote to increase debt. Thus, in this scenario equilibrium debt must be $\overline{d}$ and consumption must be given by $\left(c_1^*\left(\overline{d}\right), c_2^*\left(\overline{d}\right), c_3^*\left(\overline{d}\right)\right)$. If $d^* < \overline{d} \leq d^{**}$, then, by the same reasoning, equilibrium debt must be at least d^*. But then, by the definition of d^*, debt is higher than second-period commitment consumption, and optimal savings are at a corner: $s_{12} = s_{23} = 0$, implying that $c_2 = d$. Because $d < d^{**}$, we then have that $\beta(1+\eta)u'(c_3) < u'(c_2) < u'(c_3)$. This implies that voters vote for higher debt unless $d = \overline{d}$. Finally, If $\overline{d} \geq d > d^{**}$, then by the definition of d^{**}, $u'(d) < \beta(1+\eta)u'(c_3)$, so voters would vote to reduce debt. This proves that, for any $\overline{d} \geq d^{**}$ equilibrium debt is given by d^{**}. ■

Welfare Analysis. When consumption takes place only in periods 2 and 3, the analysis of the impact of distortions on welfare is dramatically simplified. Indeed, equilibrium consumption is essentially governed by the second-period constraint. Technically, we can

[12] Again, the Intermediate Value Theorem assures that such d^{**} always exists since $c_2^\eta(0) = c_2^U(0) \geq 0$, and $c_2^\eta(k/\eta) = 0 < k/\eta$, and the Theorem of the Maximum implies that $c_2^\eta(d)$ is continuous.

use the implicit function theorem to derive a full ranking of welfare for different distortion levels η. When consumption occurs in period 1 as well, the budget available in period 2 is endogenous and may depend on η. Nonetheless, we can still determine the detrimental effects of distortions, as well as the impacts of suffering from self-control problems. The following result provides a comparison of equilibrium welfare with and without distortions when debt limits are large (namely, $\overline{d} > d^{**}$).

Proposition D. 3 (Welfare Effects of Distortions) *Whenever $\beta < \beta(1+\eta) < 1$ the equilibrium with distortions determined by η leads to lower first period welfare than the equilibrium corresponding to no distortions, when $\eta = 0$. If $\beta(1+\eta) > 1$, then first period welfare is higher than that induced by any $\beta(1+\eta) < 1$.*

Proof. Consider the following maximization problem:

$$\begin{aligned} &\max u(c_1) + \beta\left[u(c_2) + u(c_3)\right] \\ &s.t.\ \ u'(c_2) = \beta(1+\eta)u'(c_3) \\ &c_1 + c_2 + c_3 = k - \eta c_2. \end{aligned} \tag{20}$$

This is an artificial problem corresponding to an agent who chooses the debt level and her consumption plan in tandem but consuming c_2 destroys resources just as debt does. In particular, this problem generates a higher overall utility (from period 1's perspective) than that experienced by an agent who consumes $c_1^{\eta}(d^{**}), c_2^{\eta}(d^{**}), c_3^{\eta}(d^{**})$ because such an agent takes the equilibrium level of debt as given and cannot alter it unilaterally. The latter generates the equilibrium level of welfare for distortions η. Furthermore, the two coincide when $\eta = 0$. We now show that the maximized objective of problem (20) is decreasing in η. Indeed, suppose $\eta_1 > \eta_2$. Denote the solution of (20) for distortions η_1 by (c_1, c_2, c_3). We now approximate a policy under distortions η_2 small enough that it satisfies the constraints and generates a strictly higher value for the objective.

For η_2 close enough to η_1, there exists $\varepsilon > 0, \varepsilon < c_3$ such that

$$u'(c_2) = \beta(1+\eta_2)u'(c_3 - \varepsilon).$$

Therefore,

$$u'(c_2) = \beta(1+\eta_2)\left[u'(c_3) - \varepsilon u''(c_3) + O(\varepsilon^2)\right].$$

Since (c_1, c_2, c_3) is a solution to the problem with distortions η_1, $u'(c_2) = \beta(1+\eta_1)u'(c_3)$. It follows that:

$$\varepsilon = \frac{(\eta_2 - \eta_1)\, u'(c_2)}{\beta(1+\eta_2)u''(c_3)} + O(\varepsilon^2).$$

Consider then the policy $(c_1 + \varepsilon + (\eta_1 - \eta_2)\, c_2, c_2, c_3 - \varepsilon)$ when the distortions are η_2. Notice that, by construction, this policy satisfies the two constraints in problem (20). The difference between the generated objective and the maximal value of the objective under distortions η_1 is then:

$$\Delta = [u(c_1 + \varepsilon + (\eta_1 - \eta_2)\, c_2) - u(c_1)] + \beta\, [u(c_3 - \varepsilon) - u(c_3)]\, .$$

Using a first order approximation,

$$\begin{aligned}
\Delta &= (\varepsilon + (\eta_1 - \eta_2)\, c_2)\, u'(c_1) - \beta \varepsilon u'(c_3) = \\
&= (\eta_1 - \eta_2)\, c_2 u'(c_1) + \frac{(\eta_2 - \eta_1)\, u'(c_2) u'(c_1)}{\beta(1+\eta_2) u''(c_3)} - \frac{(\eta_2 - \eta_1)\, u'(c_2) u'(c_3)}{(1+\eta_2) u''(c_3)} + O(\varepsilon^2) \\
&= \frac{(\eta_1 - \eta_2)}{(1+\eta_2)} u'(c_2) \left[\frac{u'(c_1) c_2}{u'(c_2)} - \frac{u'(c_1) - \beta u'(c_3)}{\beta u''(c_3)} \right] + O(\varepsilon^2).
\end{aligned}$$

Notice that the solution to problem (20) with distortions η_1 must satisfy $u'(c_1) = \beta\, [u'(c_2) + u'(c_3)]$ and so:

$$\Delta = \frac{(\eta_1 - \eta_2)}{(1+\eta_2)} u'(c_2) \left[\frac{u'(c_1) c_2}{u'(c_2)} - \frac{u'(c_2)}{u''(c_3)} \right] + O(\varepsilon^2),$$

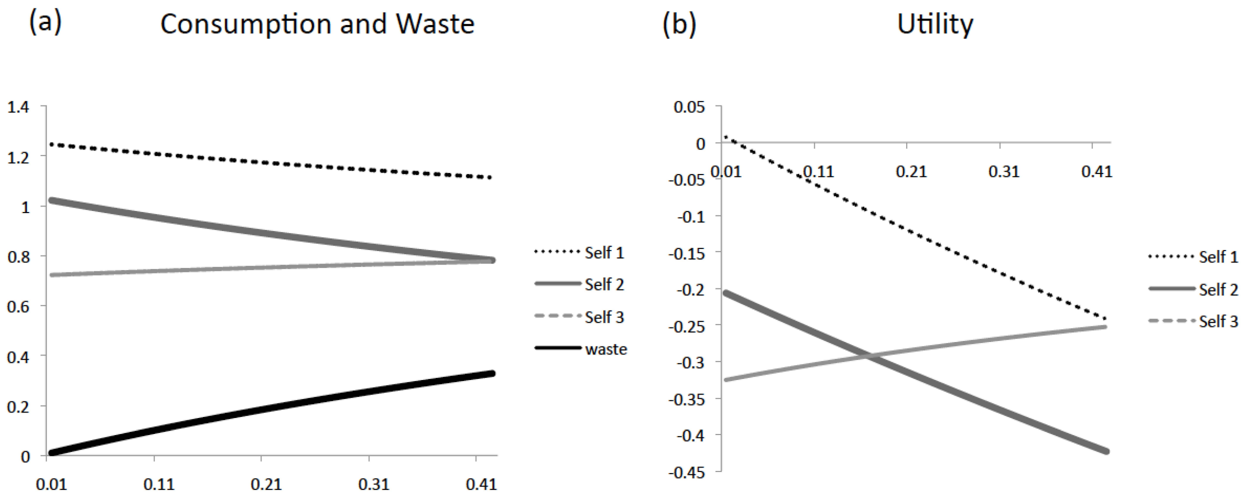
which from concavity of the instantaneous utility u, is positive whenever η_1 and η_2 are close enough. In particular, the optimal solution for problem (20) with distortions η_2 must generate a strictly higher level of the objective function than the solution with distortions η_1. It follows that welfare in our distortion economy is lower under any $\eta > 0$ relative to the case of $\eta = 0$.

Last, notice that when $\beta(1+\eta) < 1$, all agents achieve their commitment solution absent debt, an consequently the maximal period 1 utility under the budget constraint. From Proposition 2, this is no longer the case when $\beta(1+\eta) > 1$ and so period 1 utility is lower for distortions exceeding $1 - \beta$. ■

As in the model analyzed in paper, there are two contrasting effects of positive distortions. On the negative side, given that there is debt in equilibrium, the presence of distortions causes wealth destruction. On the positive side, distortions relax the commitment constraint in the artificial maximization that determines equilibrium debt. In fact, when η is very high ($\eta > 1 - \beta$), distortions serve as a full commitment device since, in equilibrium, voters do not vote for positive debt in the second period. The proposition shows that the negative effect dominates.

Figure 1 illustrates the impact of distortions in the case of instantaneous log-utility, where we take the budget to be $k = 3$ and the population time preferences to be $\beta = 0.7$. The left panel of the figure illustrates the consumption patterns and wealth destroyed. Notice

Figure 1: Outcomes for Log Instantaneous Utility ($k = 3$, $\beta = 0.7$)

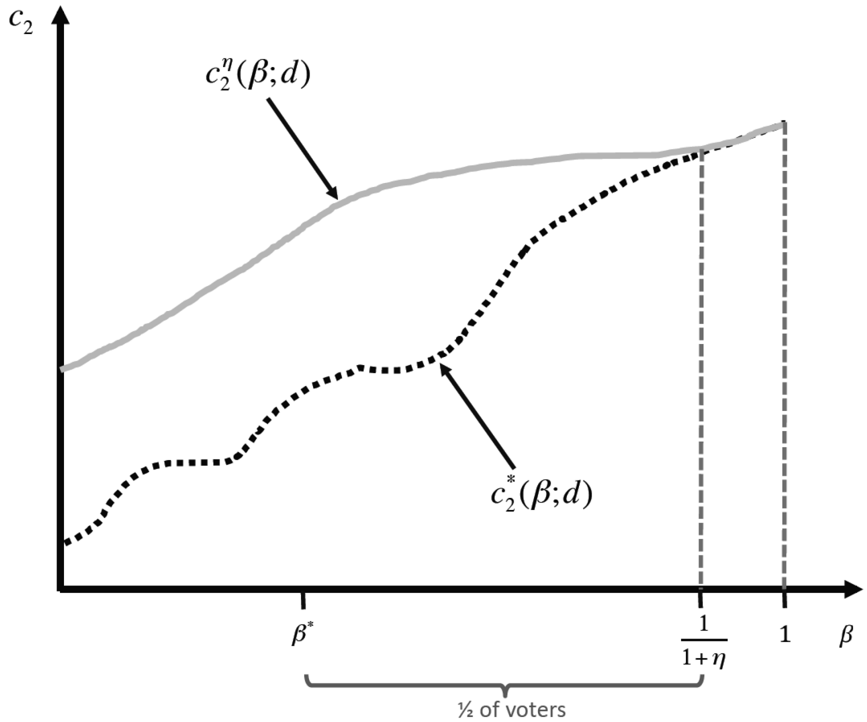


that consumption declines with η in periods 1 and 2, but is increasing in period 3. This reflects the two effects discussed above that distortions have – on the one hand, they destroy wealth, and indeed, wealth destruction increases with η; On the other hand, they relax the constraints in period 2, which allows for more delayed consumption. The right panel of the figure illustrates the impact of distortions on welfare from the perspective of each self. Welfare for period-1 and period-2 selves declines with η, in line with the statement in the proposition. This indicates that the effect of wealth destruction outweighs the benefits of smoothing derived from greater distortions, and so overall greater distortions do not help individuals early in the process. However, since period 3 consumption is increasing, so does welfare in period 3.

Heterogeneity. We now consider what happens when agents are heterogeneous in their present-bias parameter β. In analogy to our previous notation, we will denote by $c_t^*(\beta; d)$ and $c_t^\eta(\beta; d)$ the commitment solution for debt d and the solution to the constrained problem (19) for each individual of preference parameter β.

We start by assuming that second period consumption $c_2^\eta(\beta; d)$ increases monotonically in β. This holds when the utility function has sufficient curvature. We note that there are many preferences for which this does not hold. For instance, with log utility, consumption is not monotonic. However, even in such a case our initial discussion will be valid for a fairly wide class of distributions of the β parameter. We discuss the more general case below. We note that this assumption stands in stark contrast with the environment in which there is no consumption in the first period. Indeed, in that case $c_2^\eta(\beta; d)$ is decreasing and $c_t^*(\beta; d)$ is a constant function independent of β.

Figure 2: Consumption Patterns for a Given Debt Level



Let β^* be such that $G(\frac{1}{1+\eta}) - (\beta^*) = 1/2$. That is, half the population has preferences that are between β^* and $\frac{1}{1+\eta}$. Figure 3 depicts the shape of commitment and no-commitment consumption levels in period 2 as a function of preferences for a particular debt level.

The agent of type β^* turns out to be the pivotal agent for determining debt in this environment. We can now define $d^*(\beta^*)$ and $d^{**}(\beta^*)$ as the solutions of $d^* = c_2^*(\beta^*, d^*)$ and $d^{**} = c_2^\eta(\beta^*, d^{**})$.[13]

Proposition D. 4 *1. If $\beta_M(1+\eta) > 1$, then in equilibrium there is no debt, and consumption is given by $c_1^*(\beta), c_2^*(\beta), c_3^*(\beta)$.*

2. Assume that $\beta_M(1+\eta) < 1$. If $\overline{d} \leq d^{**}(\beta^*)$, then equilibrium debt is given by $\overline{d}$. If $\overline{d} > d^{**}(\beta^*)$, then debt is given by $d^{**}(\beta^*)$.

3. For any equilibrium debt level d, individual consumption for an agent of preference

[13]Existence and uniqueness of these debt levels follow the same arguments used for the case of a homogenous electorate.

parameter β, period-2 consumption level in equilibrium is given by:

$$c_2(\beta; d) = \begin{cases} c_2^{\eta}(\beta; d) & \beta \leq \beta_L(d) \\ d & \beta_L(d) \leq \beta < \beta_H(d) \\ c_2^{*}(\beta; d) & \beta \geq \beta_H(d) \end{cases}.$$

With respect to the distribution of preferences, notice that a shift in distribution changes the debt structure in the economy only when it modifies the preferences β^* of the 'pivotal agent'. As β^* increases, $c_2^*(\beta^*; d)$ and $c_2^{\eta}(\beta^*; d)$ increase for all d, and therefore both d^* and d^{**} increase.

We say G' is a *median preserving spread* of G if both share the same median β_M and for any $\beta < \beta_M$, $G'(\beta) \geq G(\beta)$, while for any $\beta > \beta_M$, $G'(\beta) \leq G(\beta)$. Intuitively, this implies that, under G', more weight is put on more extreme values of β (see Malamud and Trojani (2009) for applications to a variety of other economic phenomena).

The above discussion then implies the following corollary.

Corollary D. 1 (Distributional Shifts) *1. Assume $G(\frac{1}{1+\eta}) = G'(\frac{1}{1+\eta})$. If G' First Order Stochastically Dominates G, and the corresponding medians $\beta_M, \beta'_M < \frac{1}{1+\eta}$, then equilibrium debt under G' is (weakly) higher than that under G.*

2. If G' is a Median Preserving Spread of G, then equilibrium debt under G' is (weakly) lower than that under G.

Part 1 of this corollary says that, as the population becomes more "virtuous" or less subject to self-control problems, equilibrium debt increases. This is potentially surprising but is a natural consequence of the logic of our model. There are two ways to glean intuition for this result. The more mechanical one is to recall that equilibrium debt is equal to second period consumption. As β^* increases, so does the desired second period consumption of the pivotal agent β^*. Thus, equilibrium debt increases. Alternatively, notice that in our model debt arises because of the desire of the pivotal agent to constrain her future self, and the subsequent response of the political system undoing this commitment. The more virtuous the pivotal agent, the higher the level of debt that is required to prevent this agent from attempting to commit at an even higher level.

We now discuss the more general case in which second period consumption may not be increasing in β. For any η, denote by d^p the debt level such that:

$$G\left\{\beta \mid c_2^{\eta}(\beta; d^p) < d^p\right\} = \frac{1}{2}.$$

Proposition D.1. can now be restated with d^p playing the role of $d^{**}(\beta^*)$. If second period consumption is *decreasing* in β, then d^p will correspond to $c_2^\eta(\beta_M; d^{**})$: the median voter will be pivotal. Otherwise, there may be multiple pivotal voters.

We now discuss how the welfare of different agent types is affected by the presence of illiquid assets. Our result in Proposition 5 of the paper showing that agents would be made better off in the first period if illiquid assets were penalized obviously extends to the case where the degree of heterogeneity is limited. Furthermore, if $c_2^\eta(\beta, d)$ is increasing in β, it is possible to show that, for any degree of heterogeneity, all agents with $\beta \leq \beta^*$ as well as those with sufficiently high β are made worse off by the presence of illiquid assets: the former group because for these types, debt is higher than $c_2^\eta(\beta; d)$ and second period consumption is completely out of transfers, so the logic of Proposition 5 immediately holds for these agents; the latter group because these types do not have much of a self-control problem, so the presence of illiquid assets gains them little commitment but generates a destruction of resources through debt.

E. Private Debt with Debt Limits

In the paper, when we discuss borrowing on the private market from intermediaries such as credit card companies, we assume that there is no debt limit. This has no effect on the results generated when $\beta(1+\eta) > 1$. In this case, in equilibrium there is no debt and consumption is given by (c_2^*, c_3^*). However, when $\beta(1+\eta) < 1$, corner solutions may emerge when the debt limit is sufficiently small.

Formally, notice that $U_1(c_2^*(d), c_3^*(d))$ is decreasing in d and so there is a unique $d^C > 0$ for which $U_1\left(c_2^*\left(d^C\right), c_3^*\left(d^C\right)\right) = U_1\left(c_2^\eta(0), c_3^\eta(0)\right)$. Debt d^C is the debt level that renders the agent indifferent between borrowing d^C but perfectly smoothing utility between periods 2 and 3, and not borrowing but accepting the constrained commitment allocation. It may be the case that $c_2^*\left(d^C\right), c_3^*\left(d^C\right)$ is not feasible because it would violate the constraint that $s_{12} \geq 0$. Thus, we need to consider the case in which $s_{12} = 0$, and second period consumption is equal to $\overline{d}$. Let d^{CC} denote the debt level such that $U_1\left(d^{CC}, k - d^{CC}(1+\eta)\right) = U_1\left(c_2^\eta(0), c_3^\eta(0)\right)$. We have:

Proposition 4 (Equilibrium with Credit Cards) *Assume $\beta(1+\eta) < 1$. If $\overline{d} > \max\left\{d^C, d^{CC}\right\}$ then agents make portfolio decisions in period 1 that ensure no debt in the second period: equilibrium debt is zero, the equilibrium consumption sequence is given by $\left(c_2^\eta(0), c_3^\eta(0)\right)$, and first-period welfare is increasing in η. If $\overline{d} \leq \max\left\{d^C, d^{CC}\right\}$, then debt is $\overline{d}$, and consumption is either $\left(c_2^*\left(\overline{d}\right), c_3^*\left(\overline{d}\right)\right)$ or $\left(\overline{d}, k - \overline{d}(1+\eta)\right)$.*

Proof. Suppose $\hat{s}_{12}, \hat{s}_{13}$, and $d\left(\hat{s}_{12}, \hat{s}_{13}, \overline{d}\right) > 0$ constitute part of an equilibrium. If $d\left(\hat{s}_{12}, \hat{s}_{13}, \overline{d}\right) < \overline{d}$, then the second period first-order condition of the agent must hold, and therefore we must have:

$$u'\left(\hat{s}_{12} + d\left(\hat{s}_{12}, \hat{s}_{13}, \overline{d}\right)\right) = \beta\left(1+\eta\right) u'\left(\hat{s}_{13} - d\left(\hat{s}_{12}, \hat{s}_{13}, \overline{d}\right)\left(1+\eta\right)\right).$$

The following savings plan constitutes an improving plan in period 1: $s_{12} = \hat{s}_{12} + d\left(\hat{s}_{12}, \hat{s}_{13}, \overline{d}\right)$, $s_{13} = \hat{s}_{13} - d\left(\hat{s}_{12}, \hat{s}_{13}, \overline{d}\right)(1+\eta)$. Given this savings plan, the second period first-order conditions are satisfied with $d\left(\hat{s}_{12}, \hat{s}_{13}, \overline{d}\right) = 0$ and consumption at $t = 2$ and $t = 3$ is unchanged. However, this saving plan increases the resources available to the consumer in period 1 by $\eta d\left(\hat{s}_{12}, \hat{s}_{13}, \overline{d}\right)$. These can be distributed between periods 2 and 3, while still satisfying the first order condition. In particular, $\left(c_2^\eta(0), c_3^\eta(0)\right)$ is the resulting consumption sequence which clearly satisfies the second period first-order condition.

Let us now consider the case in which the debt limit is binding ($d\left(\hat{s}_{12}, \hat{s}_{13}, \overline{d}\right) = \overline{d}$) in equilibrium. In this case, the agent must either be consuming her commitment consumption sequence $\left(c_2^*\left(\overline{d}\right), c_3^*\left(\overline{d}\right)\right)$, or we must have $c_2 = \overline{d}$ and $s_{12} = 0$: otherwise, the agent could improve her first-period utility by reducing s_{12} and increasing s_{13} without changing debt.

Thus, we have two possible equilibria that may be induced by a first period choice: (1) $d = 0$ and consumption $\left(c_2^\eta(0), c_3^\eta(0)\right)$ or (2) $d = \overline{d}$ and consumption of either $\left(c_2^*\left(\overline{d}\right), c_3^*\left(\overline{d}\right)\right)$ or $\left(\overline{d}, k - \overline{d}(1+\eta)\right)$. These two alternative plans yield utilities of $U_1\left(c_2^\eta(0), c_3^\eta(0)\right)$ and either $U_1\left(c_2^*\left(\overline{d}\right), c_3^*\left(\overline{d}\right)\right)$ or $U_1\left(\overline{d}, k - \overline{d}(1+\eta)\right)$ respectively. We now note that $U_1\left(c_2^\eta(0), c_3^\eta(0)\right)$ is independent of $\overline{d}$ while both $U_1\left(c_2^*\left(\overline{d}\right), c_3^*\left(\overline{d}\right)\right)$ and $U_1\left(\overline{d}, k - \overline{d}(1+\eta)\right)$ are decreasing in $\overline{d}$. In either case, from the definitions of d^C and d^{CC}, it follows that for $\bar{d} \leq d^J$ (where J may be either C or CC) and the agent chooses to commit and accepts that in the second period debt will be binding. For $\bar{d} > \max\left\{d^C, d^{CC}\right\}$, the agent gives up commitment and will choose a debt of zero. ■

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