Competitive Policy Development

Alexander V. Hirsch and Kenneth W. Shotts

January 1, 2015

Online Appendix

This Online Appendix is divided into three parts. Appendix A proves results in the main text. Appendix B is a complete treatment of the general model (which allows each entrepreneur to value the quality of her opponent's policies at $(1-\beta)\,q_{-i}$) and proves Lemma 1, which contains necessary conditions for equilibrium. Appendix C is a complete treatment of the variant with dogmatic entrepreneurs, and proves Lemma 2 which contains necessary conditions for equilibrium in that variant.

A Main Proofs

We first transform strategies (y, q) to be expressed in terms of score and ideology. An entrepreneur's pure strategy (s_i, y_i) is a two-dimensional element of $\mathbb{B} \equiv \{(s, y) \in R^2 \mid s + y^2 \geq 0\}$, or the set of scores and ideologies that imply positive-quality policies. A mixed strategy σ_i is a probability measure over the Borel subsets of $\mathbb{B}$, and let $F_i(s)$ denote the CDF over scores induced by i's mixed strategy σ_i. For technical convenience we restrict attention to strategies generating score CDFs that can be written as the sum of an absolutely continuous and a discrete distribution. The decisionmaker is the last mover, so equilibrium requires that he choose a policy (s, y) with the maximum score. While a complete description also requires specifying his tie-breaking rules, equilibria are invariant to this decision so we omit the additional notation.

In Appendix B we prove the following properties of equilibrium for both the baseline model, and the variant in which each entrepreneur values the quality of her opponent's policies at $(1-\beta)\, q_{-i}$.

Lemma 1. *The following properties hold for $\beta \geq 0$.*

1. *At any score $s_i > 0$ where $-i$ has no atom, developing $(s_i, y_i^*(s_i))$ with $y_i^*(s_i) = F_{-i}(s_i)\frac{x_i}{\alpha_i}$ is strictly better than developing any other policy.*

2. *In any equilibrium, $F_k(0) = 0$ for some $k \in \{L, R\}$ and the support of the score CDFs (F_i, F_{-i}) over $s \geq 0$ is common, convex, atomless, and includes 0.*

We now prove the remaining results in the main text using this result.

Proof of Proposition 1 Since the CDFs are atomless over $(0, \infty)$ and at such scores developing $(s_i, y_i^*(s_i))$ is strictly better than any other policy (by Lemma 1), in equilibrium the probability a policy (s_i, y_i) with $s_i > 0$ satisfies $y_i = y_i^*(s_i)$ is 1. Observation 1 then follows: if $F_i(0) = 0$ then i's policies are strictly positive score with probability 1, and if $F_i(0) > 0$ then $F_{-i}(0) = 0$ (also by Lemma 1), policies with score $s_i \leq 0$ both lose for sure and never affect a tie, and therefore must be 0 quality.

Also observe that the score conditions in Lemma 1 (combined with our technical restriction on the score CDFs) immediately imply that (i) the score CDFs are absolutely continuous over $[0, \bar{s}]$, where $\bar{s} > 0$ is the maximum score and may be $= \infty$, (ii) $F_k(0) = 0$ for some $k \in \{L, R\}$, and (iii) $\lim_{s \to \bar{s}} \{F_i(s)\} = 1\ \forall i$.

Now the proof proceeds in three steps. First we derive a pair of differential equation on the score CDF (F_i, F_{-i}) that must be satisfied. Next we prove that with symmetric entrepreneurs the CDFs must be identical $F_i = F_{-i} = F$ and derive a unique solution to the system. Finally we prove that the resulting strategies yield an equilibrium.

Step 1

Define the function $\Pi_i^*(s_i;F)$ over all scores $s_i \geq 0$ to be equal to:

$$\Pi_i^*(s_i;F) = -\alpha_i\left(s_i + [y_i^*(s_i)]^2\right) + F_{-i}(s_i)\cdot V_i(s_i, y_i^*(s_i)) + \int_{s_i}^{\infty} V_i\left(s_{-i}, y_{-i}^*(s_{-i})\right) dF_{-i}. \quad \text{(A.1)}$$

This is an entrepreneur's expected utility were she always to win at a score-tie (with her opponent or the reservation policy) with the policies $y_i^*(s_i)$ substituted in.

Let σ^* denote an equilibrium strategy profile, and U_i^* denote i's equilibrium utility in that profile. At an equilibrium profile there are no atoms above 0, so entrepreneur i's utility from developing the optimal policy at any score $s_i > 0$ is exactly equal to $\Pi_i^*(s_i;F)$. The statement is not necessarily true at $s_i = 0$, but i can achieve utility arbitrarily close to $\Pi_i^*(0;F)$ by developing a score ε above. Consequently $U_i^* \geq \Pi_i^*(s_i;F)$ for all $s_i \geq 0$. In addition, $\Pi_i^*(s_i;F) \geq U_i^*$ and thus $= U_i^*$ $\forall s_i \in [0,\bar{s}]$; if instead for some $s_i \in [0,\bar{s}]$ we had $U_i^* > \Pi_i^*(s_i;F)$ then by continuity of (F_i, F_{-i}) over $s_i > 0$ (and right continuity at 0) i would be developing scores with positive probability that yield strictly lower utility than her equilibrium utility, a contradiction.

Finally, since (F_i, F_{-i}) are also absolutely continuous and strictly increasing $\forall s \in [0,\bar{s}]$ by full support, $U_i^* = \Pi_i^*(s_i;F)$ $\forall s_i \in [0,\bar{s}] \iff \frac{\partial}{\partial s_i}(\Pi_i^*(s_i;F)) = 0$ for almost all $s \in [0,\bar{s}]$, which yields the pair of differential equations

$$\alpha_i - F_{-i}(s) = f_{-i}(s)\cdot 2x_i\left(\left(\frac{x_i}{\alpha_i}\right)F_{-i}(s) - \left(\frac{x_{-i}}{\alpha_{-i}}\right)F_i(s)\right). \quad \text{(A.2)}$$

Step 2

Substituting symmetric parameters (x,α) into (A.2) yields

$$\alpha - F_{-i}(s) = f_{-i}(s)\cdot 2\frac{x^2}{\alpha}\left(F_L(s) + F_R(s)\right) \text{ for almost all } s \in [0,\bar{s}] \text{ and } i \in \{L,R\}.$$

We first prove that a solution must be symmetric. The above implies $\frac{f_L(s)}{\alpha - F_L(s)} = \frac{f_R(s)}{\alpha - F_R(s)}$ a.e., which then $\rightarrow \exists C$ s.t. $\ln C - \ln(\alpha - F_L(s)) = -\ln(\alpha - F_R(s))$ a.e. and hence everywhere (since the F's are continuous), which then $\rightarrow \frac{\alpha - F_L(s)}{\alpha - F_R(s)} = C$ $\forall s \in [0,\bar{s}]$.

We now argue that $F_L(0) = F_R(0)$, which $\rightarrow C = 1 \iff F_L(s) = F_R(s) = F(s)$ $\forall s \in [0,\bar{s}]$. Observe that $\lim_{s\rightarrow\bar{s}}\left\{\frac{\alpha - F_L(s)}{\alpha - F_R(s)}\right\} = 1$ since $\lim_{s\rightarrow\bar{s}}\{F_i(s)\} = 1$ $\forall i$. But if there were some $k \in \{L,R\}$ with $F_k(0) > 0$, then Lemma 1 requires that $F_{-k}(0) = 0$, implying that $\frac{\alpha - F_L(s)}{\alpha - F_R(s)} = C \neq 1$ $\forall s \in [0,\bar{s}]$, a contradiction. Hence, the entrepreneurs must use identical score CDFs $F(s)$ with $F(0) = 0$. The ideologies $y_i(s) = sign(x_i)\cdot\frac{x}{\alpha}F(s)$ then follow from Lemma 1. The rest of the derivation is contained in the main text; also observe the differential equation on $F^{-1}(F)$ satisfies a Lipschitz condition and so has a unique solution.

Step 3

The strategies derived (a unique common score CDF and policies equal to $y_i^*(s) = \frac{x_i}{\alpha}F(s)$ with probability 1 since $F(0) = 0$) have been shown to be necessary for equilibrium. We now show they are an equilibrium and hence the unique one. First, observe that all policies $(s_i, y_i^*(s_i))$ s.t. $s_i \in (0, \bar{s}]$ yield utility equal to a $\hat{U}_i^*$ that is strictly higher than the utility from any other policy (s_i, y_i) with $s_i > 0$, since by construction $\Pi_i^*(s_i; F)$ is constant over $[0, \bar{s}]$ and strictly decreasing above (for $s_i > \bar{s}$, $\Pi_i^*(s_i; F) - \Pi_i^*(\bar{s}; F) = -(\alpha_i - 1)(s_i - \bar{s}) < 0$). This is also i's utility from playing her strategy. Second, at negative scores $s_i \leq 0$, developing a 0-quality policy (i.e. ideology $y_i = \pm\sqrt{-s_i}$) is strictly better than developing any other policy; $F(0) = 0$ implies neither entrepreneur ever ties with the reservation policy, so a weakly negative score policy with positive quality both always loses and never influences a tie. Finally, again since $F(0) = 0$ and is right continuous, negative-score 0-quality policies (including the reservation policy) yield $\Pi_i^*(0; F) = \hat{U}_i^*$. All policies thus yield the same or strictly less utility than $\hat{U}_i^*$ and we have an equilibrium. ■

Proof of Proposition 2 The CDF over ideological extremism $G(y) = \frac{y}{x/\alpha}$ is decreasing in x and increasing in α, showing the desired effects of ideology and costs. For the remaining CDFs we must work with their inverses. Observe that for a parameter p, $F^{-1}(F(s;p);p) = s \to \frac{\partial F}{\partial p} = -\frac{\partial F^{-1}/\partial p}{\partial F^{-1}/\partial F}$; so since a CDF and its inverse $\partial F^{-1}/\partial F$ are increasing functions, a CDF $F(s;p)$ is first order stochastically increasing in p i.f.f. its inverse is increasing in p. The inverse score CDF is $F^{-1}(F) = 4x^2\left(\ln\left(\frac{\alpha}{\alpha - F}\right) - \frac{F}{\alpha}\right)$ which is clearly increasing in x and decreasing in α, showing the desired comparative statics. Finally, the inverse quality CDF is $H^{-1}(H) = F^{-1}(H) + \left(H\frac{x}{\alpha}\right)^2$; both components are increasing in x and decreasing in α. ■

Proof of Proposition 3 To see that an entrepreneur's utility for her opponent's equilibrium policies is decreasing in their extremism when $\alpha \geq 3$, observe that i's utility when her opponent develops a policy of extremism y is $V_i(s(y), -sign(x_i)y) = -x^2 + s(y) - 2xy$; differentiating yields $s'(y) - 2x = \frac{4xy}{x-y} - 2x \leq 0$ i.f.f. $y \leq \frac{x}{3}$. Since ideological extremism is uniformly distributed over $\left[0, \frac{x}{\alpha}\right]$, the derivative is negative for all ideologies in the support i.f.f. $\alpha \geq 3$.

To show statements about the entrepreneurs' equilibrium utility, observe it is equal to

$$-\alpha\left(\bar{s} + (x/\alpha)^2\right) + V_i(\bar{s}, x_i/\alpha) = -(1 - 1/\alpha)x^2 - (\alpha - 1)\bar{s},$$

i.e., their utility from producing score $\bar{s}$ with ideology $\frac{x_i}{\alpha}$ and winning for sure. From this it immediately follows that each entrepreneur is strictly worse off with competition than as a monopolist, because her utility as a monopolist is $-\alpha(x/\alpha)^2 + V_i(0, x_i/\alpha) = -(1 - 1/\alpha)x^2$ and $\bar{s} > 0$ in equilibrium.

Next, substituting in $\bar{s} = F^{-1}(1) = 4x^2\left(\ln\left(\frac{\alpha}{\alpha-1}\right) - \frac{1}{\alpha}\right)$ and rearranging yields the entrepreneurs' equilibrium utility

$$-x^2(\alpha-1)\left(4\ln\left(\frac{\alpha}{\alpha-1}\right)-\frac{3}{\alpha}\right)$$

To show the remaining desired properties we show that this function is strictly single-troughed with minimum at $\hat{\alpha}>1$, approaches 0 as α approaches 1, is strictly less than $-x^2$ (their utility from the reservation policy) at $\hat{\alpha}$, and approaches $-x^2$ as $\alpha\to\infty$. Thus, cost increases benefit the entrepreneurs for $\alpha\geq\hat{\alpha}$ (and over this set they are strictly worse off with competition than just getting the reservation policy). In addition, there $\exists\bar{\alpha}\in(1,\hat{\alpha})$ s.t. their utility is $>-x^2$ for $\alpha\in(1,\bar{\alpha})$.

Writing utility as $-x^2 f(\alpha)$, where $f(\alpha)=(\alpha-1)\left(4\ln\left(\frac{\alpha}{\alpha-1}\right)-\frac{3}{\alpha}\right)$, we see that $\lim_{\alpha\to1^+}f(\alpha)=0$ and $\lim_{\alpha\to\infty}f(\alpha)=1$. Next, we show $\exists\alpha^*$ s.t. a) $f(\alpha)$ is strictly concave below α^*, b) $f(\alpha^*)>1$, and c) $f'(\alpha)<0$ for $\alpha\geq\alpha^*$. These properties imply that $f(\alpha)$ has a unique maximum $\hat{\alpha}\in(1,\alpha^*)$ and $f(\hat{\alpha})>1$. Finally, the preceding observations imply that $f(\alpha)=1$ at some $\bar{\alpha}<\hat{\alpha}$, and that $f(\alpha)<1$ for $\alpha<\bar{\alpha}$ and >1 for $\alpha>\bar{\alpha}$.

Property a) can be shown by taking the second derivative $f''(\alpha)$ and setting equal to 0; the solution is $\alpha^*=3$. For property b) just evaluate at α^*. Property c) can be shown by rearranging the first derivative to be

$$\frac{1}{\alpha^2}\left(4\alpha^2\ln\left(\frac{\alpha}{\alpha-1}\right)-(3+4\alpha)\right)=\frac{1}{\alpha^2}\left(\int_0^1\frac{4\alpha^2}{\alpha-q}dq-2\int_0^1(3+4\alpha)\,q\,dq\right)$$
$$=\frac{1}{\alpha^2}\left(\int_0^1\frac{-4\alpha^2-6\alpha+q(6+8\alpha)}{\alpha-q}dq\right).$$

The numerator is clearly <0 $\forall q\in[0,1]$ when $\alpha\geq\alpha^*=3$. ■

Proof of Proposition 4 First note that our arguments about the form of equilibria do not apply here since they are based on the assumption that $x_i,x_{-i}\neq0$. Now let $\Pi_i(s_i,y_i)$ denote i's expected utility from developing policy (s_i,y_i) (suppressing the dependence on the other players' strategies). Suppose $x_k=0$ and $x_{-k}\neq0$; then $V_k(s,y)=s$. Since k has the same utility as the DM she no longer cares about how score ties are broken, so it is simple to verify that her utility from developing any policy (s_k,y_k) is equal to,

$$\Pi_k(0,0)-\alpha_k\left(s_k+y_k^2\right)+F_{-k}(\max\{s_k,0\})\cdot(s_k-E[\max\{s_{-k},0\}\mid s_{-k}\leq s_k]).$$

The above is $=\Pi_k(0,0)$ for $s_k+y_k^2=0$ and $<\Pi_k(0,0)$ otherwise. So regardless of $-k$'s strategy, k's $0-$quality policies are all equivalent, and strictly dominate all other policies.

Now consider $-k$; if her opponent is developing only 0-quality (and thus ≤0 score policies) then she can win for sure with score ε and achieve utility arbitrarily close to her monopoly utility from developing $\left(0,\frac{x_{-k}}{\alpha_{-k}}\right)$. It is also simple to verify that this utility is strictly higher than that from developing any other policy. Thus, if the DM picks $\left(0, \frac{x_{-k}}{\alpha_{-k}}\right)$ with probability < 1 when developed, $-k$'s best-response correspondence is empty. Conversely, any strategy profile in which k mixes over 0-quality policies, $-k$ develops $\left(0, \frac{x_{-k}}{\alpha_{-k}}\right)$, and the DM chooses it with probability 1 is an equilibrium. (Note that unlike the baseline model, equilibrium in this variant requires specifying the DM's tie-breaking rule.) ■

Proof of Proposition 5 Begin with the two-player symmetric equilibrium. Observe that there are no atoms, and both entrepreneurs develop strictly positive-score policies with probability 1. Developing any 0-quality policy is therefore exactly equivalent, and strictly better than developing a weakly negative score policy with positive quality. We henceforth refer to developing a 0-quality policy as being "inactive" and a strictly positive-score policy as being "active." Also observe that since $F(0) = 0$, both entrepreneurs are active with probability 1, and would achieve their equilibrium utility by being inactive.

Now consider the game with N additional weakly more moderate and less skilled entrepreneurs, and the strategy profile in which the original two entrepreneurs keep their strategies from the two-player symmetric equilibrium, and the additional entrepreneurs are inactive. First observe that the payoffs to the two active entrepreneurs from developing any policy are unchanged from the original game (in theory the behavior of the additional inactive entrepreneurs could be used to break ties between the active entrepreneurs or the reservation policy, but with symmetric active entrepreneurs such ties never occur). Thus, the active entrepreneurs have no profitable deviation.

To see that the inactive entrepreneurs also have no profitable deviation, first observe that the net gain to an active entrepreneur i from producing $(s_i \geq 0, y_i)$ above inactivity is

$$-\alpha\left(s_i + y_i^2\right) + F(s_i) \cdot \left(\left(s_i - E\left[s_{-i} \,|\, s_{-i} \leq s_i\right]\right) + 2x\left(y_i + E\left[y(s_{-i}) \,|\, s_{-i} \leq s_i\right]\right)\right), \qquad \text{(A.3)}$$

where s_{-i} denotes the score of the other active entrepreneur. Because the active entrepreneurs' strategies are an equilibrium absent the other players, the maximum over all $(s_i \geq 0, y_i)$ is 0. To compute the net gain to an inactive entrepreneur j from producing $(s_j \geq 0, y_j)$ above inactivity, first observe that the expected ideological outcome conditional on both active entrepreneurs producing scores $\leq s_j$ is 0 by symmetry. Second, let $s_{\max}$ denote the maximum score policy developed; the probability that $s_{\max}$ is $\leq s_j$ is $[F(s_j)]^2$ where $F(\cdot)$ is the score CDF of the active entrepreneurs. Finally, $E\left[s_{-i} \,|\, s_{-i} \leq s_j\right] \leq E\left[s_{\max} \,|\, s_{\max} \leq s_j\right]$. Combining, the net gain to an inactive entrepreneur j from deviating to a policy $(s_j \geq 0, y_j)$ is

$$-\alpha_j\left(s_j + y_j^2\right) + \left[F(s_j)\right]^2 \cdot \left(\left(s_j - E\left[s_{\max} \,|\, s_{\max} \leq s_j\right]\right) + 2x_j y_j\right) \qquad \text{(A.4)}$$

By the previous observations, Equation A.3 is $\geq$ than Equation A.4 $\forall (s_j \geq 0, y_j)$ when $\alpha_j \geq \alpha$ and $|x_j| \leq x$; so the maximum net gain from deviating to activity for an inactive entrepreneur is ≤ 0, and we have an equilibrium. ■

Proof of Proposition 6 Before beginning the main proof we prove an accessory lemma.

Lemma A.1. *Consider a continuous function $h(x)$ that is almost-everywhere differentiable, and let $h^i(x)$ denote the i'th derivative of h (with $h^0(x) = h(x)$). Then the following two conditions imply that $h(x)$ is increasing in $x \geq 0$.*

1. $h^k(0) = 0$ $\forall$ *integer* $k \in [0, i]$, *and* $h^{i+1}(0) > 0$
2. $h(x) > 0 \to h'(x) > 0$ *wherever h is differentiable*

Proof. We first argue that property (1) implies $h(x) > 0$ in a neighborhood above 0. First observe that $sign\{h(x)\} = sign\left\{\frac{h(x)}{x^{i+1}}\right\}$ for $x > 0$. Next, repeated applications of L'Hopital's rule imply that $\lim_{x\to 0}\left\{\frac{h(x)}{x^{i+1}}\right\} = \frac{h^{i+1}(0)}{(i+1)!} > 0$, which then implies $h(x) > 0$ in a neighborhood above 0. Next, when the preceding holds then by property 2 $h(x)$ is also strictly increasing in that neighborhood almost everywhere; it therefore must remain positive and strictly increasing thereafter. ■

The proof now proceeds in two steps. In Step 1 we derive the unique symmetric equilibrium, and in Step 2 we show it has the desired properties.

Step 1

Beginning with the necessary conditions in Lemma 1, the proof proceeds identically as the proof of Proposition 1 through the end of Step 1, yielding a modified differential equation

$$\alpha_i - F_{-i}(s) = f_{-i}(s) \cdot 2x_i \left(\left(\frac{x_i}{\alpha_i}\right) F_{-i}(s) - \left(\frac{x_{-i}}{\alpha_{-i}}\right) F_i(s) + \beta\left(s + \left(\frac{x_{-i}}{\alpha_{-i}} F_i(s)\right)^2\right)\right). \quad \text{(A.5)}$$

that must be satisfied a.e. in the support $[0, \bar{s}]$ and generates a constant $\Pi_i^*(s_i; F)$ for $s \in [0, \bar{s}]$. Relative to Equation A.2, losing now entails an additional cost of $\beta\left(s + \left(\frac{x_{-i}}{\alpha_{-i}} F_i(s)\right)^2\right)$, which is the share of an opponent's quality that isn't valued.

In the baseline model we proved that the equilibrium with symmetric parameters is unique and symmetric; for this variant we make the weaker assertion that there is a unique symmetric equilibrium, but do not rule out other asymmetric equilibria. Substituting the symmetric parameters and common score CDF $F(s)$ yields the differential equation $\alpha - F(s) = f(s)\left(4\frac{x^2}{\alpha}F(s) + \beta\left(s + \left(\frac{x}{\alpha}F(s)\right)^2\right)\right)$. For notational simplicity we will henceforth use $s(F)$ to denote the inverse $F^{-1}(F)$, and again as in the argument for Proposition 1 we may derive a simpler differential equation for the inverse $s(F)$,

$$s'(F) = \frac{4\frac{x^2}{\alpha}F + \beta\left(s(F) + \left(\frac{x}{\alpha}F\right)^2\right)}{\alpha - F}. \quad \text{(A.6)}$$

The differential equation satisfies a Lipschitz condition and thus has a unique solution with the boundary condition $s(0)=0$ which is necessary. It can be verified that this solution is:

$$s(F;x,\alpha,\beta)=x^2\left(\frac{(8+6\beta)\cdot\left(\left(\frac{\alpha}{\alpha-F}\right)^{\beta}-1-\frac{\beta}{\alpha}F\right)-(1+\beta)\left(\frac{\beta F}{\alpha}\right)^2}{\beta(1+\beta)(2+\beta)}\right)$$

and yields a well defined $\bar{s}=s(1)<\infty\ \forall\beta>0$. Finally, the argument that these strategies are indeed an equilibrium is the same as in step 3 in the proof of Proposition 1. The fact that the distribution of ideologies is identical to the baseline model follows directly from Lemma 1, using the fact that in a symmetric equilibrium $y_i^*(s_i)=F_{-i}(s_i)\frac{x_i}{a_i}$ and $F(0)=0$.

Step 2

We show that $s(F;x,\alpha,\beta)$ is increasing in F, and $\forall F\in(0,1]$ is increasing in x and β, decreasing in α, and satisfies increasing differences in (x,β). By the arguments in the proof of Proposition 2 this suffices to show the desired first-order stochastic changes in both the score and quality CDFs from (x,β,α). It also suffices to show that the cross partial in (x,β) of the DM's equilibrium utility is positive, since (from footnote 4) the DM's equilibrium utility is $\int_0^1 2F\cdot s(F;x,\alpha,\beta)\,dF$, and the cross partial in (x,β) is $\int_0^1 2F\cdot\frac{\partial^2(s(F;x,\alpha,\beta))}{\partial x\partial\beta}dF>0$.

Despite a closed form solution, it is nevertheless easier to derive comparative statics directly from the differential equation in $(A.6)$. It will also be helpful to have the first, second, and third derivatives of $s(F)$ evaluated 0; repeated differentiation of $(A.6)$ and employing the boundary condition $s(0)=0$ yields that $s'(0)=0$, $s''(0)=4\frac{x^2}{\alpha^2}$, and $s'''(0)=\left(\frac{2x^2}{\alpha^3}\right)\cdot(3\beta+4)$.

We now show the desired properties of $s(F;x,\alpha,\beta)$ by repeatedly employing Lemma A.1.

- To see $s(F)$ increasing in F, observe that property 1 follows immediately from the derivatives evaluated at 0, and property 2 is easily verified from Equation $A.6$.
- To see $s(F;x)$ increasing in x for $F>0$, we show that $s(F;\hat{x})-s(F;x)$ is >0 $\forall\hat{x}>x$ and $F>0$ (it is equal to 0 for $F=0$ and any values of $x,\hat{x}$). To do this, we show the stronger property that $s(F;\hat{x})-s(F;x)$ is increasing in F by showing it satisfies properties 1 and 2 of Lemma A.1. Differentiating using Equation A.6 yields $s'(F;\hat{x})-s'(F;x)=\frac{1}{\alpha-F}\cdot\left(\left(\frac{4F}{\alpha}+\beta\frac{F^2}{\alpha^2}\right)(\hat{x}^2-x^2)+\beta(s(F;\hat{x})-s(F;x))\right)$, which is positive when $s(F;\hat{x})-s(F;x)>0$ and therefore satisfies property 2. Property 1 follows from observing (using the previously derived derivatives) that the first derivative of the difference at $F=0$ is 0 and the second derivative is $\frac{4}{\alpha^2}(\hat{x}^2-x^2)>0$.
- For identical reasons we show $s(F;\hat{\beta})-s(F;\beta)$ satisfies properties 1 and 2 for $\hat{\beta}>\beta\geq 0$. Differentiating using Equation A.6 yields $s'(F;\hat{\beta})-s'(F;\beta)=\frac{(\hat{\beta}-\beta)\left(\frac{x}{\alpha}F\right)^2+\left(\hat{\beta}s\left(F;\hat{\beta}\right)-\beta s(F;\beta)\right)}{\alpha-F}$.

To see the function satisfies property 2 observe that $\hat{\beta}s(F;\hat{\beta})-\beta s(F;\beta)>\hat{\beta}\left(s\left(F;\hat{\beta}\right)-s(F;\beta)\right)$. To see it satisfies property 1 observe that the first and second derivatives at 0 are 0, and the third derivative is $\left(\frac{6x^2}{\alpha^3}\right)\cdot\left(\hat{\beta}-\beta\right)>0$.

- We show $s(F;\alpha)-s(F;\hat{\alpha})$ satisfies properties 1 and 2 for $1\leq\alpha<\hat{\alpha}$. The function satisfies property 2 since $s'(F;\alpha)$ is increasing in $s(F;\alpha)$ and decreasing in α (holding $s(F;\alpha)$ fixed), so $s(F;x,\alpha)>s(F;x,\hat{\alpha})\to s'(F;x,\alpha)>s'(F;x,\hat{\alpha})$. It also satisfies property 1 since $s''(0;\alpha)-s''(0;\hat{\alpha})=4x^2\left(\frac{\hat{\alpha}^2-\alpha^2}{\alpha^2\hat{\alpha}^2}\right)>0$.

- We show that $(s(F;\hat{x},\hat{\beta})-s(F;x,\hat{\beta}))-(s(F;\hat{x},\beta)-s(F;x,\beta))$ satisfies properties 1 and 2 for $\hat{x}>x$ and $\hat{\beta}>\beta\geq0$. Differentiating using Equation A.6 yields,

$$\frac{1}{\alpha-F}\left(\hat{\beta}\left(s\left(F;\hat{x},\hat{\beta}\right)-s\left(F;x,\hat{\beta}\right)\right)-\beta\left(s\left(F;\hat{x},\beta\right)-s\left(F;x,\beta\right)\right)+(\hat{\beta}-\beta)\frac{F^2\left(\hat{x}^2-x^2\right)}{\alpha^2}\right).$$

This is $>\frac{\hat{\beta}}{\alpha-F}\left(\left(s\left(F;\hat{x},\hat{\beta}\right)-s\left(F;x,\hat{\beta}\right)\right)-\left(s\left(F;\hat{x},\beta\right)-s\left(F;x,\beta\right)\right)\right)$ so the function satisfies property 2. It also satisfies property 1 since first and second derivatives at 0 are 0, but the third derivative is $\left(\frac{6}{\alpha^3}\right)\cdot\left(\hat{x}^2-x^2\right)\cdot\left(\hat{\beta}-\beta\right)>0$. ■

Proof of Proposition 7 Suppose we are at an equilibrium strategy profile satisfying Lemma 1 and nobody engages in sabotage. Consider a deviation by i to a profile (s_i,y_i,q_i^s) with sabotage $q_i^s>0$. Then the probability that $-i$'s policy yields utility $\leq s$ is $F_{-i}(s+q_i^s)$. Since by Lemma 1 neither entrepreneur has an atom above 0 in equilibrium, with some sabotage an opponent's policy never ties with the reservation policy. So all the negative score 0 quality policies $s_i<0$, $y_i=\pm\sqrt{-s_i}$ are strictly better than the positive quality ones, and yield the same utility as the reservation policy with sabotage $(0,0,q_i^s)$. In addition, winning outright when $s_{-i}\leq s_i+q_i^s$ and $s_i=0$ is at least as good as tying with the reservation policy (since a tie generates a mixture between i's policy and the reservation policy). We therefore restrict attention to deviations with $q_i^s>0$ and $s_i>0$.

Now i's utility if she develops such a policy is,

$$-\alpha_i\left(s_i+y_i^2\right)-\alpha_i^s q_i^s+F_{-i}\left(s_i+q_i^s\right)V_i\left(s_i,y_i\right)+\int\limits_{s_{-i}>s_i+q_i^s}\left(V_i\left(s_{-i},y_{-i}\right)-\beta\left(s_{-i}+y_{-i}^2\right)-(1-\beta)\,q_i^s\right)d\sigma_{-i}.$$

If instead she reallocated her sabotage to productive effort $(s_i+q_i^s,y_i,0)$ then it is easy to verify that her net gain is weakly positive i.f.f.

$$F_{-i}\left(s_i+q_i^s\right)\cdot q_i^s+\left(1-F_{-i}\left(s_i+q_i^s\right)\right)\cdot(1-\beta)\,q_i^s\geq\left(\alpha_i-\alpha_i^s\right)q_i^s.$$

The inequality is most difficult to satisfy if $F_{-i}(s_i + q_i^s) = 0$ (since $1 - \beta \leq 1$), and holds in this case $\iff$ $\alpha_i^s \geq \alpha_i - (1-\beta)$. Thus, if this condition holds then some strategy without sabotage is at least as good as any strategy with sabotage, and the no sabotage equilibrium holds. Conversely, if this condition fails then the inequality fails for $F_{-i}(s_i + q_i^s)$ sufficiently small. Since by Lemma 1 $F_{-i}(0) = 0$ for some i in any equilibrium, there $\exists q_i^s$ and s_i sufficiently small s.t. every policy $(s_i + q_i^s, y_i, 0)$ with only productive effort (including the one in i's support) is strictly worse than the policy (s_i, y_i, q_i^s) with sabotage; hence she has a profitable deviation and no sabotage is not an equilibrium. ■

Proof of Proposition 8 In this variant, the definition of a score and the DM's best responses remain unchanged. With the entrepreneurs' modified utility function, $i's$ utility for a policy (s, y) being implemented is $V_i(s, y) = (s + y^2) + 1_{y=x_i} \cdot B|x_i|$. Observe that the entrepreneurs have dogmatic preferences over ideology, but nevertheless still value quality on their opponent's policy; the model therefore still has rank-order spillovers. We make this assumption to preserve comparability to the main model (although it stands to reason that a dogmatic entrepreneur would also discount her opponent's quality).

In Appendix C we prove the following lemma.

Lemma 2. *Developing a policy (s_i, y_i) with $s_i > 0$ and either (i) $y_i \neq x_i$, or (ii) $y_i = x_i$ and $F_{-i}(s_i) \leq \frac{\alpha_i |x_i|}{|x_i| + B}$, is strictly worse than developing (0,0). In addition, the support of the equilibrium score CDFs over $s \geq 0$ is common, convex, atomless above 0, and includes* 0.

Since the CDFs are atomless over $(0, \infty)$ and at such scores developing (s_i, x_i) is strictly better than any other policy, in equilibrium (i) the probability a policy (s_i, y_i) with $s_i > 0$ satisfies $y_i = x_i$ is 1, (ii) the CDFs are absolutely continuous over $[0, \bar{s}]$ with $\bar{s} > 0$ and satisfy $\lim_{s \to \bar{s}} \{F_i(s)\} = 1 \; \forall i$. The proof now proceeds in two steps. First, we derive an equilibrium and show it must yield utility at least as high as any other equilibrium. Second, we prove that the decisionmaker's utility in the equilibrium described satisfies the conditions of Proposition 8.

Step 1

First we argue that in any equilibrium the entrepreneurs use a common score CDF $F_{-i}(s) = F_i(s) = F(s)$ for $s \geq 0$. Since the CDFs have common convex support, either $F_i(0) = 1 \; \forall i$ (in which case the properties hold trivially), or $F_i(0) < 1 \; \forall i$. As in the proof of Proposition 1 define $\Pi_i^*(s_i; F)$ over all scores $s_i \geq 0$ as:

$$\Pi_i^*(s_i; F) = -\alpha_i \left(s_i + x_i^2\right) + F_{-i}(s_i) \cdot \left(s_i + x_i^2 + B|x_i|\right) + \int_{s_i}^{\infty} \left(s_{-i} + x_{-i}^2\right) dF_{-i}, \qquad \text{(A.7)}$$

and by identical arguments $\Pi_i^*(s_i; F) = U_i^* \; \forall s_i \in [0, \bar{s}]$. Substituting in the symmetric parameters and differentiating, this condition holds i.f.f. for almost all $s \in [0, \bar{s}]$,

$$\alpha - F_{-i}(s) = f_{-i}(s) \cdot Bx \quad \forall i. \tag{A.8}$$

The above implies that $c_i - \ln(\alpha - F_i(s)) = \frac{s}{Bx}$ $\forall s \in [0, \bar{s}]$ with $\bar{s} < \infty$; since $F_i(\bar{s}) = F_{-i}(\bar{s}) = 1$ we have $c_i = c_{-i}$ and $F_i(s) = F(s)$ $\forall s \in [0, \bar{s}]$, i.e., a common score CDF over this range. Inserting the starting value $F(0)$ yields $\ln\left(\frac{\alpha - F(0)}{\alpha - F(s)}\right) = \frac{s}{Bx}$ in any equilibrium. In addition, $F(0) \geq \min\left\{\frac{\alpha x}{x+B}, 1\right\}$; if instead $F(0) < \frac{\alpha x}{x+B}$ then $\bar{s} > 0$ and by Lemma 2 both entrepreneurs are developing policies with scores close to 0 that yield strictly less utility than developing the reservation policy, a contradiction.

Second we argue that for distinct equilibria yielding score CDFs $\left(F, \hat{F}\right)$, the DM's utility is strictly greater in the former i.f.f. $F(0) < \hat{F}(0)$. In any equilibrium $\ln\left(\frac{\alpha - F(0)}{\alpha - F(s)}\right) = \frac{s}{Bx}$ $\forall s \in [0, \bar{s}]$; so if there are two distinct equilibrium score CDFs $\left(F, \hat{F}\right)$ with $F(0) < \hat{F}(0)$ then it is easily verified that $\hat{F}(s) \geq F(s)$ $\forall s > 0$ with positive measure strict; so the DM's utility $\int_{s \geq 0} \frac{\partial}{\partial s}\left([F(s)]^2\right) ds$ is strictly higher with F than $\hat{F}$ (and equal if $F(0) = \hat{F}(0)$).

Finally we assert that the following symmetric strategies are an equilibrium (i) each entrepreneur develops either $(0,0)$ or (s, x_i) with $s > 0$, (ii) each entrepreneur uses the CDF $F(s) = 0$ for $s < 0$, $F(s)$ solving $\ln\left(\frac{\alpha - \frac{\alpha x}{x+B}}{\alpha - F(s)}\right) = \frac{s}{Bx}$ for $s \in \left[0, Bx \ln\left(\frac{\alpha - \frac{\alpha x}{x+B}}{\alpha - 1}\right)\right]$, and $F(s) = 1$ above. Using similar arguments to those in step 3 of the proof of Proposition 1, it is easily established that (i) all policies (s_i, x_i) s.t. $s_i \in (0, \bar{s}]$ yield utility equal to some value $\hat{U}_i^*$ that is strictly higher than the utility from any other policy (s_i, y_i) with $s_i > 0$, and (ii) at strictly negative scores $s_i < 0$ developing a 0-quality policy (i.e. ideology $y_i = \pm\sqrt{-s_i}$) is strictly better than developing any other policy, and yields the same utility as developing the reservation policy.[16] It is also easy to verify that developing the reservation policy yields $\hat{U}_i^*$; thus both entrepreneurs get $\hat{U}_i^*$ by playing their strategies.

It remains only to show that developing $(0, y_i)$ with $y_i \neq 0$ is not a profitable deviation. Since $-i$ develops (0,0) at 0, the utility from actually developing $(0, y_i)$ is weakly worse than the utility from developing it and always winning when $s_{-i} \leq 0$, which in turn is $\leq \Pi_i^*(0; F) = U_i^*$. All policies thus yield the same or strictly less utility than $\hat{U}_i^*$ and we have an equilibrium; since $F(0) = \frac{\alpha x}{\alpha + B}$, its lowest possible value, the DM's utility is at least as high as in any other equilibrium.

Step 2

We now show the decisionmaker's utility in the equilibrium characterized satisfies Prop. 8. If $\frac{\alpha x}{x+B} \geq 1$ then $F(0) = 1$ and the decisionmaker's equilibrium utility is 0. If $\frac{\alpha x}{x+B} < 1$

[16] The proof of Proposition 1 relies on $F_i(0) = 0$ $\forall i$; here we have "ties" at the 0-score but they are inconsequential because both entrepreneurs develop the reservation policy at this score.

then since the DM's utility $U(x)$ is $\int_{F(0)}^{1} \frac{\partial}{\partial F}\left(F^2\right) F^{-1}(F) \cdot dF$ we have

$$U(x) = 2Bx \cdot \int_{\frac{\alpha x}{x+B}}^{1} F \ln\left(\frac{\alpha - \frac{\alpha x}{x+B}}{\alpha - F}\right) dF.$$

Clearly this is equal to 0 at $x = 0$ and $\frac{B}{\alpha - 1}$, and positive in between. We now show that this expression is strictly quasi-concave (single peaked) in between, by showing that (i) $U'(0) > 0$, (ii) $U'\left(\frac{B}{\alpha-1}\right) = 0$, and (iii) $\exists x^* \in \left(0, \frac{B}{\alpha-1}\right)$ s.t. $U''(x) < (=)(>) 0 \iff x < (=)(>) x^*$. Then (iii) implies that U' is strictly single troughed and minimized at $x^* \in \left(0, \frac{B}{\alpha-1}\right)$, (ii) implies that $U'(x^*) < 0$, and (i) implies that $\exists \hat{x} \in (0, x^*)$ with $U' > 0$ for $x \in (0, \hat{x})$ and < 0 for $x \in \left(\hat{x}, \frac{B}{\alpha-1}\right)$, yielding strict single peakedness. The first derivative is

$$2B\left(\int_{\frac{\alpha x}{x+B}}^{1} F \ln\left(\frac{\alpha - \frac{\alpha x}{x+B}}{\alpha - F}\right) dF - \left(1 - \frac{\alpha x}{x+B}\right) \cdot \left(\frac{x}{x+B}\right)\right)$$

and it is easily verified that this satisfies (i) and (ii). The second derivative is,

$$2B\left(x\frac{(B - (\alpha - 1)x) + \alpha B}{(x+B)^3} - 2\left(1 - \frac{\alpha x}{x+B}\right) \cdot \left(\frac{1}{x+B}\right)\right),$$

which is $> (=) (<) \; 0$ i.f.f. $\frac{\alpha B x}{x+2B} > (=) (<) \; B - (\alpha - 1)x$. The l.h.s. is strictly increasing in x, the r.h.s. is strictly decreasing in x, and it is easily verified that l.h.s.<r.h.s. when $x = 0$ and l.h.s.>r.h.s. when $x = \frac{B}{\alpha-1}$. ■

B Complete Treatment of Baseline Model

We prove Lemma 1 as a sequence of four lemmas. From the main text, an entrepreneur's utility for her own policy is $V_i(s_i, y_i)$, while for her opponent's policy is $V_i(s_{-i}, y_{-i}) - \beta\left(s_{-i} + y_{-i}^2\right)$ (since she discounts its quality if $\beta \geq 0$). Let $\Pi_i(s_i, y_i; \sigma_{-i})$ denote i's expected utility from developing policy (s_i, y_i) (suppressing the dependence on the DM's tie-breaking rules). At any score $s_i > 0$ where $-i$ has no atom, this expected utility is equal to

$$-\alpha_i\left(s_i + y_i^2\right) + F_{-i}(s_i) \cdot V_i(s_i, y_i) + \int\limits_{s_{-i} > s_i} \left(V_i(s_{-i}, y_{-i}) - \beta\left(s_{-i} + y_{-i}^2\right)\right) d\sigma_{-i}. \tag{B.1}$$

Taking the first order condition with respect to y_i yields the following.

Lemma B.1. *At any score $s_i > 0$ where $-i$ has no atom, developing $(s_i, y_i^*(s_i))$ with $y_i^*(s_i) = F_{-i}(s_i) \cdot \frac{x_i}{a_i}$ is strictly better than developing any other policy.*

Next we show that in equilibrium there is 0 probability of a tie at a strictly positive score.

Lemma B.2. *In equilibrium there is 0-probability of a tie at scores $s > 0$.*

Proof: Suppose not, i.e., each player's strategy generates an atom of size p_i at some common $s > 0$. Let $\bar{y}_t$ denote the expected ideological outcome conditional on the tie at score s (which depends on both players' strategies and the decisionmaker's tiebreaking rule). Let $F_{-i}^-(s_i) = \lim\limits_{s \to s_i^-} \{F_{-i}(s)\}$, and let $w_i(y_i, y_{-i}; s)$ denote the probability the DM chooses i's policy when the entrepreneurs develop (s, y_i) and (s, y_{-i}).

Entrepreneur k's utility from playing according to her strategy conditional on a tie (which may involve mixing over ideologies) is

$$\begin{aligned}
&-\alpha_k Var\left[y_k \mid s\right] - \alpha_k\left(s + \left(E\left[y_k \mid s\right]\right)^2\right) + F_{-k}^-(s) \cdot V_k\left(s, E\left[y_k \mid s\right]\right) \\
&+p_{-k} \cdot \left(V_k(s, \bar{y}_t) - \beta \int\limits_{s_k = s} \int\limits_{s_{-k} = s} w_{-k}(y_k, y_{-k}; s) \cdot \left(s_{-k} + y_{-k}^2\right) \frac{d\sigma_k}{p_k} \frac{d\sigma_{-k}}{p_{-k}}\right) \\
&+ \int\limits_{s_{-k} > s} \left(V_k(s_{-k}, y_{-k}) - \beta\left(s_{-k} + y_{-k}^2\right)\right) d\sigma_{-k}
\end{aligned}$$

Since the entrepreneurs want to move ideology in strictly opposite directions conditional on a score, $V_k(s, 0) \geq V_k(s, \bar{y}_t)$ for at least one k. In addition, tying also involves a potential quality discount cost when $-k$'s policy wins. Rearranging, k's utility from playing according to her strategy conditional on a tie (which may involve mixing over ideologies), is $\leq$

$$
\begin{aligned}
&-\alpha_k \left(s + E\left[y_k^2 \,|\, s\right]\right) + F_{-k}^{-}(s) \cdot V_k(s, E[y_k \,|\, s]) \\
&+p_{-k} \cdot V_k(s,0) + \int\limits_{s_{-k}>s} \left(V_k(s_{-k}, y_{-k}) - \beta\left(s_{-k} + y_{-k}^2\right)\right) d\sigma_{-k}.
\end{aligned}
$$

Now let $\underline{y} = \lim\limits_{s_k \to s^-} \{y_k^*(s_k)\}$; by definition $E[y_k \,|\, s] = \underline{y}$ with zero variance maximizes the first line. Moreover since $\underline{y}$ is weakly better than 0 for k, the above is $\leq \lim_{s_k \to s^+} \left\{\Pi_k\left(s_k, \underline{y}; \sigma_{-k}\right)\right\}$; but this in turn is strictly $< \lim_{s_k \to s^+} \{\Pi_k(s_k, y_k^*(s); \sigma_{-k})\}$ (because $p_{-k} > 0$ implies $\underline{y} \neq \lim\limits_{s_k \to s^+} \{y_k^*(s_k)\}$, so tying must be strictly worse than just winning with $(s, y_k^*(s))$. ■

Having ruled out ties at strictly positive scores, we now show one of the entrepreneurs k must always be *active*, in the sense of developing a policy strictly better for the decisionmaker than the reservation policy ($F_k(0) = 0$ for some k). An immediate implication is that the decisionmaker is strictly better off with competition with probability 1.

Lemma B.3. *In equilibrium $F_k(0) = 0$ for some $k \in \{L, R\}$.*

Proof: Denote entrepreneur i's equilibrium utility as U_i^*, and suppose not, i.e., $F_i(0) > 0$ $\forall i$; this could be due to atoms at 0, developing scores lower than 0, or both. Let $\bar{y}_t$ denote the expected ideological outcome conditional on both players developing scores ≤ 0, which could be a complicated function of the players' strategies and the decisionmaker's tie-breaking rule. We will show equilibrium implies $\bar{y}_t = 0$, implying both entrepreneurs have a strict incentive to produce score ε and win with strictly positive probability bounded away from 0.

Entrepreneur i can achieve U_i^* by mixing according to her strategy conditional on generating score ≤ 0, and utility arbitrarily close to $\lim_{s_i \to 0^+} \{\Pi_i(s_i, 0; \sigma_{-i})\}$ by developing a policy with ε score and ideology at 0. Equilibrium thus requires that $U_i^* - \lim_{s_i \to 0^+} \{\Pi_i(s_i, 0; \sigma_{-i})\} \geq 0$. But also observe that

$$
U_i^* - \lim_{s_i \to 0^+} \{\Pi_i(s_i, 0; \sigma_{-i})\} \leq -\alpha_i \int\limits_{s_i \leq 0} \left(s_i + y_i^2\right) \frac{d\sigma_i}{F_i(0)} + F_{-i}(0) \cdot (V_i(0, \bar{y}_t) - V_i(0,0))
$$

(and the l.h.s. = r.h.s. when $\beta = 0$). Equilibrium thus requires that the r.h.s. be ≥ 0 for both entrepreneurs $\{i, -i\}$. But this then implies that both $\bar{y}_t \geq 0$ and $\bar{y}_t \leq 0$, so $\bar{y}_t = 0$ and $U_i^* = \lim_{s_i \to 0^+} \{\Pi_i(s_i, 0; \sigma_{-i})\}$. But then by Lemma B.1 $\lim_{s_i \to 0^+} \{\Pi_i(s_i, y_i^*(0); \sigma_{-i})\}$ is strictly higher for both entrepreneurs since $y_i^*(0) = F_{-i}(0) \cdot \frac{x_i}{\alpha_i} \neq 0$, and they each have a strict incentive to deviate to an ε-higher score and produce their optimal ideology. ■

Note that the decisionmaker's access to the reservation policy is irrelevant for the proof; in the competitive model what matters about the reservation policy is that it is "free" to develop. This contrasts with the monopoly model, where the policy that the DM can unilaterally implement matters crucially, because the monopolist behaves as an agenda setter.

Last, we show additional natural properties of the equilibrium score CDFs.

Lemma B.4. *The support of the equilibrium score CDFs over $s \geq 0$ is common, convex, and includes* 0*. In addition, both CDFs are atomless* $\forall s > 0$.

Proof: We first show that if $\hat{s} > 0$ is in the support of F_i then $F_{-i}(\hat{s}) - F_{-i}(\hat{s} - \varepsilon) > 0$ $\forall \varepsilon > 0$. Suppose not; then $\exists \varepsilon > 0$ such that $F_{-i}(s)$ is constant over $[\hat{s} - \varepsilon, \hat{s}]$ and $-i$ has no atom at $\hat{s} - \varepsilon$ or $\hat{s}$. Intuitively, this can't happen because i would be playing scores above $\hat{s} - \varepsilon$ without getting a higher probability of victory. Formally $\Pi_i(\hat{s}, y_i; \sigma_{-i}) - \Pi_i(\hat{s} - \varepsilon, y_i; \sigma_{-i}) = -(\alpha_i - F_{-i}(\hat{s})) \cdot \varepsilon < 0 \; \forall y_i$, implying by an envelope argument that i's utility from developing $(\hat{s} - \varepsilon, y^*(\hat{s} - \varepsilon))$ is strictly higher than developing $(\hat{s}, y^*(\hat{s}))$; if $\hat{s}$ were in the support she could do strictly better by deviating to $(\hat{s} - \varepsilon, y^*(\hat{s} - \varepsilon))$, a contradiction.

Now the preceding argument implies several of the desired properties. If the players' score CDFs did not have common support over $s > 0$, then one player would have support at a score where the other player's CDF is constant below, violating the condition. If the common support did not include 0 or were not convex, then there would exist a score $s'' > 0$ in the common support and a strictly lower score $s' \geq 0$ such that $F_i(s)$ was constant $\forall i$ over $[s', s'')$, at least one k had $F_k(s'') = F_k(s')$ (since both cannot have atoms at s'' by Lemma B.2), and the condition would again be violated.

Finally we show that no entrepreneur has an atom above 0 by contradiction. Suppose $-i$ has an atom at $\hat{s} > 0$ of size p_{-i}; then i does not (by Lemma B.2) which implies $E[y_{-i} \mid \hat{s}] = y^*_{-i}(\hat{s})$ (by Lemma B.1). By the argument in the preceding paragraph, i's support includes $[0, \hat{s}]$, which implies $F_i(\hat{s}) > 0$ and $y^*_{-i}(\hat{s}) \neq 0$. In addition, $\lim_{s_i \to \hat{s}^-} \{\Pi_i(s_i, y^*_i(s_i); \sigma_{-i})\} \geq U^*_i$ (since otherwise i would be putting positive probability on scores yielding strictly less utility than her equilibrium utility). Now let $\hat{y}_i = \lim_{s_i \to \hat{s}^-} \{y^*_i(s_i)\} \neq 0$, i.e., i's optimal ideology if she developed score $\hat{s}$ and expected to always lose a tie. It is easily verified that

$$\begin{aligned} & \lim_{s_i \to \hat{s}^+} \{\Pi_i(s_i, \hat{y}_i; \sigma_{-i})\} - \lim_{s_i \to \hat{s}^-} \{\Pi_i(s_i, y^*_i(s_i); \sigma_{-i})\} \\ = \;& p_{-i}\left(V_i(\hat{s}, \hat{y}_i) - V_i\left(\hat{s}, y^*_{-i}(\hat{s})\right) + \beta\left(\hat{s} + \left(y^*_{-i}(\hat{s})\right)^2\right)\right) > 0, \end{aligned}$$

meaning it would yield utility strictly higher than i's equilibrium utility to develop ideology $\hat{y}_i$ and score just above $\hat{s}$ to win for sure, which is a contradiction. ■

C Complete Treatment of Variant with Dogmatists

We prove Lemma 2 via two lemmas. Using previous notation, $\Pi_i(s_i, y_i; \sigma_{-i})$ for any $s_i > 0$ is

$$-\alpha_i\left(s_i + y_i^2\right) + F_{-i}^{-}(s_i)\cdot V_i(s_i, y_i) + \int\limits_{s_{-i}>s_i} V_i(s_{-i}, y_{-i})\, d\sigma_{-i}.$$

$$+\int\limits_{s_{-i}=s_i}\left(w_i(y_i, y_{-i}; s_i)\cdot V_i(s_i, y_i) + w_{-i}(y_i, y_{-i}; s_i)\cdot V_i(s_i, y_{-i})\right) d\sigma_{-i} \tag{C.1}$$

With some manipulation this may be rewritten as,

$$\begin{aligned}
&\Pi_i(0,0;\sigma_{-i}) - (\alpha_i - F_{-i}(s_i))\, s_i - (\alpha_i - F_{-i}(s_i))\, y_i^2 + F_{-i}(s_i)\, B\,|x_i| \cdot 1_{y_i=x_i}\\
&-\int\limits_{s_{-i}=s_i}\left(w_{-i}(y_i, y_{-i}; s_i)\cdot V_i(s_i, y_i)\right) d\sigma_{-i} - \int\limits_{s_{-i}=s_i}\left(w_i(y_i, y_{-i}; s_i)\cdot V_i(s_i, y_{-i})\right) d\sigma_{-i}\\
&-\int\limits_{s_{-i}=0} w_{-i}(0, y_{-i}; 0)\cdot V_i(0, y_{-i})\, d\sigma_{-i} - \int\limits_{s_{-i}\in(0,s_i)} V_i(s_{-i}, y_{-i})\, d\sigma_{-i}.
\end{aligned} \tag{C.2}$$

This is i's utility $\Pi_i(0,0;\sigma_{-i})$ from developing the reservation policy, plus a sequence of terms. The remaining terms in the first line are i's policy utility (net of costs) if she were to always win when $s_{-i} \leq s_i$. The negative terms in the second line arise from the fact that $-i$ may have an atom at s_i; the first term captures the fact that i sometimes loses to $-i$ when they tie at $s_{-i} = s_i$ and thus doesn't get $V_i(s_i, y_i)$, while the second term nets off the foregone utility from sometimes defeating $-i's$ policies with score $s_{-i} = s_i$. The third line nets off the foregone policy utility from defeating $-i$'s policies with score $s_{-i} \in [0, s_i)$; since $V_i(s, y) > 0\ \forall s > 0$ or $s = 0, y \neq 0$, the second and third lines are weakly negative. It is then straightforward to show the following.

Lemma C.1. *Developing a policy* (s_i, y_i) *with* $s_i > 0$ *and either (i)* $y_i \neq x_i$*, or (ii)* $y_i = x_i$ *and* $F_{-i}(s_i) \leq \frac{\alpha_i |x_i|}{|x_i|+B}$*, is strictly worse than developing the reservation policy.*

Proof: For $s_i > 0$, it is easy to verify from Equation C.2 that $-(\alpha_i - F_{-i}(s_i))\cdot y_i^2 + F_{-i}(s_i)\, B\,|x_i| \cdot 1_{y_i=x_i} \leq 0 \rightarrow \Pi_i(s_i, y_i; \sigma_{-i}) < \Pi_i(0,0;\sigma_{-i})$; this is exactly the case when $y_i \neq x_i$ or $y_i = x_i$ and $F_{-i}(s_i) \leq \frac{\alpha_i |x_i|}{|x_i|+B}$. ■

Using this, we prove the remaining parts of Lemma 2.

Lemma C.2. *The support of the equilibrium score CDFs over* $s \geq 0$ *is common, convex, and includes* 0*. In addition, both CDFs are atomless* $\forall s > 0$*.*

Proof: We first rule out ties at scores $s > 0$. Suppose not so that both entrepreneurs have an atom at $s > 0$ in equilibrium; by Lemma C.1 they must be developing (s, x_i) and

(s, x_{-i}), so $\lim_{s_i \to s^+} \{\Pi_i (s_i, x_i; \sigma_{-i})\} - \Pi_i (s_i, x_i; \sigma_{-i}) = p_{-i}(s) \cdot B |x_i| \cdot w_{-i}(x_i, x_{-i}; s)$ (where $p_i(s)$ denotes the size of i's atom at s). Since $w_i + w_{-i} = 1$ this is > 0 for at least one entrepreneur k, who has a strict incentive to deviate and produce ε-higher score.

Next, it is simple to verify using identical arguments as in the proof of Lemma B.4 that $\hat{s} > 0$ in support of $F_i \to F_{-i}(\hat{s}) - F_{-i}(\hat{s} - \varepsilon) > 0 \ \forall \varepsilon > 0$, which then implies support over $s \geq 0$ is common, convex, and includes 0. Last we rule out atoms above 0 by contradiction. Suppose $-i$ has an atom at $\hat{s} > 0$ of size $p_{-i}(\hat{s})$; by Lemma C.1 she develops $(\hat{s}, x_{-i})$. By preceding arguments i's support includes $[0, \hat{s}]$ so $\lim_{s_i \to \hat{s}^-} \{\Pi_i (s_i, x_i; \sigma_{-i})\} \geq U_i^*$, but

$$\lim_{s_i \to \hat{s}^+} \{\Pi_i (s_i, x_i; \sigma_{-i})\} - \lim_{s_i \to \hat{s}^-} \{\Pi_i (s_i, x_i; \sigma_{-i})\} = p_{-i}(\hat{s}) \cdot B |x_i| > 0,$$

so i can do strictly better than her equilibrium utility by developing $(s_i + \varepsilon, x_i)$ for sufficiently small ε, a contradiction. ■