Online Appendix to "The Limits of Price Discrimination"

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This Appendix provides a formal analysis of the example of Section IV. We observe that optimal menus have $q_2 = 1$, and thus we define the quantity of low type for simplicity as $q \triangleq q_1$, and as individual rationality binds for the low type, it must be that $p_1 = q + \epsilon q (1-q)$. The benefit of the high type from pretending to be the low type is thus:

$$2q + \epsilon q (1-q) - p_1 = q.$$

Since the high type has a binding incentive constraint, his payment is simply $p_2 = 2 - q$. When there is a proportion $x_1 \triangleq x$ of low types, profits when allocating q to the low type are:

$$(q + \epsilon q (1-q)) x + (2-q)(1-x).$$

Differentiating with respect to q, the first order condition is:

$$0 = (1 + \epsilon (1-2q)) x - (1-x) \Leftrightarrow q(x) = \frac{1}{2} - \frac{1-2x}{2\epsilon x}.$$

If this number is in $[0,1]$, then the optimal menu sells this quantity to the low type. If it is negative, for which the condition is:

$$\frac{1}{2} \leq \frac{1-2x}{2\epsilon x} \Leftrightarrow x \leq \frac{1}{2+\epsilon},$$

then since profits are concave in q, it is optimal to exclude the low type with an allocation of $q = 0$. Note that when $x < \frac{1}{2}$, excluding the low type also yields a strictly higher payoff than pooling.

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Similarly, if the solution to the first-order condition is greater than 1, for which the condition is:

$$-\frac{1}{2} \geq \frac{1-2x}{2\epsilon x} \Leftrightarrow x \geq \frac{1}{2-\epsilon},$$

then it is optimal to pool the high type and the low type at the efficient output. Note that when $x > \frac{1}{2}$, pooling yields strictly higher profit than exclusion. Hence, output is:

$$q(x) = \begin{cases} 0, & \text{if} \quad 0 \leq x < \frac{1}{2+\epsilon}; \\ \frac{1}{2} - \frac{1-2x}{2\epsilon x}, & \text{if} \quad \frac{1}{2+\epsilon} \leq x < \frac{1}{2-\epsilon}; \\ 1, & \text{if} \quad \frac{1}{2-\epsilon} \leq x \leq 1. \end{cases}$$

Producer surplus is given by:

$$\pi(x) = q(x)(1+\epsilon(1-q(x)))x + (2-q(x))(1-x);$$

and consumer surplus is given by:

$$u(x) = (1-x)q(x),$$

which are illustrated in Figure 7.

We will solve for the surplus set when the aggregate market is $x^* = \frac{1}{2}$. We can write:

$$w_\lambda(x) \triangleq \lambda\pi(x) + u(x)$$

for $\lambda \in \mathbb{R}$. The support function of the surplus set at a given direction $(1, \lambda)$ is given by the concavification of $w_\lambda(x)$ at x^*. Similarly, the support function at directions $(-1, -\lambda)$ is given by the concavification of $-w_\lambda(x)$ at x^*.

Eastern Frontier The concavification of w_λ at x^* falls into four "regimes", defined relative to three cutoff values of λ which are $\underline{\lambda} < \widehat{\lambda} < \overline{\lambda}$. We depict examples for each of these regimes in Figure 1. For each of four values of λ, we plot scaled producer surplus, consumer surplus, the sum, as well as their respective concavifications. The plots are scaled to show the relative magnitudes of $\lambda\pi$ and u, with λ decreasing as we progress downwards through the figure. For λ extremely large, i.e., $\lambda \in (\overline{\lambda}, \infty)$, we are close to maximizing π. This is accomplished by perfect price discrimination, in which the market is segmenting into $x = 0$ and $x = 1$, as depicted in the top row of Figure 1. For this range of λ, the extreme point of the welfare set is the efficient point where $u = 0$, which the northernmost point in Figure 8.

As λ decreases, consumer surplus becomes more important, and the concavification of

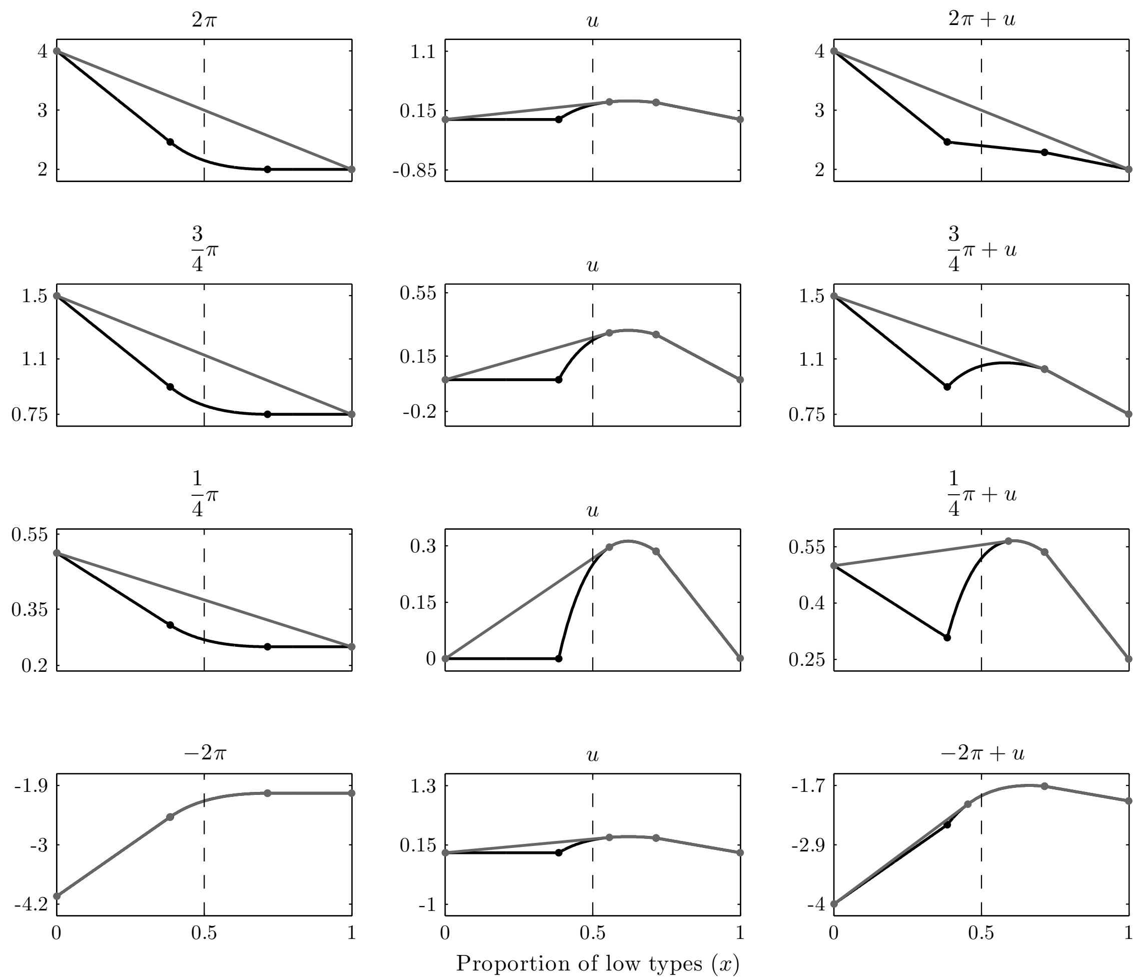

Figure 1: Examples for the concavification along the eastern frontier.

w_λ comes closer to the line segment between $x = 1/(2-\epsilon)$ and $x = 1$, where both u and π are linear. At a critical value $\overline{\lambda}$, the two coincide, in particular when:

$$\frac{1}{2-\epsilon} w_\lambda(1) + \left(1 - \frac{1}{2-\epsilon}\right) w_\lambda(0) = w_\lambda\left(\frac{1}{2-\epsilon}\right).$$

The solution to this equation is given by $\overline{\lambda} = 1$. At this point, the concavification at $x = 1/2$ becomes the line segment connecting $(0, w_\lambda(0))$ and $(1/(2-\epsilon), w_\lambda(1/(2-\epsilon)))$. In other words, for the next range of directions, it is optimal to segment between markets with $x = 0$ and $x = 1/(2-\epsilon)$. This case is depicted in the second row of Figure 1, where $\lambda = 3/4$. Note that as λ decreases, this corresponds to the direction we are maximizing in rotating clockwise from due north. At some point, the optimum switches from the northernmost point to the easternmost point on the efficient frontier in Figure 8.

As λ decreases further towards zero, $w_\lambda(0) = 2\lambda$ falls relative to w_λ in $[1/(2+\epsilon), 1/(2-\epsilon)]$. Eventually, the tangent between $(0, w_\lambda(0))$ and the graph of w_λ moves from being at $1/(2-\epsilon)$ to a point in $[1/2, 1/(2-\epsilon)]$. The tangent just moves to the left of $1/(2-\epsilon)$ when:

$$\frac{1}{2-\epsilon} \lim_{x \uparrow \frac{1}{2-\epsilon}} w'_\lambda(x) = \left(w_\lambda\left(\frac{1}{2-\epsilon}\right) - w_\lambda(0) \right).$$

The solution is:

$$\widehat{\lambda} = \frac{3}{2} - \frac{1}{2\epsilon}.$$

In this regime, the concavification at $x = 1/2$ is given by the line connecting $(0, w_\lambda(0))$ and $(x(\lambda), w_\lambda(x(\lambda)))$, where:

$$x\, w'_\lambda(x) = w_\lambda(x) - w_\lambda(0) \Leftrightarrow x(\lambda) = \frac{2-\lambda}{3+\epsilon(1-\lambda)-2\lambda}.$$

This is the case in the third row of Figure 1, where $\lambda = 1/4$. For λ in this range, we trace out the curved portion of the eastern-southeastern frontier of Figure 8. The final cutoff $\underline{\lambda}$ is the solution to:

$$\frac{1}{2} w'_\lambda\left(\frac{1}{2}\right) = w_\lambda\left(\frac{1}{2}\right) - w_\lambda(0) \Leftrightarrow \underline{\lambda} = 1 - \frac{1}{\epsilon}.$$

It is at this point that the tangent point from $(0, w_\lambda(0))$ to the graph of w_λ moves to the left of $1/2$, so that the concavification of w_λ at $x = 1/2$ is just $w_\lambda(1/2)$, which is true for all $\lambda \in (-\infty, \underline{\lambda})$. In this range, the weight on minimizing producer surplus is so large that the solution is no segmentation, as in the bottom row of Figure 1. For these values, the direction $(1, \lambda)$ points southerly enough that the optimum is no information, i.e., the southern corner of Figure 8.

Western Frontier For the western frontier, we find the concavification of $-w_\lambda(x)$ at $x = 1/2$ for $\lambda \in \mathbb{R}$. As before, there are four regimes, and we depict examples in Figure 2. For λ sufficiently large, again we are close to minimizing π, and no segmentation is optimal, as in the top row. Note that large λ corresponds to maximizing a direction $(-1, -\lambda)$ close to due south, so again we are at the southern corner of Figure 8.

As λ decreases, the weight on u becomes relatively large compared to the weight on π, and eventually the concavification at $x = 1/2$ switches to the tangent line between $(1, w_\lambda(1)) =$

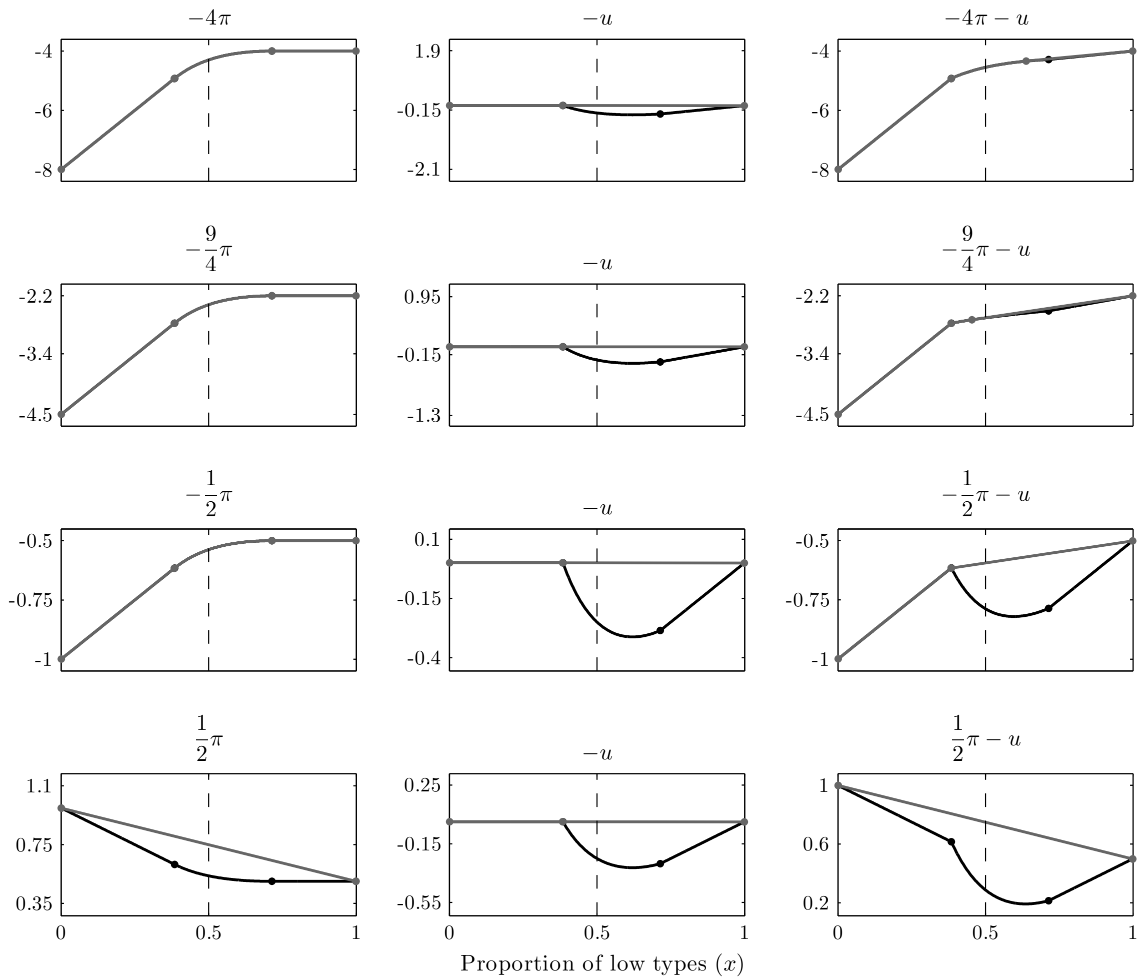

Figure 2: Examples for the concavification along the western frontier.

$(1, \lambda)$ and the graph of $w_\lambda(x)$. Let $x(\lambda)$ again denote the point of tangency, which solves:

$$(1-x)\,w'_\lambda(x) = 2\lambda - w_\lambda(x) \implies x(\lambda) = 1/\left(1 - \frac{\sqrt{(1-\epsilon(6-\epsilon))\,\lambda(\lambda-2)}}{\lambda - 2}\right).$$

The critical $\underline{\lambda}$ occurs when $x(\lambda) = 1/2$, which is:

$$\underline{\lambda} = \frac{2}{\epsilon(6-\epsilon)}.$$

As λ increases above $\underline{\lambda}$, $x(\lambda)$ decreases from $1/2$ until it eventually hits $1/(2+\epsilon)$. The critical λ is:

$$\widehat{\lambda} = \frac{(1+\epsilon)^2}{4\epsilon}.$$

For $\lambda \in \left[\underline{\lambda}, \widehat{\lambda}\right]$, the solution is to segment using markets $x(\lambda)$ and $x = 1$, as in the second row of Figure 2. Thus, in Figure 8, there is in fact a subtle curve to the southwestern frontier as $x(\lambda)$ moves smoothly from 1/2 to $1/(2+\epsilon)$. At $\widehat{\lambda}$, the regime changes to segmenting between $x = 1/(2+\epsilon)$ and $x = 1$, as in the third row of Figure 2. This generates the southernmost point along the western frontier where $u = 0$. This continues until we hit $\overline{\lambda}$ at which:

$$w_\lambda \left(\frac{1}{2+\epsilon}\right) = \frac{1}{2+\epsilon} w_\lambda (1) + \left(1 - \frac{1}{2+\epsilon}\right) w_\lambda (0),$$

which occurs precisely at $\overline{\lambda} = 0$, when we are minimizing consumer surplus. Finally, for $\lambda \in (\overline{\lambda}, \infty)$, when we have negative weight on consumer surplus and a non-negative weight on producer surplus, the optimum is again perfect price discrimination, and we are back to the northernmost corner of Figure 8.

Limit as $\epsilon \to 0$ This characterization of the frontier can be used to study the limit of the surplus set as $\epsilon \to 0$. In particular, we know that for ϵ sufficiently small, there are directions for which segmenting $x^* = 1/2$ into $\{0, 1\}$, $\{0, 1/(2-\epsilon)\}$, and $\{1/(2+\epsilon), 1\}$ are respectively optimal.

For every ϵ, segmentation into $\{0, 1\}$ induces consumer surplus of zero and producer surplus equal to 3/2, so this is also the limit as $\epsilon \to 0$.

In the limit as $\epsilon \to 0$, the segmentation of 1/2 into $\{0, 1/(2-\epsilon)\}$ puts probability approaching 1 on the segment with $x = 1/(2-\epsilon)$. In this segment, the monopolist is indifferent between screening and selling the efficient quantity to both types at a price of 1. We break indifference in favor of the latter so that the monopolist receives a profit of 1 and the allocation is efficient. Asymptotically, this segment dominates so that total surplus is efficient and profit is $\pi^* = 1$.

In the limit as $\epsilon \to 0$, the segmentation of 1/2 into $\{1/(2+\epsilon), 1\}$ puts probability approaching 1 on the segment with $x = 1/(2+\epsilon)$. In this segment, the monopolist only sells to the high type and garners a profit of $2(1-\epsilon)/(2+\epsilon)$, which is converging to 1 as $\epsilon \to 0$. In the limit, consumer surplus is 0 and profit is $\pi^* = 1$.

Finally, we briefly comment on what happens in the limit for $x^* \neq 1/2$. If $x^* > 1/2$, then for ϵ sufficiently small it is optimal to pool in the aggregate market, so that no segmentation exactly achieves point C in Figure 1. Point D can be attained by segmenting into $\{1/(2+\epsilon), 1\}$, and having the monopolist "pool" in the market with $x = 1$ and exclude in the market with $1/(2+\epsilon)$ and sell only to the high types. The weight on the market with $x = 1/(2+\epsilon)$ is $(1-x^*)(2+\epsilon)/(1-\epsilon)$, so that the monopolist's profit is:

$$\frac{2+\epsilon}{1-\epsilon}(1-x^*)2\frac{1}{2+\epsilon} + 1 - \frac{2+\epsilon}{1-\epsilon}(1-x^*),$$

which approaches 1 as $\epsilon \to 1$. Similarly, if $x^* < 1/2$, then it is optimal to exclude in the aggregate market for ϵ sufficiently small, so no segmentation achieves D. And by a similar calculation as before, segmentation into $\{0, 1/(2-\epsilon)\}$ achieves point C in the limit.

Thus, the three extreme points of the surplus triangle are attained in the limit, and by convexity the entire triangle must be attained as well. We can then summarize these results as follows (and as reflected in Figure 8).

Proposition A1 (Surplus Set Close to Linearity)
In the linear-quadratic model (7), the set of attainable profit and consumer surplus pairs converges to the surplus triangle of the uniform price monopoly as $\epsilon \to 0$.