# Online Appendix

"Price Reaction to Information with Heterogeneous Beliefs and Wealth Effects: Underreaction, Momentum, and Reversal"
by Marco Ottaviani and Peter Norman Sørensen

**Proof of Proposition 8**

We verify that the described outcome is an equilibrium. For the final equilibrium condition, note that the market clears because trader positions are the same as in the static equilibrium. The remainder of the proof verifies that this constant position is indeed optimal in the individual dynamic optimization problem.

Let $\Delta x_{it}(L_t)$ denote the contingent net position of trader $i$ in period $t$ after information realization $L_t$. By convention, $\Delta x_{i0} = 0$. The trader's wealth evolves randomly over time as $w_{it}(L_t) = w_{it-1}(L_{t-1}) + (p_t(L_t) - p_{t-1}(L_{t-1}))\Delta x_{it-1}(L_{t-1})$ for $t = 1, \ldots, T+1$, with $w_{i0} > 0$ given as before. If constrained, the trader's net position choice at $t-1$ must satisfy $\Delta x_{it-1}(L_{t-1}) \in [-w_{it-1}(L_{t-1})/(1 - p_{t-1}(L_{t-1})), w_{it-1}(L_{t-1})/p_{t-1}(L_{t-1})]$.

Suppose at period $t$, information $L_t$ has been realized. To save notation, write $p_t$ for the realization of $p_t(L_t)$ and $w_{it}$ for $w_{it}(L_t)$. Two observations are essential. First, $\Delta x_{it}$ is at the upper bound (interior, lower bound) of the constraint set $[-w_{it}/(1-p_t), w_{it}/p_t]$ if and only if, for all $L_{t+1}$, $\Delta x_{it+1}$ is on the upper bound (interior, lower bound) of constraint set $[-w_{it+1}(L_{t+1})/(1 - p_{it+1}(L_{t+1})), w_{it+1}(L_{t+1})/p_{it+1}(L_{t+1})]$. Second, for all realizations of the string $(L_{t+1}, \ldots, L_T)$, the feasible choice $\Delta x_{iT}(L_T) = \ldots = \Delta x_{it+1}(L_{t+1}) = \Delta x_{it}$ implies

$$\frac{u_i'(w_{iT}(A))}{u_i'(w_{iT}(A^c))} = \frac{u_i'(w_{it} + (1-p_t)\Delta x_{it})}{u_i'(w_{it} - p_t \Delta x_{it})}.$$

Both observations follow from the wealth evolution equation $w_{i\tau}(L_t) = w_{i\tau-1}(L_{\tau-1}) + (p_\tau(L_\tau) - p_{\tau-1}(L_{\tau-1}))\Delta x_{i\tau-1}(L_{\tau-1})$ for periods $\tau = t+1, \ldots, T$.

To prove our claim that the trader in every period selects the same position $\Delta x_{it} = \Delta x_{i1}(L_1)$ as in the static model given price $p_1(L_1)$, we proceed by backwards induction. The induction hypothesis $t$ states that the agent in period $t$ given price $p_t(L_t)$ (i) chooses $\Delta x_{it}$ to satisfy the static first-order condition

$$\frac{p_t(L_t)}{1-p_t(L_t)} = \frac{\pi_i(L_t)}{1-\pi_i(L_t)} \frac{u_i'(w_{it}(L_t) + (1-p_t(L_t))\Delta x_{it})}{u_i'(w_{it}(L_t) - p_t(L_t)\Delta x_{it})}$$

if feasible, or (ii) chooses $\Delta x_{it} = w_{it}(L_t)/p_t(L_t)$ if the left-hand side of this static condition is below the right-hand side at this choice, and (iii) chooses $\Delta x_{it} = -w_{it}(L_t)/(1-p_t(L_t))$ if the left-hand side of this static condition exceeds the right-hand side at this choice.

Note from the previous two essential observations, that once we have proved the induction hypothesis for all $t$, we have $\Delta x_{iT}(L_T) = \ldots = \Delta x_{i1}(L_1)$, and $\Delta x_{i1}(L_1)$ is the solution to the individual problem in Proposition 4.

The induction hypothesis $T$ is satisfied because the static first-order condition characterizes the solution to the remaining one-period problem. We now assume that the induction hypothesis is true at $t+1, \ldots, T$, and will prove that induction hypothesis $t < T$ is true. Suppose at period $t$, information $L_t$ is realized. Final wealth levels are then

$$w_{iT}(A) = w_{it} + (p_{t+1}(L_{t+1}) - p_t)\Delta x_{it} + (1 - p_{t+1}(L_{t+1}))\Delta x_{it+1}(L_{t+1})$$

and

$$w_{iT}(A^c) = w_{it} + (p_{t+1}(L_{t+1}) - p_t)\Delta x_{it} - p_{t+1}(L_{t+1})\Delta x_{it+1}(L_{t+1})$$

where $\Delta x_{it+1}(L_{t+1})$ is the reaction prescribed by induction hypothesis $t+1$. The time $t$ problem is

$$\max_{\Delta x_{it} \in [-w_{it}/(1-p_t), w_{it}/p_t]} \pi_i(L_t) E[u_i(w_{iT}(A)) | A] + (1 - \pi_i(L_t)) E[u_i(w_{iT}(A^c)) | A^c]$$

where the expectations are taken over the realization of $L_{t+1}$. In case (i), the static first-order condition can be satisfied with an interior choice of $\Delta x_{it}$. Evaluated at this choice, the derivative of the time $t$ objective function is, by the envelope theorem,

$$\begin{aligned} & \pi_i(L_t) E[(p_{t+1}(L_{t+1}) - p_t) u_i'(w_{iT}(A)) | A] \\ & + (1 - \pi_i(L_t)) E[(p_{t+1}(L_{t+1}) - p_t) u_i'(w_{iT}(A^c)) | A^c] \\ = \; & p_t E\left[\tfrac{\pi_i(L_t) u_i'(w_{iT}(A))}{p_t} (p_{t+1}(L_{t+1}) - p_t) | A\right] \\ & + (1 - p_t) E\left[\tfrac{(1-\pi_i(L_t)) u_i'(w_{iT}(A^c))}{1-p_t} (p_{t+1}(L_{t+1}) - p_t) | A^c\right]. \end{aligned}$$

Here $w_{iT}(A)$ and $w_{iT}(A^c)$ are constant across realizations of $L_{t+1}$. The static first-order condition then allows us to rewrite the derivative with respect to the control variable as

$$\frac{\pi_i(L_t) u_i'(w_{iT}(A))}{p_t} \{p_t E[p_{t+1}(L_{t+1}) - p_t | A] + (1 - p_t) E[(p_{t+1}(L_{t+1}) - p_t) | A^c]\}.$$

By the martingale property of Bayes-updated prices at market belief $p_t$, we have

$$p_t E[p_{t+1}(L_{t+1}) - p_t | A] + (1 - p_t) E[(p_{t+1}(L_{t+1}) - p_t) | A^c] = 0.$$

Thus the first-order condition for optimality of $\Delta x_{it}$ is satisfied at the choice resulting from the static model. The other two cases (with constrained choices) follow likewise.