Claim Validation

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Online Appendix

This section is organized as follows. In subsection 7.1 we study more in detail the definition of claim validation and show how to construct measures that violate this property. Subsection 7.2 contains different characterizations of definition 1. In Subsection 7.3 we provide extensions of Theorems 1 and 2. Finally, in subsection 7.4 we consider an alternative approach to defining claim validation.

## 3.1 Claim Validation

### 3.1.1 Guides to the future

Let a *guide for the future* be a function

$$f : \{\Omega\} \cup \bigcup_{t=1}^{\infty} \{0,1\}^t \longrightarrow [0,1]$$

which takes as input the observed data $\omega^t$ (or, at period 0, the empty history $\{\Omega\}$) and returns as an output the probability of observing outcome 1 in the next period. As is well known, given a path $\omega = (\omega_1, ..., \omega_n, ...)$ a guide to the future $f$ induces a unique probability $P_f(\omega^t)$ for each cylinder $\omega^t$, defined as

$$P_f\left(\omega^t\right) = \prod_{n=1}^{t} f(\omega|n)^{\omega_n} (1 - f(\omega|n))^{1-\omega_n}$$

where $\omega|1 = \{\Omega\}$ and $\omega|n = (\omega_1, ..., \omega_{n-1})$ for all $n \geq 2$. We denote by $E(f)$ the set of all measures $P$ that satisfy $P(\omega^t) = P_f(\omega^t)$ for every cylinder $\omega^t$. This is the set of all measures extending the guide to the future $f$. The definition of claim validation can be restated as follows.

**Remark 3** For every guide to the future $f$, a measure $P \in E(f)$ satisfies claim validation if and only if for every path $\omega \in \Omega$,

$$\lim_{t\to\infty} P_f\left(\omega^t\right) > 0 \implies \lim_{t\to\infty} P\left(\{\omega\}\,|\omega^t\right) = 1. \tag{1}$$

Consider the case where $\omega = (1, 1, ...)$. If $\lim_{t\to\infty} P_f\left(\omega^t\right) > 0$ then Bob's predictions do not rule out the possibility that indefinitely many white swans will appear. Whether the evidence can convince Bob the law "all swans are white" is correct will depend on whether the extension $P$ satisfies $\lim_{t\to\infty} P\left(\{\omega\}\,|\omega^t\right) = 1$.

The definition of claim validation can be satisfied vacuously when $\inf_t P_f\left(\omega^t\right) = 0$ for every $\omega$ (e.g. if $f$ is equal to $\frac{1}{2}$ for every cylinder). In this case, we say that $f$ *rules out deterministic laws.*

### 3.1.2 A characterization

We now provide a characterization of the set of measures that satisfy claim validation. Let $\tilde{\mathcal{A}}$ be the smallest algebra containing all cylinders and all singletons $\{\omega\}$, $\omega \in \Omega$.

**Theorem 4** Let $\mathcal{A} = \tilde{\mathcal{A}}$. Given a guide to the future $f$ and for each path $\omega$, let $p_\omega \in [0, 1]$ satisfy $p_\omega \leq P_f\left(\omega^t\right)$ for every $t$. There exists a unique measure $P \in E\left(f\right)$ such that $P\left(\{\omega\}\right) = p_\omega$ for all $\omega \in \Omega$. It satisfies claim validations if and only if $p_\omega = \lim_{t\to\infty} P_f\left(\omega^t\right)$ for every $\omega \in \Omega$.

The results show that a measure $P$ on $\tilde{\mathcal{A}}$ can be defined by first considering a guide for the future $f$ and then choosing the ex-ante probability of each claim $\omega$ (i.e. $\left(p_\omega\right)_{\omega\in\Omega}$ ). As shown in the proof of Theorem 4, once the probabilities of all singletons are fixed, the odds of all events in the algebra are derived by elementary computation.

**Theorem 5** Let $\mathcal{A}$ be an algebra such that $\tilde{\mathcal{A}} \subseteq \mathcal{A}$. Every measure $P$ defined on $(\Omega, \tilde{\mathcal{A}})$ can be extended to a measure $Q$ defined on $(\Omega, \mathcal{A})$.

By construction, the extension $Q$ validates claim if and only if the original measure $P$ validates claims. Therefore, the characterization above applies to any algebra $\mathcal{A}$ containing all cylinders and all singletons.

If $f$ does not rule out deterministic laws then there exists at least one path $\omega$ such that $\lim_{t\to\infty} P_f(\omega^t) > 0$. In this case, depending on how the probabilities $(p_\omega)_{\omega\in\Omega}$ are chosen, the extension $P$ may or may not satisfy claim validation.

**Corollary 6** Let $f$ be a guide to the future that does not rule out deterministic claims. There exist an extension $P \in E(f)$ that satisfies claim validation and an extension $Q \in E(f)$ that does not satisfy claim validation.

## 3.2 Equivalent definitions

The idea of claim validation can be formalized in different, equivalent, ways. Suppose that after observing several consecutive white swans, all swans appearing in an increasingly large number of future periods are predicted to be white. A measure validates claims if and only if, whenever the latter is true, the statement that "all swans are white" eventually becomes a virtual certainty. In order to formalize this idea, call a sequence of natural numbers $(\kappa_t)_{t=1}^{\infty}$ an *horizon* if it is increasing and unbounded.

**Theorem 7** A measure $P \in \Delta$ validates claims if and only if for every path $\omega \in \Omega$, whenever

$$\lim_{t\to\infty} P\left(\omega^{t+\kappa_t} | \omega^t\right) = 1 \text{ for every horizon } (\kappa_t)_{t=1}^{\infty} \tag{2}$$

then

$$\lim_{t\to\infty} P\left(\{\omega\} \mid \omega^t\right) = 1.$$

The next result provides a different characterization of claim validation.

**Theorem 8** A measure $P \in \Delta$ validates claims if and only if for every path $\omega$,

$$P(\{\omega\}) > 0 \implies \lim_{t\to\infty} P\left(\{\omega\} | \omega^t\right) = 1 \tag{3}$$

and

$$P(\{\omega\}) = 0 \implies \lim_{t\to\infty} P\left(\omega^t\right) = 0. \tag{4}$$

So, by (3), a claim is eventually seen as a virtual certainty if evidence in its favor accumulates, provided that the claim was initially deemed possible. By (4), if a claim $\omega$ is deemed impossible then overwhelming evidence in its favor ($\omega^t$, where $t$ is large) is considered unlikely.

**Corollary 9** A measure $P \in \Delta$ validates claims if and only if for every path $\omega$,

$$P(\{\omega\}) > 0 \implies \lim_{t\to\infty} P\left(\{\omega\} \,|\omega^t\right) = 1$$

and if $P(\{\omega\}) = 0$ there exists an horizon $(\kappa_t)_{t=1}^{\infty}$ such that $\limsup_t P\left(\omega^{t+\kappa_t}|\omega^t\right) < 1$.

Hence, claim validation rules out the case in which a claim is denied as impossible and predictions are eventually made as if the denied claim were true.

## 3.3 Extensions

### 3.3.1 Deterministic laws

Let $\Delta_+(\Omega)$ be the set of measures $P$ for which there exists at least one path $\omega$ such that $\lim_{t\to\infty} P(\omega^t) > 0$. These measures are extensions of guides to the future that do not rule out deterministic claims. The main results in the paper continue to hold if we restrict the attention to measures in $\Delta_+(\Omega)$.

**Theorem 10** Consider the case where Bob can announce any measure in $\Delta_+(\Omega)$. Let $T$ be a test that $\Delta_+(\Omega)$ −controls for Type-I errors with probability $1-\epsilon$. The test $T$ can be manipulated with probability $1-\epsilon-\delta$, for every $\delta \in (0, 1-\varepsilon]$.

**Theorem 11** Fix $\epsilon \in (0, 1]$. Consider the case where Bob's measure is required to validate claims. There exists a rejection test $T$ that $\Delta_+(\Omega) \cap \Delta_v(\Omega)$ −controls for Type-I error with probability $1-\epsilon$ and is non-manipulable.

The proof of Theorem 10 follows the proof of Theorem 1. Any measure $P$ in $\Delta(\Omega)$ can be approximated by a measure $\tilde{P}$ belonging to $\Delta_+(\Omega)$. This implies that test $T$ which $\Delta_+(\Omega)$-control for type-I error can be passed with probability close to

$1-\epsilon$, according to $P$, by reporting the measure $\tilde{P}$. Thus, from the perspective of a strategic expert the test $\Delta(\Omega)$-controls for type-I error with probability $1-\epsilon$. It then follows as a corollary of Theorem 1 that the test is manipulable. Theorem 10 is an immediate consequence of Theorem 2.

### 3.3.2 Random Tests

The conclusions of Theorem 1 continue to hold if Bob does not know for sure what test he will face, but knows the odds according to which Alice will choose her test. Let $\mathbb{T}$ be the set of all tests, and denote by $\Delta_o\mathbb{T}$ the collection of all probability measures with finite support defined on $\mathbb{T}$. We say that $\mu$ *prevents Type-I error with probability* $1-\epsilon$ if

$$\sum_{T\in\mathbb{T}} \mu(T)\, P(\{\omega : T(\omega, P) = \text{pass}\}) \geq 1-\epsilon \text{ for every } P \in \Delta(\Omega).$$

That is, under $\mu$ the average probability of passing the test is larger than $1-\epsilon$.[1] For the next result, given a randomization $\mu$ and a strategy $\zeta$, we denote their product by $\mu \otimes \zeta$ .

**Theorem 12** Let $\mu$ prevent Type-I error with probability $1-\epsilon$. For every $\delta > 0$ there exists a strategy $\zeta$ such that

$$(\zeta \otimes \mu)\{(Q,T) : T(\omega, Q) = \textit{pass}\} \geq 1-\epsilon-\delta$$

for every $\omega \in \Omega$.

Given the randomization $\mu$, there exists a strategy $\zeta$ which guarantees Bob an high probability of passing the test.

[1] It is convenient to restrict the attention to randomizations with finite support. Without this assumption, the order of integration will matter for the definition of Type-I error.

### 3.3.3 Strategies

In this subsection we consider the possibility for the expert to randomize according to more general mixed strategies. We first consider the case where a strategy is any $\sigma$-additive probability measure on $\Delta(\Omega)$. We endow $\Delta(\Omega)$ with the Borel $\sigma$-algebra induced by the weak* topology and denote by $\Delta_\sigma\Delta(\Omega)$ the set of $\sigma$-additive measures defined on $\Delta(\Omega)$. A test $T$ is *measurable in* $\Delta(\Omega)$ if for every $\omega \in \Omega$ the set $\{P : T(\omega, P) = \text{pass}\}$ is Borel.

Given a path $\tilde{\omega} \in \Omega$, recall that a measure $P$ *validates claim* $\tilde{\omega}$ if $\lim_{t\to\infty} P\left(\tilde{\omega}^t\right) > 0$ implies $\lim_{t\to\infty} P\left(\tilde{\omega}|\tilde{\omega}^t\right) = 1$. Let $\Delta_{\tilde{\omega}}(\Omega)$ be the set of measures that validate claim $\tilde{\omega}$.

**Theorem 13** Fix $\epsilon \in (0, 1]$. Consider the case where Bob's measure is required to validate claim $\tilde{\omega}$. There exists a rejection test $T$ that $\Delta_{\tilde{\omega}}(\Omega)-$controls for Type-I error with probability $1 - \epsilon$, is measurable in $\Delta(\Omega)$ and such that for every $\zeta \in \Delta_\sigma\Delta(\Omega)$ and every $\delta > 0$ *t*here exists a cylinder $C_{\delta,\zeta}$ such that for every path $\omega \in C_{\delta,\zeta}$,

$$\zeta\left(\left\{P \in \Delta(\Omega) : \omega \in A^P\right\}\right) \leq \delta.$$

If Bob is restricted to measures that validate claim $\tilde{\omega}$, then there exists a test such that for some realization of the data the probability of passing the test is arbitrarily small. So, the result confirms the conclusions of Theorem 2 (and follows a similar proof).

For completeness, we also consider the case where the expert can randomize according to any finitely additive probability. Denote by $\Delta_o(\Omega)$ the set of probabilities on $\Omega$ with finite support. Let $\Delta\Delta_o(\Omega)$ be the set of finitely additive probabilities defined on $\Delta_o(\Omega)$, when the latter is endowed with the discrete $\sigma$-algebra.

**Theorem 14** Let $T$ be a test measurable in $\Delta(\Omega)$ that $\Delta_o(\Omega)-$controls for Type-I errors with probability $1 - \epsilon$. There exists a strategy $\zeta \in \Delta\Delta_o(\Omega)$ such that

$$\zeta\left(\left\{P \in \Delta_o(\Omega) : \omega \in A^P\right\}\right) \geq 1 - \epsilon$$

for every $\omega \in \Omega$.

If the expert can randomize according to a finitely additive probability then any test that controls for Type-I error for probabilities with finite support can be manipulated. Because measures with finite support validate claims, the result shows that under this more permissive notion of mixed strategy claim validation does not allow to discredit strategic experts. The result is based on Fan's minmax Theorem, with one important difference with respect to Theorem 2. In Theorem 2 the conditions of Fan's minmax theorem are satisfied because of the weak* compactness of $\Delta(\Omega)$. In the proof of Theorem 14, it is the weak* compactness of $\Delta\Delta_o(\Omega)$ which plays a crucial role.

## 3.4 An Alternative Approach

In this subsection we provide an alternative approach to defining claim validation, based on the characterization of Theorem 8. In this alternative approach, property (4) is a background assumption, and a measure validates claims if it satisfies (3) for every path.

### 3.4.1 Natural tests

The property

$$\text{for all } \omega, \text{ if } P(\{\omega\}) = 0 \text{ then } \lim_{t\to\infty} P\left(\omega^t\right) = 0 \tag{5}$$

has an appeal that is independent of strategic considerations. Let $\omega = (1, 1, ...)$ be the claim "all swans are white". It is natural to consider a test $T_\omega$ such that if Bob considers the claim $\omega$ to be impossible, then there exists a long enough sequence of white swans after which he is rejected. Formally, the test $T_\omega$ is such that for every measure $P$ for which $P(\{\omega\}) = 0$ there exist a time $t_P$ such that $\omega^{t_P} \subseteq R^P$. For this test to have arbitrarily small Type-I error with respect to a measure $P$ such that $P(\{\omega\}) = 0$, $P$ must satisfy $P(\omega^t) \to 0$ as $t \to \infty$. Thus, (5) allows for natural ways to test and reject Bob if he asserts that some claim $\omega$ is impossible.

### 3.4.2 Claim validation

The relationship between (5) and claim validation can be further clarified if Alice has a probabilistic belief about at least some aspects of the problem at hand. Fix a path $\omega$ and let Alice be endowed with a probability measure $P_a$ defined on the algebra generated by the collection of events $\{\Omega, \{\omega\}, \{\omega^t : t > 0\}\}$. The idea is that Alice has, compared to Bob, a very limited theory about how future data will evolve.

Given Alice's measure $P_a$, a basic property for a test to satisfy is that for every measure $P$ of Bob that satisfies $P(\{\omega\}) = 0$ there exists a time $t_P$ such that

$$P_a\left(\{\omega\} | \omega^{t_P}\right) > 1 - \epsilon \text{ and } \omega^{t_P} \subseteq R^P.$$

That is, if Bob claims that the claim $\omega$ is impossible and Alice becomes convinced the claim is indeed true, then Bob is discredited. For the test to have arbitrarily small type-I error with respect to a measure $P$ such that $P(\{\omega\}) = 0$, it must be that $P$ satisfies $P(\omega^t) \to 0$ as $t \to \infty$. Therefore, property (5) allows to construct test that, from Alice's perspective, are consistent with claim validation.

### 3.4.3 Results

For the next result, denote by $\Delta_*(\Omega)$ the set of measures that satisfy property (5).

**Theorem 15** Consider the case where Bob can announce any measure in $\Delta_*(\Omega)$. Let $T$ be a test that $\Delta_*(\Omega)$ −controls for Type-I errors with probability $1 - \epsilon$. The test $T$ can be manipulated with probability $1 - \epsilon - \delta$, for every $\delta \in (0, 1 - \varepsilon]$.

Restricting the attention to measures that satisfy the background assumption (5) does not affect the conclusion of Theorem 1. Conversely, suppose that Bob is required to announce measures in $\Delta_*(\Omega)$ that also satisfy (5). In this case, Theorem 8 implies that Bob is restricted to announce measures that satisfy claim validation (i.e. that satisfy Definition 1). Hence, by Theorem 2, Bob can be discredited by a non-manipulable test.

# 4 Appendix

**Proof of Theorem 4.** Let $P_f$ be the probability induced by $f$ on $(\Omega, \mathcal{F})$. By Lemma 1, $E \in \tilde{\mathcal{A}}$ if and only if there exists a (unique) event $F_E \in \mathcal{F}$ such that $E \triangle F_E$ is finite. Define a function $P : \tilde{\mathcal{A}} \to [0,1]$ as

$$P(E) = P_f(F_E) + \sum_{\omega \in E \setminus F} p_\omega - \sum_{\omega \in F \setminus E} p_\omega \tag{6}$$

for every $E \in \tilde{\mathcal{A}}$.

We now show that $P$, as defined in (6), is additive. Let $E_1, E_2 \in \tilde{\mathcal{A}}$ be disjoint. By definition,

$$P(E_1 \cup E_2) = P_f(F_{E_1 \cup E_2}) + \sum_{\omega \in E \setminus F} p_\omega - \sum_{\omega \in F \setminus E} p_\omega$$

Moreover, $F_{E_1 \cup E_2} = F_{E_1} \cup F_{E_2}$ and $F_{E_1} \cap F_{E_2} = \emptyset$ (see footnote 5). Therefore $P_f(F_{E_1 \cup E_2}) = P_f(F_{E_1}) + P_f(F_{E_2})$. Let $E = E_1 \cup E_2$ and $F = F_{E_1} \cup F_{E_2}$. Simple computations imply

$$\begin{aligned}
\sum_{\omega \in E \setminus F} p_\omega - \sum_{\omega \in F \setminus E} p_\omega &= \sum_{\omega \in E_1 \setminus F_{E_1}} p_\omega - \sum_{\omega \in E_1 \cap F_{E_1}^c \cap F_{E_2}} p_\omega + \sum_{\omega \in E_2 \setminus F_{E_2}} p_\omega - \sum_{\omega \in E_2 \cap F_{E_2}^c \cap F_{E_1}} p_\omega \\
&\quad - \sum_{\omega \in F_{E_1} \setminus E_1} p_\omega + \sum_{\omega \in F_{E_1} \cap E_1^c \cap E_2} p_\omega - \sum_{\omega \in F_{E_2} \setminus E_2} p_\omega + \sum_{\omega \in F_{E_2} \cap E_2^c \cap E_1} p_\omega \\
&= \sum_{\omega \in E_1 \setminus F_{E_1}} p_\omega + \sum_{\omega \in E_2 \setminus F_{E_2}} p_\omega - \sum_{\omega \in F_{E_1} \setminus E_1} p_\omega - \sum_{\omega \in F_{E_2} \setminus E_2} p_\omega \\
&= \sum_{\omega \in E_1 \setminus F_{E_1}} p_\omega - \sum_{\omega \in F_{E_1} \setminus E_1} p_\omega + \sum_{\omega \in E_2 \setminus F_{E_2}} p_\omega - \sum_{\omega \in F_{E_2} \setminus E_2} p_\omega
\end{aligned}$$

Hence $P(E_1 \cup E_2) = P(E_1) + P(E_1)$.Therefore $P$ is a measure (and it belongs to $E(f)$).

It follows from the proof of Theorem 2 that $P$ validates claims if and only it satisfies $P(\omega^t) \to P(\{\omega\})$ for every $\omega$. Equivalently, $P$ validates claims if and only if $\lim_{t\to\infty} P_f(\omega^t) = p_\omega$ for every $\omega$. ■

**Proof of Theorem 5.** See, e.g., Corollary 3.3.4 in Rao and Rao (1983). ■

**Proof of Corollary 6.** By Theorem 5, it is enough to consider the case where $\mathcal{A} = \tilde{\mathcal{A}}$. Let $f$ and $\tilde{\omega}$ be such that $\lim_{t\to\infty} P(\tilde{\omega}^t) > 0$. Let $P$ be an extension such that $\lim_{t\to\infty} P_f(\omega^t) = P(\omega^t)$ for every $\omega$. Then $P$ validates claims. Conversely, fix $p_{\tilde{\omega}} \in [0,1]$ such that $p_{\tilde{\omega}} < \inf_t P(\tilde{\omega}^t)$. By Theorem 4 we can find an extension $Q \in E(f)$ such that $Q(\{\tilde{\omega}\}) = p_{\tilde{\omega}}$. By construction, it does not validate claims. ■

**Proof of Theorem 7.** Let $P$ be a measure that validates claims, and fix $\omega \in \Omega$. From the proof of Theorem 2, we know that $P$ satisfies $P(\omega^t) \to P(\{\omega\})$. Hence, for every $t$

$$P\left(\omega^{t+n}|\omega^t\right) = \frac{P(\omega^{t+n})}{P(\omega^t)} \to \frac{P(\{\omega\})}{P(\omega^t)} = P\left(\{\omega\}|\omega^t\right) \text{ as } n \to \infty \tag{7}$$

Now suppose (7) holds with respect to $\omega$. By (7), for each $t$ we can choose $\kappa_t$ large enough such that

$$\left|P\left(\{\omega\}|\omega^t\right) - P\left(\omega^{t+\kappa_t}|\omega^t\right)\right| < 2^{-t}$$

for every $t$ and the resulting sequence $(\kappa_t)$ is increasing and unbounded. By assumption, $\lim_{t\to\infty} P(\omega^{t+\kappa_t}|\omega^t) = 1$. Hence $\lim_{t\to\infty} P(\{\omega\}|\omega^t) = 1$.

Conversely, let $P$ be such that for each path $\omega$, if (7) holds then $\lim_{t\to\infty} P(\{\omega\}|\omega^t) = 1$. We now prove $P$ validates claims. Let $\omega \in \Omega$ be such that $\delta = \lim_{t\to\infty} P(\omega^t) > 0$. We have $P(\omega^t) = \delta + \xi_t$ for some sequence $(\xi_t)_{t=1}^{\infty}$ such that $\xi_t \downarrow 0$. For every $t$ and $\kappa \in \mathbb{N}$,

$$P\left(\omega^{t+\kappa}|\omega^t\right) = \frac{P(\omega^{t+\kappa})}{P(\omega^t)} = \frac{\delta + \xi_{t+\kappa}}{\delta + \xi_t} \geq \frac{\delta}{\delta + \xi_t}$$

In particular, for every horizon $(\kappa_t)_{t=1}^{\infty}$

$$P\left(\omega^{t+\kappa_t}|\omega^t\right) \geq \frac{\delta}{\delta + \xi_t}$$

therefore $\lim_{t\to\infty} P\left(\omega^{t+\kappa_t}|\omega^t\right) = 1$. Therefore (7) holds. Hence $\lim_{t\to\infty} P\left(\{\omega\}|\omega^t\right) = 1$. We can conclude that $P$ validates claims. ■

**Proof of Theorem 8.** A measure $P$ validates claims if and only if for every path $\omega$ it satisfies $P\left(\omega^t\right) \to P\left(\{\omega\}\right)$. The latter is equivalent to (9) in the case where $P\left(\{\omega\}\right) = 0$ and, by Bayes' rule, is equivalent to (8) in the case $P\left(\{\omega\}\right) > 0$. ■

**Proof of Corollary 9.** Given Theorem 8, it is enough to prove that for a measure $P$ that satisfies (9) and a path $\omega$ such that $P\left(\{\omega\}\right) = 0$, $\lim_{t\to\infty} P\left(\omega^t\right) = 0$ if and only if there exists an horizon $(\kappa_t)$ such that $\limsup_{t\to\infty} P\left(\omega^{t+\kappa_t}|\omega^t\right) < 1$. Suppose $\lim_{t\to\infty} P\left(\omega^t\right) = 0$. Then, for every $t$, $\lim_{n\to\infty} P\left(\omega^{t+n}|\omega^t\right) = 0$. Hence, for every $t$, we can inductively choose a number $\kappa_t$ such that the resulting sequence $(\kappa_t)$ is increasing and the sequence $\left(P\left(\left(\omega^{t+\kappa_t}|\omega^t\right)\right)\right)$ is bounded away from 1.

Conversely, assume there exists an horizon $(\kappa_t)$ such that $\limsup_t P\left(\omega^{t+\kappa_t}|\omega^t\right) < 1$. Suppose, by way of contradiction, that $\delta = \lim_{t\to\infty} P\left(\omega^t\right) > 0$. By replicating the argument in the proof of Theorem 7 we can conclude that $\lim_{t\to\infty} P\left(\omega^{t+\kappa_t}|\omega^t\right) = 1$. A contradiction. Therefore, $\lim_{t\to\infty} P\left(\omega^t\right) = 0$. ■

**Proof of Theorem 10.** Consider a test $T$ that $\Delta_+(\Omega)$-prevents Type-I error with probability $1-\epsilon$. Fix $\alpha \in (0, 1-\epsilon)$ and $\tilde{\omega} \in \Omega$. For every $P \in \Delta(\Omega)$ define the measure $\tilde{P} = \alpha\delta_{\tilde{\omega}} + (1-\alpha) P$. Then $\tilde{P} \in \Delta_+(\Omega)$ and

$$P(A^{\tilde{P}}) \geq \tilde{P}(A^{\tilde{P}}) - \alpha \geq 1 - \epsilon - \alpha$$

The rest of the proof now follows the proof of Theorem 1, with one difference. The function $V$ now satisfies

$$\begin{aligned}
\min_{P\in\Delta(\Omega)} \sup_{\zeta\in\Delta_o\Delta(\Omega)} V\left(P,\zeta\right) &\geq \min_{P\in\Delta(\Omega)} V\left(P,\delta_{\tilde{P}}\right) \\
&= \min_{P\in\Delta(\Omega)} P\left(A^{\tilde{P}}\right) \\
&\geq 1 - \epsilon - \alpha
\end{aligned}$$

It follows from Fan's Minmax Theorem that for every $\delta > 0$ there exists a strategy $\zeta$ such that $\zeta\left(\left\{Q : \omega \in A^Q\right\}\right) \geq 1 - \epsilon - \alpha - \delta$. Because $\alpha$ and $\delta$ can be chosen to be arbitrarily small, the proof is concluded. ■

**Proof of Theorem 12.** Consider the function

$$V(P, \zeta) = \sum_{Q \in \Delta(\Omega)} \sum_{T \in \mathbb{T}} \zeta(Q)\, \mu(T)\, P\{\omega : T(\omega, Q) = \text{pass}\}$$

As in the proof of Theorem 1, we can apply Fan's Minmax Theorem and conclude that for every $\delta > 0$ there exists $\zeta \in \Delta_f \Delta(\Omega)$ such that $V(\delta_\omega, \zeta) \geq 1 - \epsilon - \delta$ for every $\omega \in \Omega$. Equivalently,

$$(\zeta \otimes \mu)\{(Q, T) : T(\omega, Q) = \text{pass}\} = \sum_{Q \in \Delta(\Omega)} \sum_{T \in \mathbb{T}} \zeta(Q)\, \mu(T)\, 1_{\{(Q,T):T(\omega,Q)=\text{pass}\}}(T, Q) \geq 1 - \epsilon - \delta$$

for every $\omega \in \Omega$, where $1_{\{(Q,T):T(\omega,Q)=\text{pass}\}}$ is the indicator function of the set $\{(Q, T) : T(\omega, Q) = \text{pass}\}$. ■

**Proof of Theorem 13.** Fix a path $\tilde{\omega} \in \Omega$. Let $\mathcal{B}$ be the Borel $\sigma$-algebra generated by the weak* topology. For each $\xi \in [0, 1]$ and each $A \in \mathcal{A}$, the event $\{P \in \Delta(\Omega) : P(A) > \xi\}$ belongs to $\mathcal{B}$. We first show that $\Delta_{\tilde{\omega}}(\Omega) \in \mathcal{B}$. Let $E_\omega = \left\{P \in \Delta(\Omega) : \lim_{t \to \infty} P\left(\tilde{\omega}^t\right) > 0\right\}$ and notice that

$$E_\omega = \bigcup_{q \in \mathbb{Q}_+} \bigcap_{t \geq 1} \left\{P \in \Delta(\Omega) : P\left(\tilde{\omega}^t\right) > q\right\}$$

Since $\left\{P \in \Delta(\Omega) : P\left(\tilde{\omega}^t\right) > q\right\} \in \mathcal{B}$ and $\mathcal{B}$ is a $\sigma$-algebra, then $E_\omega \in \mathcal{B}$. Moreover

$$\Delta_{\tilde{\omega}}(\Omega) = E_\omega \cap \left\{P \in \Delta(\Omega) : \lim_{t \to \infty} \frac{P(\{\tilde{\omega}\})}{P(\omega^t)} = 1\right\}$$

The function $\phi_t : E_\omega \to \mathbb{R}$ defined as $\phi_t(P) = \frac{P(\{\tilde{\omega}\})}{P(\omega^t)}$ is $\mathcal{B}$-measurable. By standard arguments, the event $\Delta_{\tilde{\omega}}(\Omega) = E_\omega \cap \{P : \lim_t \phi_t(P) = 1\}$ belongs to $\mathcal{B}$.

We now show that the test $T$ constructed in the proof of Theorem 2 can be chosen to be measurable in $\Delta(\Omega)$. Fix $\epsilon > 0$ and for each $P \in \Delta_{\tilde{\omega}}(\Omega)$ let $t_P$ be the smallest number such that $P\left(\tilde{\omega}^{t_P} - \{\tilde{\omega}\}\right) \leq \epsilon$. Define $T$ such that $R^P = \tilde{\omega}^{t_P} - \{\tilde{\omega}\}$ for every $P \in \Delta_{\tilde{\omega}}(\Omega)$. For each $t$, the sets $\left\{P \in \Delta(\Omega) : P\left(\tilde{\omega}^t - \{\tilde{\omega}\}\right) > \epsilon\right\}$ and $\left\{P \in \Delta(\Omega) : P\left(\tilde{\omega}^t - \{\tilde{\omega}\}\right) \leq \epsilon\right\}$ belong to $\mathcal{B}$. Hence, for each $t$, $\{P \in \Delta_{\tilde{\omega}}(\Omega) : t_P \geq t\}$ belongs to $\mathcal{B}$. Therefore $\{P \in \Delta_{\tilde{\omega}}(\Omega) : t_P = t\} \in \mathcal{B}$.

For each $\omega$, consider the set $\left\{t : \omega \in \tilde{\omega}^t - \{\tilde{\omega}\}\right\}$ and denote by $\bar{t}$ its maximum (set $\bar{t} = 0$ if the set is empty). Then

$$\{P \in \Delta(\Omega) : T(\omega, P) = \text{pass}\} = \{P \in \Delta_{\tilde{\omega}}(\Omega) : t_P > \bar{t}\} \in \mathcal{B}$$

Therefore the test is measurable in $\Delta(\Omega)$.

Fix $\delta > 0$. Because $\{P \in \Delta_{\tilde{\omega}}(\Omega) : t_P \leq t\} \uparrow \Delta_{\tilde{\omega}}(\Omega)$ there is a $t'$ such that

$$\zeta\left(\{P \in \Delta_{\tilde{\omega}}(\Omega) : t_P \leq t'\}\right) \geq 1 - \delta$$

For every $\omega \in \tilde{\omega}^{t'+1} - \{\tilde{\omega}\}$ we have

$$\zeta\left(\{P \in \Delta_{\tilde{\omega}}(\Omega) : T(\omega, P) = \text{pass}\}\right) < \delta$$

Hence the test is non manipulable. ■

**Proof of Theorem 14.** Consider the function $V : \Delta_o(\Omega) \times \Delta\Delta_o(\Omega) \to [0,1]$ defined as

$$V(P,\zeta) = \sum_{\omega \in \Omega} P(\{\omega\})\, \zeta\left(\left\{Q : \omega \in A^Q\right\}\right)$$

for every $(P,\zeta) \in \Delta_o(\Omega) \times \Delta\Delta_o(\Omega)$. The function is affine in each variable. Endow $\Delta\Delta_o(\Omega)$ with the weak* topology. Under this topology, for each $\omega \in \Omega$ the function

$$\zeta \mapsto \zeta\left(\left\{Q : \omega \in A^Q\right\}\right)$$

is continuous. Finally, the set $\Delta\Delta_o(\Omega)$ is compact. Fan's minmax theorem implies

$$\inf_{P\in\Delta_o(\Omega)} \max_{\zeta\in\Delta\Delta_o(\Omega)} V(P,\zeta) = \max_{\zeta\in\Delta\Delta_o(\Omega)} \inf_{P\in\Delta_o(\Omega)} V(P,\zeta)$$

For every $P \in \Delta_o(\Omega)$, $V(P,\delta_P) \geq 1-\epsilon$. Hence there exists $\zeta \in \Delta\Delta_o(\Omega)$ such that $V(P,\zeta) > 1-\epsilon$ for every $P \in \Delta_o(\Omega)$. In particular,

$$V(\delta_\omega,\zeta) = \zeta\left(\left\{Q : \omega \in A^Q\right\}\right) > 1-\epsilon$$

for every $\omega \in \Omega$. ■

**Proof of Theorem 15.** We first claim that for every $P \in \Delta(\Omega)$ and for every $\delta \in (0,1)$ there exists $P' \in \Delta_*(\Omega)$ such that $\sup_{E\in\mathcal{A}} |P(E) - P'(E)| \leq \epsilon$. To this end, for every $P \in \Delta(\Omega)$ define the set

$$\Omega_P = \left\{\omega \in \Omega : \lim_{t\to\infty} P\left(\omega^t\right) > 0\right\}.$$

We claim that the set is countable. To see this, endow $\Omega$ with the product topology, and let $P_\sigma$ be the Borel $\sigma$-additive measure that agrees with $P$ on every cylinder. If $\omega \in \Omega_P$ then $P_\sigma(\{\omega\}) > 0$. Hence $\Omega_P$ is at most countable. Let $\Omega_P = \{\omega_1, \omega_2, ...\}$. Define

$$P' = (1-\delta)P + \delta\sum_{i=1}^{\infty} 2^{-i}\delta_{\omega_i}$$

and note that $\sup_{A\in\mathcal{A}} |P(E) - P'(E)| \leq \delta$. It remains to prove that $P'$ belongs to $\Delta_*(\Omega)$. If $P'(\{\omega\}) = 0$ then by construction $\omega \notin \Omega_P$. Hence $\lim_{t\to\infty} P(\omega^t) = 0$. Therefore $\lim_{t\to\infty} P'(\omega^t) = 0$. Hence $P' \in \Delta_*(\Omega)$

Now let $T$ be a test that $\Delta_*(\Omega)$-controls for type-I error with probability $1-\epsilon$. Then for every $P \in \Delta(\Omega)$, we have

$$P\left(A^{P'}\right) \geq P'\left(A^{P'}\right) - \delta \geq 1-\epsilon-\delta$$

where $\delta$ can be chosen to be arbitrarily small. The result now follows the proof of Theorem 10. ■

# References


[1] Rao, K.P.S. and M. Rao (1983): *Theory of Charges*, Academic Press, New York.