Mandatory Versus Discretionary Spending: the Status Quo Effect

Online Appendix

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B1. Pareto efficient allocations

PROPOSITION B.1: *Consider a T-period problem where $T \in \{0, 1, ..., \infty\}$. For any Pareto efficient allocation in the T-period problem such that $x_L^{t'} > 0$ for some $t' \leq T$ and $x_H^{t''}$ for some $t'' \leq T$, we have $g^t = \theta_H + \theta_L$ for all $t \leq T$.*

Proof: A Pareto efficient allocation in the T-period problem solves the following problem:

$$\begin{aligned}
\max_{\{\mathbf{b}^t\}_{t=0}^{\infty}} & \sum_{t=0}^{T} \delta^t (x_L^t + \theta_L \ln(g^t)) \\
\text{s.t.} & \sum_{t=0}^{T} \delta^t (x_H^t + \theta_H \ln(g^t)) = \bar{U}, \\
& x_L^t + x_H^t + g^t \leq 1,\ x_L^t \geq 0,\ x_H^t \geq 0,\ g^t \geq 0 \text{ for all } t \leq T.
\end{aligned}$$

We first prove the result for any finite T. The Lagrangian of this problem is

$$\sum_{t=0}^{T} \delta^t (x_L^t + \theta_L \ln(g^t)) - \lambda_1 \left(\bar{U} - \sum_{t=0}^{T} \delta^t (x_H^t + \theta_H \ln(g^t)) \right) - \sum_{t=0}^{T} \lambda_{2t} (x_L^t + x_H^t + g^t - 1).$$

The first order conditions with respect to x_L^t, x_H^t and g^t are:

$$\text{(B1)} \qquad \delta^t - \lambda_{2t} \leq 0, \qquad (\delta^t - \lambda_{2t}) x_L^t = 0,$$

$$\text{(B2)} \qquad \lambda_1 \delta^t - \lambda_{2t} \leq 0, \qquad (\lambda_1 \delta^t - \lambda_{2t}) x_H^t = 0,$$

$$\text{(B3)} \qquad \frac{\delta^t \theta_L}{g^t} + \frac{\lambda_1 \delta^t \theta_H}{g^t} - \lambda_{2t} \leq 0, \qquad \left(\frac{\delta^t \theta_L}{g^t} + \frac{\lambda_1 \delta^t \theta_H}{g^t} - \lambda_{2t} \right) g^t = 0.$$

Suppose there exist t' and t'' such $x_L^{t'} > 0$ and $x_H^{t''} > 0$. Since $x_L^{t'} > 0$, we have $\lambda_{2t'} = \delta^{t'}$ from (B1). It follows from (B2) that $\lambda_1 \leq \frac{\lambda_{2t'}}{\delta^{t'}} = 1$. Similarly, since $x_H^{t''} > 0$, we have $\lambda_1 = \frac{\lambda_{2t''}}{\delta^{t''}}$. Since $\frac{\lambda_{2t''}}{\delta^{t''}} \geq 1$ from (B1), it follows that $\lambda_1 \geq 1$. Hence $\lambda_1 = 1$.

Note that $g^t = 0$ violates (B3), hence $g^t > 0$. We next show that $g^t \neq 1$ for any $t \leq T$, which implies that for any $t \leq T$, at least one of x_H^t and x_L^t is strictly positive. Suppose $g^t = 1$ for some t, then, since $\lambda_1 = 1$, (B3) implies that $\lambda_{2t} = \delta^t(\theta_H + \lambda_1 \theta_L) < \delta^t$, which contradicts (B1). Since $\lambda_1 = 1$ and at least one of x_L^t and x_H^t is strictly positive for any $t \leq T$, it follows that $\lambda_{2t} = \delta^t$ for all $t \leq T$. Substituting in (B3), we get $g^t = \theta_H + \theta_L$ for all $t \leq T$.

Now suppose $T = \infty$. Consider an allocation $\mathbf{A} = \{\mathbf{b}^t\}_{t=0}^{\infty}$ in which $x_L^{t'} > 0$ for some t', $x_H^{t''}$ for some t'' and $g^{\tilde{t}} \neq \theta_H + \theta_L$ for some $\tilde{t}$. We next show that this allocation is not Pareto efficient. Let $\tau = \max\{t', t'', \tilde{t}\}$. Then, as we have shown above, $\{\mathbf{b}^t\}_{t=0}^{\tau}$ is not a Pareto efficient allocation in the τ-period problem. Let $\{\mathbf{b}_t'\}_{t=0}^{\tau}$ be an allocation that Pareto dominates $\{\mathbf{b}^t\}_{t=0}^{\tau}$ in the τ-period problem. Let $\mathbf{A}'' = \{\mathbf{b}_t''\}_{t=0}^{\infty}$ be such that $\mathbf{b}_t'' = \mathbf{b}_t'$ for $t \leq \tau$ and $\mathbf{b}_t'' = \mathbf{b}^t$ for $t > \tau$. It follows that $\mathbf{A}''$ Pareto dominates $\mathbf{A}$. Hence $\mathbf{A}$ is not Pareto efficient, which establishes the result for $T = \infty$. ∎

B2. Non-Markov equilibrium with discretionary public spending

Consider the following strategy profile: (i) In period 0, if i is the proposer, then it proposes $\mathbf{z} = (g', x'_H, x'_L)$ with $g' = \theta_H + \theta_L$, $x'_i = 1 - g'$ and $x'_j = 0$; (ii) if the level of public good spending proposed in all previous periods is equal to $\theta_H + \theta_L$, then proposer i in period t proposes $g' = \theta_H + \theta_L, x'_i = 1 - g', x'_j = 0$; (iii) if the level of public good spending proposed in some previous period is not equal to $\theta_H + \theta_L$, then proposer i in period t proposes $g' = \theta_i, x'_i = 1 - g', x'_j = 0$; (iv) party j accepts a proposal if and only if its dynamic payoff from accepting is weakly higher than its dynamic payoff from rejecting.

Off the equilibrium path, proposer i has no incentive to deviate because the strategies induce a Markov perfect equilibrium in subgames off the equilibrium path by Proposition 1.

On the equilibrium path, proposer i's payoff from following the strategy is given by

$$V_i^* = \frac{(1-\delta p)(1-(\theta_H+\theta_L))+\theta_i \ln(\theta_H+\theta_L)(1+\delta-2\delta p)}{(1-\delta)(1+\delta-2\delta p)}.$$

If i deviates, then the most profitable deviation is to propose $g' = \theta_i, x'_i = 1 - g', x'_j = 0$ and the payoff from this deviation is given by

$$V_i = \frac{(1-\delta p)(1-\theta_i)+\theta_i \ln(\theta_i)(1-\delta p)+\theta_i \ln(\theta_j)\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}.$$

Proposer has i has no incentive to deviate if $V_i^* \geq V_i$. This is satisfied when

$$\ln\left(\frac{\theta_H+\theta_L}{\theta_i}\right) + \frac{\delta(1-p)}{1-\delta p}\ln\left(\frac{\theta_H+\theta_L}{\theta_j}\right) \geq \frac{\theta_j}{\theta_i},$$

which in turn is satisfied when $\delta \geq \underline{\delta}_i$ where $\underline{\delta}_i = \frac{Z_i}{Z_i p+1-p}$ and $Z_i = \frac{\frac{\theta_j}{\theta_i} - \ln\left(\frac{\theta_H+\theta_L}{\theta_i}\right)}{\ln\left(\frac{\theta_H+\theta_L}{\theta_j}\right)}$.

It remains to find conditions under which $Z_i < 1$ for $i = H, L$ which guarantees that $\underline{\delta}_i < 1$ for $i = H, L$ for some $p < 1$. Note that $Z_i < 1$ iff $\frac{\theta_j}{\theta_i} < \ln(\frac{\theta_H+\theta_L}{\theta_H}) + \ln(\frac{\theta_H+\theta_L}{\theta_L})$. Since $\theta_L \leq \theta_H$, if $Z_L < 1$, then $Z_H < 1$. Thus the conjectured equilibrium exists if $Z_L < 1$, i.e. if $\frac{\theta_H}{\theta_L} < 2\ln(\theta_H + \theta_L) - \ln(\theta_H) - \ln(\theta_L)$, which holds if θ_H and θ_L are sufficiently close.

Note that the threshold $\underline{\delta}_i$ is increasing in p, so it is easier to sustain the efficient level through these non-Markov strategies when persistence of power is lower.

B3. Proof of Lemma 7 (Low-polarization equilibrium characterization)

We proceed by first conjecturing an equilibrium strategy-payoff pair and then verifying that it satisfies guesses (G1)-(G3), equilibrium conditions (E1)-(E3), and our assumption on α^i that all proposals made on the equilibrium path are accepted when $\frac{\theta_H}{\theta_L} \leq \frac{1-\delta p}{\delta(1-p)}$ and condition $(*)$ holds.

We conjecture an equilibrium strategy-payoff pair such that for any $i, j \in \{H, L\}$ with $j \neq i$, the acceptance strategy $\alpha^i(g, \mathbf{z})$ satisfies (E1), the proposal strategies are

$$\gamma^i(g) = \begin{cases} g_i^* & \text{for } g \le g_i^*, \\ g & \text{for } g_i^* \le g \le \theta_H + \theta_L, \\ \theta_H + \theta_L & \text{for } \theta_H + \theta_L \le g, \end{cases}$$

$$\chi_j^i(g) = \begin{cases} 0 & \text{for } g \le \theta_H + \theta_L, \\ \frac{\theta_j(1-\delta p) - \theta_i \delta(1-p)}{(1-\delta)(1+\delta-2\delta p)} \ln(\frac{g}{\theta_H+\theta_L}) & \text{for } \theta_H + \theta_L \le g, \end{cases}$$

and $\chi_i^i(g) = 1 - \gamma^i(g) - \chi_j^i(g)$, where $g_L^* = \theta_L$ and $g_H^* = \frac{1+\delta-2\delta p}{1-\delta p}\theta_H$, and the associated payoff functions are

$$V_L(g) = \begin{cases} V_L^* & \text{for } g < g_L^*, \\ \frac{1}{1-\delta p}[1 - g + \theta_L \ln(g) + \delta(1-p)W_L^*] & \text{for } g_L^* \le g \le g_H^*, \\ \frac{(1-\delta p)(1-g)}{(1+\delta-2\delta p)(1-\delta)} + \frac{\theta_L}{1-\delta}\ln(g) & \text{for } g_H^* \le g \le \theta_H + \theta_L, \\ C_L \ln(g) + D_L & \text{for } \theta_H + \theta_L < g, \end{cases}$$

$$W_L(g) = \begin{cases} W_L^* & \text{for } g \le g_H^*, \\ \frac{1}{1-\delta p}[\theta_L \ln(g) + \delta(1-p)V_L(g)] & \text{for } g_H^* \le g, \end{cases}$$

$$V_H(g) = \begin{cases} V_H^* & \text{for } g < g_H^*, \\ \frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{1-\delta}\ln(g), & \text{for } g_H^* \le g \le \theta_H + \theta_L, \\ C_H \ln(g) + D_H & \text{for } \theta_H + \theta_L \le g, \end{cases}$$

$$W_H(g) = \begin{cases} W_H^* & \text{for } g \le g_L^*, \\ \frac{1}{1-\delta p}[\theta_H \ln(g) + \delta(1-p)V_H(g)] & \text{for } g_L^* \le g, \end{cases}$$

where

$$C_i = \frac{-(1-\delta p)\theta_j + \delta(1-p)\theta_i}{(1-\delta)(1+\delta-2\delta p)}, \; D_i = \frac{(1-\delta p)(1-\theta_L-\theta_H+(\theta_H+\theta_L)\ln(\theta_H+\theta_L))}{(1-\delta)(1+\delta-2\delta p)},$$

and

$$
\begin{aligned}
W_L^* &= \tfrac{\delta(1-p)}{(1+\delta-2\delta p)(1-\delta)}(1-g_H^*) + \tfrac{\theta_L}{1-\delta}\ln(g_H^*),\\
V_L^* &= \tfrac{1}{1-\delta p}[1-\theta_L+\theta_L\ln(\theta_L)+\delta(1-p)W_L^*],\\
V_H^* &= \tfrac{(1-\delta p)(1-g_H^*)}{(1+\delta-2\delta p)(1-\delta)} + \tfrac{\theta_H}{1-\delta}\ln(g_H^*),\\
W_H^* &= \tfrac{1}{1-\delta p}[\theta_H\ln(g_L^*)+\delta(1-p)V_H^*].
\end{aligned}
$$

This conjecture satisfies (G2) and (G3). (Note that by substituting W_j in (6), we can verify that $W_j(g) = K_j(g)$ for $g \geq g_i^*$.) So we only need to verify that (G1) is satisfied; in particular, that $g_i^* \in \arg\max f_i(g)$ where $f_i(g) = 1 - g + \theta_i \ln(g) + \delta[pV_i(g) + (1-p)W_i(g)]$.

Since V_i and W_i are continuous under our conjecture of the equilibrium strategy-payoff pair, f_i is continuous. It is also piecewise differentiable. Specifically,

$$
f_L'(g) = \begin{cases}
-1 + \frac{\theta_L}{g} & \text{for } g < g_H^*,\\
-\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{(1-\delta)g} & \text{for } g \in (g_H^*, \theta_H+\theta_L),\\
-1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\frac{C_L}{g} & \text{for } g > \theta_H+\theta_L,
\end{cases}
$$

$$
f_H'(g) = \begin{cases}
-1 + \frac{\theta_H}{g} & \text{for } g < g_L^*,\\
-1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} & \text{for } g \in (g_L^*, g_H^*),\\
-\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{(1-\delta)g} & \text{for } g \in (g_H^*, \theta_H+\theta_L),\\
-1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\frac{C_H}{g} & \text{for } g > \theta_H+\theta_L.
\end{cases}
$$

CLAIM B.1: *Under our conjecture of the equilibrium strategy-payoff pair, $g_i^* \in \arg\max f_i(g)$ for all $i \in \{H, L\}$.*

Proof: Consider $i = L$ first. Given f_L described above, $f_L'(g) > 0$ if $g < g_L^*$, $f_L'(g) = 0$ if $g = g_L^*$, and $f_L'(g) < 0$ if $g \in (g_L^*, g_H^*)$.

Since $f_L'(g)$ is decreasing for $g \in (g_H^*, \theta_H+\theta_L)$, and at $g = g_H^*$, $-\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{(1-\delta)g} = -\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L(1-\delta p)}{(1-\delta)(1+\delta-2\delta p)\theta_H} < 0$, it follows that $f_L'(g) < 0$ for $g \in (g_H^*, \theta_H+\theta_L)$.

If $\frac{(1+\delta-2\delta p)\theta_L+\delta(p+\delta-2\delta p)C_L}{1-\delta p} \leq 0$, then $f_L'(g) < 0$ for $g > \theta_H+\theta_L$. If $\frac{(1+\delta-2\delta p)\theta_L+\delta(p+\delta-2\delta p)C_L}{1-\delta p} > 0$, then $f_L'(g)$ is decreasing in g for $g \geq \theta_H+\theta_L$. Since at $g = \theta_H+\theta_L$, $f_L'(g) = \frac{-(1-\delta p)\theta_H+(\delta-\delta p)\theta_L}{(\theta_H+\theta_L)(1+\delta-2\delta p)(1-\delta)} < 0$, it follows that $f_L'(g) < 0$ for $g > \theta_H+\theta_L$.

To summarize, $f_L'(g) > 0$ for $g < g_L^*$, $f_L'(g) = 0$ if $g = g_L^*$, $f_L'(g) > 0$ for $g > g_L^*$, and therefore $g_L^* \in \arg\max f_L(g)$.

Now consider $i = H$. Given f_H described above, $f_H'(g) > 0$ for $g < g_H^*$. By a similar argument as for party L, $f_H(g)$ is decreasing for $g > g_H^*$. Therefore $g_H^* \in \arg\max f_H(g)$. ∎

Claim B.1 shows that (G1) is satisfied. We next verify that equilibrium conditions (E1)-(E3) are satisfied. Condition (E1) is satisfied by construction.

The values V_L^*, W_L^*, V_H^* and W_H^* satisfy

$$\begin{aligned}
V_L^* &= 1 - g_L^* + \theta_L \ln(g_L^*) + \delta[pV_L^* + (1-p)W_L^*],\\
W_L^* &= \theta_L \ln(g_H^*) + \delta[(1-p)V_L^* + pW_L^*],\\
V_H^* &= 1 - g_H^* + \theta_H \ln(g_H^*) + \delta[pV_H^* + (1-p)W_H(g_H^*)],\\
W_H^* &= \theta_H \ln(g_L^*) + \delta[(1-p)V_H^* + pW_H^*].
\end{aligned}$$

These together with Lemmas 1, 3 and 5 show that (E3) is satisfied, i.e., these payoff functions are consistent with the strategy profile.

The remainder of the proof shows that (E2) is satisfied. The next claim establishes that $K_i(g)$ is increasing in g, which is useful later in the proof.

CLAIM B.2: *Under our conjecture of the equilibrium strategy-payoff pair, $K_i(g)$ is strictly increasing in g for all $i \in \{H, L\}$.*

Proof: Suppose $g \leq g_L^*$. Then $K_i(g) = \theta_i \ln(g) + \delta[(1-p)V_i^* + pW_i^*]$ and $K_i'(g) > 0$.

Suppose $g \in [g_L^*, g_H^*]$. Then $K_L(g) = \theta_L \ln(g) + \delta[(1-p)V_L(g) + pW_L^*]$ where $V_L(g) = \frac{1}{1-\delta p}[1 - g + \theta_L \ln(g) + \delta(1-p)W_L^*]$. Hence,

$$K_L'(g) = \tfrac{1+\delta-2\delta p}{1-\delta p}\tfrac{\theta_L}{g} - \tfrac{\delta(1-p)}{1-\delta p}.$$

Since $\frac{\theta_H}{\theta_L} < \frac{1-\delta p}{\delta(1-p)}$, in the low-polarization case we have $K_L'(g) > 0$.

Also, since $K_H(g) = \theta_H \ln(g) + \delta(1-p)V_H^* + \delta p[\frac{\theta_H}{1-\delta p}\ln(g) + \delta(1-p)V_H^*]$, it follows that $K_H(g)$ is increasing in g.

Suppose $g \in [g_H^*, \theta_H + \theta_L]$. Then $K_i(g) = \frac{\theta_i}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_i(g)$. Substituting for $V_i(g)$ and taking the derivative, we get

$$K_i'(g) = \tfrac{1}{1-\delta}\left[\tfrac{-\delta(1-p)}{1+\delta-2\delta p} + \tfrac{\theta_i}{g}\right].$$

Since $\frac{\theta_H}{\theta_L} < \frac{1-\delta p}{\delta(1-p)}$, it follows that $K_i'(g) > 0$ for $g \in [g_H^*, \theta_H + \theta_L]$.

Suppose $g \geq \theta_H + \theta_L$. Then again $K_i(g) = \frac{\theta_i}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_i(g)$. Substituting for $V_i(g)$ and taking the derivative, we get

$$K_i'(g) = \tfrac{\theta_i}{(1-\delta p)g} + \tfrac{\delta(1-p)}{1-\delta p}\tfrac{C_i}{g} = \tfrac{\theta_i(1-\delta p) - \theta_j\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)g}$$

where $j \in \{H, L\}, j \neq i$.

For $i = H$, $\theta_H(1-\delta p) - \theta_L\delta(1-p) > 0$ and therefore $K_H'(g) > 0$. For $i = L$, since $\frac{\theta_H}{\theta_L} < \frac{1-\delta p}{\delta(1-p)}$ in the low-polarization case, it follows that $\theta_L(1-\delta p) - \theta_H\delta(1-p) > 0$ and therefore $K_L'(g) > 0$. ∎

The claim immediately implies that the responder accepts any proposal with g' higher than the status quo g and if the responder accepts a proposal with g' lower than the status quo, then the responder must receive a positive transfer. A formal statement is as follows.

COROLLARY B.1: *Consider* $\mathbf{z}' = (g', x'_H, x'_L) \in \mathcal{B}$. *For any* $i \in \{H, L\}$, *(i) if* $g' \geq g$, *then* $\alpha^i(g, \mathbf{z}') = 1$; *(ii) if* $g' < g$ *and* $\alpha^i(g, \mathbf{z}') = 1$, *then* $x'_i > 0$.

For notational convenience, let $U_i^P(\mathbf{z}) = x_i + \theta_i \ln(g) + \delta[pV_i(g) + (1-p)W_i(g)]$ and $U_i^R(\mathbf{z}) = x_i + \theta_i \ln(g) + \delta[(1-p)V_i(g) + pW_i(g)]$. That is, $U_i^P(\mathbf{z})$ ($U_i^R(\mathbf{z})$) denotes party i's dynamic payoff when the implemented budget is $\mathbf{z}$ in the current period and party i is the proposer (responder). The next claim establishes that all equilibrium proposals are accepted.

CLAIM B.3: *Under our conjecture of the equilibrium strategy-payoff pair, if condition* $(*)$ *holds, then* $\alpha^j(g, \pi^i(g)) = 1$ *for all* g *and all* $i, j \in \{H, L\}$, $j \neq i$.

Proof: Consider $j = H$ first.

If $g \leq g_L^*$, then $U_H^R(\pi^L(g)) = \theta_H \ln(g_L^*) + \delta[(1-p)V_H^* + pW_H^*] \geq K_H(g) = \theta_H \ln(g) + \delta[(1-p)V_H^* + pW_H^*]$ and therefore $\alpha^H(g, \pi^L(g)) = 1$.

If $g \in [g_L^*, \theta_H + \theta_L]$, then $\gamma^L(g) = g$ and $\chi_H^L(g) = 0$, which implies that $U_H^R(\pi^L(g)) = K_H(g)$ and therefore $\alpha^H(g, \pi^L(g)) = 1$.

If $g > \theta_H + \theta_L$, then $\gamma^L(g) = \theta_H + \theta_L$ and $\chi_H^L(g) = K_H(g) - \theta_H \ln(\theta_H + \theta_L) - \delta[(1-p)V_H(\theta_H + \theta_L) + pW_H(\theta_H + \theta_L)]$. Note that $\chi_H^L(g) \leq 1 - (\theta_H + \theta_L)$ if condition $(*)$ holds. With this proposal, $U_H^R(\pi^L(g)) = K_H(g)$ and therefore $\alpha^H(g, \pi^L(g)) = 1$.

Now consider $j = L$.

If $g \leq g_H^*$, then $U_L^R(\pi^H(g)) = \theta_L \ln(g_H^*) + \delta[(1-p)V_L(g_H^*) + pW_L(g_H^*)]$. Since $K'_L(g) > 0$ by Claim B.2 and $U_L^R(\pi^H(g)) = K_L(g_H^*)$, it follows that $U_L^R(\pi^H(g)) \geq K_L(g)$ and therefore $\alpha^L(g, \pi^H(g)) = 1$ for $g \leq g_H^*$.

If $g \geq g_H^*$, then an argument similar to the case of $j = H$ shows that $U_L^R(\pi^H(g)) = K_L(g)$ and therefore $\alpha^L(g, \pi^H(g)) = 1$. ■

We next show that the proposer has no profitable one-shot deviation. Consider the following three cases for party L.

- $g \leq g_L^*$: Since $g_L^* = \arg\max f_L(g)$, party L has no incentive to deviate from proposing $\gamma^L(g) = g_L^*$ and $\chi_H^L(g) = 0$.

- $g_L^* < g \leq \theta_H + \theta_L$: We first show that proposing $\pi^L(g)$ is better than proposing $(\hat{g}, \hat{x}_H, \hat{x}_L)$ with $\hat{g} > g$ and then show that it is better than proposing $(\hat{g}, \hat{x}_H, \hat{x}_L)$ with $\hat{g} < g$.

  - $\hat{g} > g$: Consider $\hat{\mathbf{z}} = (\hat{g}, 0, 1 - \hat{g})$. Then $U_L^P(\hat{\mathbf{z}}) = f_L(\hat{g})$. As shown in the proof of Claim B.1, $f_L(\hat{g})$ is decreasing for $\hat{g} > g_L^*$. Since $\pi^L(g) = (g, 0, 1-g)$, this implies that $U_L^P(\pi^L(g)) > U_L^P(\hat{\mathbf{z}})$ for any $\hat{g} > g > g_L^*$. Since party L's payoff is decreasing in x_H, $U_L^P(\hat{\mathbf{z}}) \geq U_L^P((\hat{g}, \hat{x}_H, \hat{x}_L))$ for any $(\hat{g}, \hat{x}_H, \hat{x}_L) \in \mathcal{B}$, it follows that $U_L^P(\pi^L(g)) > U_L^P((\hat{g}, \hat{x}_H, \hat{x}_L))$ for any $\hat{g} > g > g_L^*$. Also, since $\alpha^H(g, \pi^L(g)) = 1$ by Claim B.3, and $U_L^P(\pi^L(g)) > U_L^P((g, 0, 0))$, the status quo payoff, it follows that proposing $\pi^L(g)$ is better than proposing any $(\hat{g}, \hat{x}_H, \hat{x}_L) \in \mathcal{B}$ with $\hat{g} > g$.
  - $\hat{g} < g$: If $\hat{g} < g$, then by Corollary B.1, $\alpha^H(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_H > 0$. Since party L's payoff is strictly decreasing in x_H, we only need to consider proposals

such that the responder's acceptance constraint (5) is binding. From (5),

(B4) $$\hat{x}_H = K_H(g) - \theta_H \ln(\hat{g}) - \delta[(1-p)V_H(\hat{g}) + pW_H(\hat{g})].$$

Consider $\hat{\mathbf{z}} = (\hat{g}, \hat{x}_H, \hat{x}_L)$ such that (B4) holds. Substituting for $\hat{x}_H$ from (B4) and taking the derivative, we get

(B5)
$$\frac{\partial U_L^P}{\partial \hat{g}} = -1 + \frac{\theta_H+\theta_L}{\hat{g}} + \delta[(1-p)V_H'(\hat{g}) + pW_H'(\hat{g})] + \delta[pV_L'(\hat{g}) + (1-p)W_L'(\hat{g})]$$

For $\hat{g} < g_L^*$, $\frac{\partial U_L^P}{\partial \hat{g}} = -1 + \frac{\theta_H+\theta_L}{\hat{g}} > 0$.

For $g_L^* < \hat{g} < g_H^*$, $\frac{\partial U_L^P}{\partial \hat{g}} = -1 + \frac{\theta_H+\theta_L}{\hat{g}} + \frac{\delta p}{1-\delta p}\frac{\theta_H}{\hat{g}} + \frac{\delta p}{1-\delta p}(-1 + \frac{\theta_L}{g}) = \frac{1}{1-\delta p}(-1 + \frac{\theta_H+\theta_L}{\hat{g}}) > 0$.

For $g_H^* < \hat{g} < g \le \theta_H + \theta_L$, $\frac{\partial U_L^P}{\partial \hat{g}} = -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{\hat{g}} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}V_L'(\hat{g}) + \frac{1}{1-\delta p}\frac{\theta_H}{\hat{g}} + \frac{\delta(1-p)}{1-\delta p}V_H'(\hat{g}) = \frac{1}{1-\delta}(-1 + \frac{\theta_H+\theta_L}{g}) > 0$.

So $U_L^P(\hat{\mathbf{z}})$ is increasing in $\hat{g}$ for $\hat{g} < g$, and therefore the proposer has no incentive to make any proposal with $\hat{g} < g$.

- $g > \theta_H + \theta_L$: Consider $\hat{\mathbf{z}} = (\hat{g}, 0, 1-\hat{g})$ with $\hat{g} > g$. By Corollary B.1, $\alpha^H(g, \hat{\mathbf{z}}) = 1$. Since $U_L^P(\hat{\mathbf{z}}) = f_L(\hat{g})$ and $f_L(\hat{g})$ is decreasing in $\hat{g}$ for $\hat{g} > g > \theta_H + \theta_L$, it follows that $U_L^P((g, 0, 1-g)) \ge U_L^P((\hat{g}, 0, 1-\hat{g}))$ if $\hat{g} \ge g$.

  Now consider $\hat{\mathbf{z}} = (\hat{g}, \hat{x}_H, \hat{x}_L)$ such that $\hat{g} \le g$ and $\alpha^H(g, \hat{\mathbf{z}}) = 1$. By Corollary B.1, $\hat{x}_H > 0$ if $\hat{g} > g$. Again we only need to consider proposals such that the acceptance constraint binds. As before, we obtain (B5). Substituting for $V_L'(\hat{g}), W_L'(\hat{g}), V_H'(\hat{g}), W_H'(\hat{g})$, we get

$$\frac{\partial U_L^P}{\partial \hat{g}} = -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{\hat{g}} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\frac{C_L}{\hat{g}} + \left(\frac{1}{1-\delta p}\frac{\theta_H}{\hat{g}} + \frac{\delta(1-p)}{1-\delta p}\frac{C_H}{\hat{g}}\right) = -1 + \frac{\theta_H+\theta_L}{\hat{g}}.$$

  Since $\gamma^L(g) = \theta_H + \theta_L$, it follows that $U_L^P(\pi^L(g)) \ge U_L^P(\hat{\mathbf{z}})$ for any $\hat{\mathbf{z}} = (\hat{g}, \hat{x}_H, \hat{x}_L)$ such that $\hat{g} < g$ and $\alpha^H(g, \hat{\mathbf{z}}) = 1$. Combined with $U_L^P((g, 0, 1-g)) \ge U_L^P((\hat{g}, 0, 1-\hat{g}))$ if $\hat{g} \ge g$, $\pi^L(g)$ is optimal for party L to propose.

Party H also has no incentive to deviate. We omit the details of the proof because the argument is similar to that for party L. ∎

B4. Low-polarization equilibrium when condition (∗) fails:

Denote by $\mathbf{z}_j^e$ the proposal $(\theta_H + \theta_L, x_H, x_L)$ where $x_i = 0$ and $x_j = 1 - \theta_H - \theta_L$. Recall that in Lemma 7, we assume condition (∗) holds, which ensures the responder j accepts the proposal $\mathbf{z}_j^e$ even when the status quo g is high. What happens if condition (∗) fails, that is, if $\alpha^j(g, \mathbf{z}_j^e) = 0$ for g sufficiently high? Then, instead of proposing $g' = \theta_H + \theta_L$, party i

proposes $g' > \theta_H + \theta_L$, $x'_i = 0$, and $x'_j = 1 - g'$ such that party j is just willing to accept. Figure B1 illustrates the parties' proposal strategies when condition $(*)$ fails. In the figure (G1)-(G2) are still satisfied, but for very high status quos, (G3) is violated. As shown in Section V.B, the failure of condition $(*)$ does not affect the set of steady states.

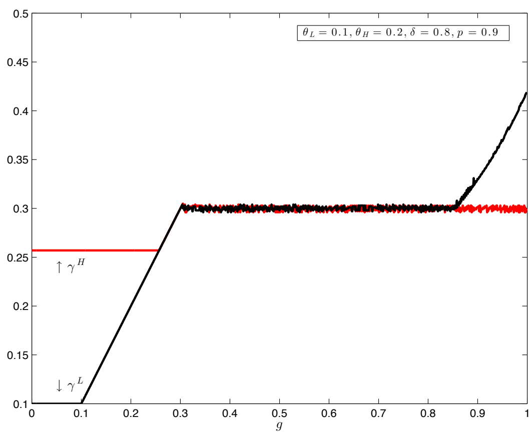


Figure B1. : $\gamma^i(g)$ in low-polarization case when $(*)$ fails

B5. Proof of Proposition 4

We formally characterize an equilibrium with the properties given in Propostion 4 in the high-polarization case where $\frac{\theta_H}{\theta_L} > \frac{1-\delta p}{\delta(1-p)}$. Recall that $f_i(g)$ defined in (8) is party i's dynamic payoff when the public spending in the current period is g and party i receives the remaining surplus. Motivated by Figure 5, we make the following guesses about an equilibrium strategy-payoff pair.

(G1′) There exist g_L^* and g_H^* with $g_L^* < \theta_H + \theta_L < g_H^*$ such that $g_i^* \in \arg\max f_i(g)$ for $i \in \{H, L\}$.

(G2′) If $g \leq g_L^*$, then $\pi^L(g) = \pi^L(g_L^*)$ and specifically $\gamma^L(g) = g_L^*$; if $g \in [g_L^*, \theta_H + \theta_L]$, then $\gamma^L(g) = g$; if $g \geq \theta_H + \theta_L$, then $\gamma^L(g) = \theta_H + \theta_L$. If $g \geq g_L^*$, then $W_H(g) = K_H(g)$.

(G3′) There exist $\underline{g}_H$ and $\tilde{g}_H$ that satisfy $g_L^* \leq \underline{g}_H < \tilde{g}_H < \theta_H + \theta_L$ such that (i) $\pi^H(g) = \pi^H(g_H^*)$ for $g \leq \underline{g}_H$ and $g \geq g_H^*$; (ii) if $g \in [\underline{g}_H, g_H^*]$ then $W_L(g) = K_L(g)$; (iii) if $g \leq \tilde{g}_H$ or if $g \geq \theta_H + \theta_L$, then $\chi_L^H(g) = 0$; and (iv)

$$\gamma^H(g) = \begin{cases} g_H^* & \text{for } g \leq \underline{g}_H, \\ g' \in [\theta_H + \theta_L, g_H^*] & \text{for } \underline{g}_H \leq g \leq \tilde{g}_H, \\ \theta_H + \theta_L & \text{for } \tilde{g}_H \leq g \leq \theta_H + \theta_L, \\ g & \text{for } \theta_H + \theta_L \leq g \leq g_H^*, \\ g_H^* & \text{for } g_H^* \leq g. \end{cases}$$

where g' is a function of g satisfying $\theta_L \ln(g') + \delta[(1-p)V_L(g') + pW_L(g')] = K_L(g)$.

(G4′) If $\gamma^i(g) = \theta_H + \theta_L$, then $V_i(g)$ is piecewise linear in g and $\ln(g)$.

We first establish some properties of an equilibrium strategy-payoff pair that satisfies (G1′)-(G4′). Suppose $\sigma = ((\pi^H, \alpha^H), (\pi^L, \alpha^L))$ and (V_H, W_H, V_L, W_L) is an equilibrium strategy-payoff pair that satisfies (G1′)-(G4′). Recall that $V_i^* = \max_g f_i(g)$ is proposer i's highest payoff without the responder's constraint (5). As in the low-polarization case, we denote $W_L(g_H^*)$ by W_L^* and $W_H(g_L^*)$ by W_H^*.

LEMMA B.1: *Under (G1′) and (G2′), if $g \le g_L^*$, then $V_L(g) = V_L^*$, $\chi_L^L(g) = 1 - g_L^*$, $\chi_H^L(g) = 0$, and $W_H(g) = W_H^*$. Under (G3′), if $g \le \underline{g}_H$ or $g \ge g_H^*$, then $V_H(g) = V_H^*$, $\chi_H^H(g) = 1 - g_H^*$, $\chi_L^H(g) = 0$, and $W_L(g) = W_L^*$.*

We omit the proof since it is similar to that of Lemma 1.

LEMMA B.2: *Under (G1′)-(G3′), (i) if $g \in [g_L^*, \underline{g}_H]$, then*

$$V_L(g) = \tfrac{1}{1-\delta p}[1 - g + \theta_L \ln(g) + \delta(1-p)W_L^*],$$

(ii) if $g \in [\underline{g}_H, \theta_H + \theta_L]$, then

$$V_L(g) = \tfrac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_L}{1-\delta}\ln(g), \tag{B6}$$

and (iii) if $g \in [\theta_H + \theta_L, g_H^*]$, then

$$V_H(g) = \tfrac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{1-\delta}\ln(g). \tag{B7}$$

We omit the proof since it is similar to that of Lemma 3.

To simplify notation, define

$$\theta_i^* \equiv \tfrac{1+\delta-2\delta p}{1-\delta p}\theta_i.$$

LEMMA B.3: *Under (G1′)-(G3′), $g_L^* = \theta_L$ and $g_H^* = \theta_H^*$.*

Proof: We omit the proof for g_L^* since it is the same as that of Lemma 4.

Now consider g_H^*. If $g > g_H^*$, then $V_H'(g) = 0$ by Lemma B.1 and $W_H'(g) = \frac{\theta_H}{(1-\delta p)g}$ by Lemma 2, and therefore

$$f_H'(g) = -1 + \tfrac{\theta_H}{g} + \delta(1-p)W_H'(g) = -1 + \tfrac{\theta_H^*}{g}. \tag{B8}$$

If $g_H^* < \theta_H^*$, then (B8) implies that $f_H'(g) > 0$ for $g \in (g_H^*, \theta_H^*)$, contradicting that $g_H^* \in \arg\max f_H(g)$. Hence $g_H^* \ge \theta_H^*$.

If $g \in (\theta_H + \theta_L, g_H^*)$, then by (G3′), $f_H(g) = V_H(g)$, and by (B7)

$$f_H'(g) = -\tfrac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{(1-\delta)g}. \tag{B9}$$

If $g_H^* > \theta_H^*$, then (B9) implies that $f_H'(g) < 0$ for $g \in (\theta_H^*, g_H^*)$, contradicting that $g_H^* \in \arg\max f_H(g)$. Hence $g_H^* \leq \theta_H^*$.

Since $g_H^* \leq \theta_H^*$ and $g_H^* \geq \theta_H^*$, it follows that $g_H^* = \theta_H^*$. ∎

Recall that we guess in (G4′) that V_i is piecewise linear in g and $\ln(g)$ if $\gamma^i(g) = \theta_H + \theta_L$. Specifically, suppose that for $g \in [\theta_H + \theta_L, g_H^*]$, $V_L(g)$ takes the form $V_L(g) = B_L^1 g + C_L^1 \ln(g) + D_L^1$; for $g \geq g_H^*$ such that $\gamma_L(g) = \theta_H + \theta_L$, $V_L(g)$ takes the form $V_L(g) = B_L^2 g + C_L^2 \ln(g) + D_L^2$; for $g \in [\tilde{g}_H, \theta_H + \theta_L]$, $V_H(g)$ takes the form $V_H(g) = B_H^1 g + C_H^1 \ln(g) + D_H^1$.

LEMMA B.4: *Under (G1′)-(G4′), $B_i^1 = \frac{\delta(1-p)(1-\delta p)}{(1-\delta)(1+\delta-2\delta p)}$ and $C_i^1 = -\frac{\theta_j}{1-\delta}$ for $i, j \in \{H, L\}$ with $j \neq i$, and $B_L^2 = 0$, $C_L^2 = -\frac{\theta_H}{1-\delta p}$.*

Proof: Similar to the proof of Lemma 5, we can write

$$V_i(g) = \chi_i^i(g) + \tfrac{1+\delta-2\delta p}{1-\delta p}\theta_i \ln(\theta_H + \theta_L) + \tfrac{\delta(p+\delta-2\delta p)}{1-\delta p} V_i(\theta_H + \theta_L),$$

where

$$\chi_j^i(g) = K_j(g) - \tfrac{\theta_j}{1-\delta p}\ln(\theta_H + \theta_L) - \tfrac{\delta(1-p)}{1-\delta p} V_j(\theta_H + \theta_L),$$
$$K_j(g) = \tfrac{\theta_j}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p} V_j(g).$$

If $g \in [\theta_H + \theta_L, g_H^*]$, then $V_H(g) = \frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{1-\delta}\ln(g)$ by Lemma B.2. Substituting in $K_H(g)$, we get

$$K_H(g) = \tfrac{\delta(1-p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{1-\delta}\ln(g).$$

Substituting in $V_L(g)$ and matching coefficients, we get $B_L^1 = \frac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}$ and $C_L^1 = -\frac{\theta_H}{1-\delta}$. A similar argument shows that $B_H^1 = \frac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}$ and $C_H^1 = -\frac{\theta_L}{1-\delta}$.

To find B_L^2 and C_L^2, note that if $g \geq g_H^*$, then by Lemma B.1, $V_H(g) = V_H^*$. By Lemma 2, $K_H(g) = \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_H^*$. Matching coefficients gives $B_L^2 = 0$ and $C_L^2 = -\frac{\theta_H}{1-\delta p}$. ∎

We next find the thresholds $\underline{g}_H$ and $\tilde{g}_H$ that are consistent with (G1′)-(G4′).

LEMMA B.5: *Under (G1′)-(G4′), the threshold $\underline{g}_H \in (0, \theta_H + \theta_L)$ is given by $\underline{g}_H = \psi$ where*

$$\begin{aligned}\psi = \min\{g \geq 0 : &\tfrac{\delta(1-p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_L}{1-\delta}\ln(g) \\ = &\tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{1}{1-\delta p}\left[\theta_L - \tfrac{\delta(1-p)}{1-\delta}\theta_H\right]\ln(\theta_H^*) \\ &+ \tfrac{\delta(1-p)}{(1-\delta p)(1-\delta)}\left[(\theta_H + \theta_L)[\ln(\theta_H + \theta_L) - 1] + \tfrac{\delta(1-p)}{1+\delta-2\delta p}\theta_H^*\right]\},\end{aligned} \tag{B10}$$

and ψ is a decreasing function of θ_H.

Proof: By (G3′) (ii) and (iv), the threshold $\underline{g}_H$ satisfies

(B11) $$\theta_L \ln(g_H^*) + \delta[(1-p)V_L(g_H^*) + pW_L(g_H^*)] = W_L(\underline{g}_H) = K_L(\underline{g}_H).$$

By(G3′)(ii), $W_L(g_H^*) = K_L(g_H^*)$. Hence by Lemma 2, we can rewrite the left-hand side of the above equation as

(B12) $$\tfrac{\theta_L}{1-\delta p} \ln(g_H^*) + \tfrac{\delta(1-p)}{1-\delta p} V_L(g_H^*).$$

By (G1′), $g_H^* > \theta_H + \theta_L$. Hence $\gamma^L(g_H^*) = \theta_H + \theta_L$ by (G2′). So $V_L(g_H^*)$ can be written as

$$V_L(g_H^*) = \chi_L^L(g_H^*) + \tfrac{1+\delta-2\delta p}{1-\delta p}\theta_L \ln(\theta_H+\theta_L) + \tfrac{\delta(p+\delta-2\delta p)}{1-\delta p} V_L(\theta_H+\theta_L),$$

where $\chi_L^L(g_H^*) = 1 - \chi_H^L(g_H^*) - \gamma^L(g_H^*) = 1 - \chi_H^L(g_H^*) - \theta_H - \theta_L$, and

$$\begin{aligned} \chi_H^L(g_H^*) =& K_H(g_H^*) - \tfrac{\theta_H}{1-\delta p}\ln(\theta_H+\theta_L) - \tfrac{\delta(1-p)}{1-\delta p} V_H(\theta_H+\theta_L), \\ K_H(g_H^*) =& \tfrac{\theta_H}{1-\delta p}\ln(g_H^*) + \tfrac{\delta(1-p)}{1-\delta p} V_H(g_H^*). \end{aligned}$$

By Lemma B.2,

$$V_i(\theta_H+\theta_L) = \tfrac{(1-\delta p)(1-\theta_H-\theta_L)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_i}{1-\delta}\ln(\theta_H+\theta_L), \qquad V_H(g_H^*) = \tfrac{(1-\delta p)(1-g_H^*)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_H}{1-\delta}\ln(g_H^*).$$

Substituting in all expressions, (B12) becomes

$$\tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{1}{1-\delta p}\left[\theta_L - \tfrac{\delta(1-p)}{1-\delta}\theta_H\right]\ln(g_H^*) + \tfrac{\delta(1-p)}{(1-\delta p)(1-\delta)}\left[(\theta_H+\theta_L)[\ln(\theta_H+\theta_L)-1] + \tfrac{\delta(1-p)}{1+\delta-2\delta p} g_H^*\right].$$

By (G3′)(ii) and Lemma 2, we can write $K_L(\underline{g}_H)$ as

$$K_L(\underline{g}_H) = \tfrac{\theta_L}{1-\delta p}\ln(\underline{g}_H) + \tfrac{\delta(1-p)}{1-\delta p} V_L(\underline{g}_H).$$

By Lemma B.2 this becomes

$$K_L(\underline{g}_H) = \tfrac{\theta_L}{1-\delta}\ln(\underline{g}_H) + \tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}(1-\underline{g}_H).$$

By Lemma B.3 $g_H^* = \theta_H^*$, hence $\underline{g}_H$ is given by

(B13) $$\begin{aligned} \tfrac{\theta_L}{1-\delta}\ln(\underline{g}_H) \quad + \quad & \tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}(1-\underline{g}_H) = \\ & \tfrac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{1}{1-\delta p}\left[\theta_L - \tfrac{\delta(1-p)}{1-\delta}\theta_H\right]\ln(\theta_H^*) \\ & + \tfrac{\delta(1-p)}{(1-\delta p)(1-\delta)}\left[(\theta_H+\theta_L)[\ln(\theta_H+\theta_L)-1] + \tfrac{\delta(1-p)}{1+\delta-2\delta p}\theta_H^*\right]. \end{aligned}$$

Let $l(x) = \frac{\theta_L}{1-\delta}\ln(x) + \frac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}(1-x)$, and denote the right-hand side of (B13) by R. At most two values of $\underline{g}_H$ satisfy (B13) since $l(x)$ is strictly concave. We show below only one is lower than $\theta_H + \theta_L$ and hence it is a candidate for $\underline{g}_H$ by (G3′). Note that

$$l(\theta_H+\theta_L) - R = \frac{\theta_L(1-\delta)-\theta_H\delta(1-p)}{(1-\delta p)(1-\delta)}\ln(\theta_H+\theta_L) + \frac{\delta^2(1-p)^2}{(1-\delta)(1+\delta-2\delta p)(1-\delta p)}(\theta_H+\theta_L)$$

$$\text{(B14)} \qquad - \left[\frac{\theta_L(1-\delta)-\theta_H\delta(1-p)}{(1-\delta p)(1-\delta))}\ln(\theta_H^*) + \frac{\delta^2(1-p)^2}{(1-\delta)(1+\delta-2\delta p)(1-\delta p)}(\theta_H^*)\right].$$

Define $h(x) = \frac{\theta_L(1-\delta)-\theta_H\delta(1-p)}{(1-\delta p)(1-\delta)}\ln(x) + \frac{\delta^2(1-p)^2}{(1-\delta)(1+\delta-2\delta p)(1-\delta p)}x$, then $l(\theta_H+\theta_L) - R = h(\theta_H + \theta_L) - h(\theta_H^*)$. It is straightforward to show $h'(x) < 0$. Since $\theta_H + \theta_L < \theta_H^*$, it follows $l(\theta_H+\theta_L) - R > 0$. Given $l(\theta_H+\theta_L) - R > 0$, the value that satisfies (B13) such that $\underline{g}_H < \theta_H + \theta_L$ must be the minimum of the solutions to (B13).

At ψ, $l(x)$ is strictly increasing, and it is straightforward to show that R is decreasing in θ_H in the high-polarization case. Hence ψ is decreasing in θ_H. ∎

LEMMA B.6: *Under (G1′)-(G4′), the threshold* $\tilde{g}_H \in (0, \theta_H+\theta_L)$ *is given by*

$$\text{(B15)} \qquad \frac{\delta(1-p)(1-\tilde{g}_H)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta}\ln(\tilde{g}_H) = \frac{\delta(1-p)(1-\theta_L-\theta_H)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta}\ln(\theta_H+\theta_L).$$

Proof: By (G3′) (ii) and (iv), the threshold $\tilde{g}_H$ satisfies

$$\text{(B16)} \qquad \theta_L\ln(\theta_H+\theta_L) + \delta[(1-p)V_L(\theta_H+\theta_L) + pW_L(\theta_H+\theta_L)] = K_L(\tilde{g}_H).$$

By Lemma B.2, $V_L(g) = \frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta}\ln(g)$ for $g \in [\tilde{g}_H, \theta_H+\theta_L]$. Substituting this in (B16) and using Lemma 2, we get (B15). ∎

In (G2′), we guess that $\gamma^L(g) = \theta_H + \theta_L$ for all $g \geq \theta_H + \theta_L$. This is analogous to the low-polarization case and we need a condition similar to $(*)$ to guarantee that it holds in equilibrium.

LEMMA B.7: *Under (G1′)-(G4′), if*

$$(**) \qquad 1 - (\theta_H+\theta_L) + \frac{\theta_H}{1-\delta}\ln(\theta_H+\theta_L) \geq \frac{\delta(1-p)(\theta_H+\theta_L-\theta_H^*)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\delta(1-p)\theta_H}{(1-\delta p)(1-\delta)}\ln(\theta_H^*).$$

then $\alpha^H(g, (\theta_H+\theta_L, x_H, x_L)) = 1$ *with* $x_H = 1-\theta_H-\theta_L$, $x_L = 0$.

Proof: For any $g \geq \theta_H + \theta_L$, we have $\alpha^H(g, (\theta_H+\theta_L, x_H, x_L)) = 1$ with $x_H = 1-\theta_H-\theta_L$, $x_L = 0$ if

$$1 - (\theta_H+\theta_L) + \theta_H\ln(\theta_H+\theta_L) + \delta[(1-p)V_H(\theta_H+\theta_L) + pW_H(\theta_H+\theta_L)] \geq K_H(g).$$

Substituting for $K_H(g)$ and $W_H(g)$ using Lemma 2 and substituting for $V_H(g) = V_H^*$ for $g \geq g_H^*$ using Lemma B.1, the inequality becomes

$$\text{(B17)} \quad 1 - (\theta_H + \theta_L) + \frac{\theta_H}{1-\delta p} \ln(\theta_H + \theta_L) + \frac{\delta(1-p)}{1-\delta p} V_H(\theta_H + \theta_L) \geq \frac{\theta_H}{1-\delta p} \ln(g) + \frac{\delta(1-p)}{1-\delta p} V_H^*.$$

Note that the right-hand side of (B17) is increasing in g, implying that if the inequality holds for $g = 1$, then it holds for all $g \geq \theta_H + \theta_L$. Substituting for $V_H(\theta_H + \theta_L)$ and V_H^* using Lemma B.2 and letting $g = 1$, we can rewrite inequality (B17) as $(**)$. ∎

We are now ready to establish the following result in the high-polarization case.

LEMMA B.8: *Suppose* $\frac{\theta_H}{\theta_L} > \frac{1-\delta p}{\delta(1-p)}$, $\psi \geq \theta_L^*$ *and condition* $(**)$ *holds. Then, there exists an equilibrium strategy-payoff pair* $\sigma = ((\pi^H, \alpha^H), (\pi^L, \alpha^L))$ *and* (V_H, W_H, V_L, W_L) *that satisfies (G1′)-(G4′).*

The proof of Lemma B.8 is given in the next subsection.

The next two lemmas show that $\psi \geq \theta_L^*$ holds when polarization is not too high and condition $(**)$ holds when parties' values of the public good are not too high.

LEMMA B.9: *In the high-polarization case,* $\psi \geq \theta_L^*$ *holds when polarization is not too high.*

Proof: Recall that $\frac{\theta_H}{\theta_L} > \frac{1-\delta p}{\delta(1-p)}$ in the high-polarization case. Since ψ is continuous in θ_H and θ_L, we only need to show that $\psi > \theta_L^*$ when $\frac{\theta_H}{\theta_L} = \frac{1-\delta p}{\delta(1-p)}$, which is equivalent to $\theta_H^* = \theta_H + \theta_L$. We substitute $\theta_H^* = \theta_H + \theta_L$ into (B10). Then (B10) simplifies to $\frac{\delta(1-p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta} \ln(g) = \frac{\delta(1-p)(1-[\theta_H+\theta_L])}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta} \ln(\theta_H + \theta_L)$. This implies $\psi = \theta_H + \theta_L$. So $\psi > \theta_L^*$ is equivalent to $\frac{\theta_H}{\theta_L} > \frac{\delta(1-p)}{1-\delta p}$, which always holds since $\frac{\theta_H}{\theta_L} \geq 1 > \frac{\delta(1-p)}{1-\delta p}$. ∎

LEMMA B.10: *In the high-polarization case,* $(**)$ *holds when parties' values of the public good are not too high.*

Proof: To show that $(**)$ holds when parties' values of the public good are not too high, we show that it holds strictly as the value of θ_H and θ_L approach zero. By l'Hôpital's rule, as $\theta_H \to 0$ and $\theta_L \to 0$, the left-hand side of condition $(**)$ approaches 1 and the right-hand side of condition $(**)$ approaches 0. So $(**)$ holds strictly, and by continuity, $(**)$ also holds when θ_H and θ_L are not too high. ∎

Proposition 4 follows from Lemmas B.3 and B.8-B.10. Specifically, for part (i) of Proposition 4, notice first that (G1′)-(G4′) imply proposal strategies for party L that are the same as those given in Proposition 3. Hence party L proposes higher levels of public good spending when it is mandatory than when it is discretionary. Similarly, proposal strategies for party H given in (G3′) shows that the level of public good proposed by party H is weakly higher than $\theta_H + \theta_L$, the efficient level, and therefore higher than what it proposes when public spending is discretionary.

For Proposition 4 (ii), observe again in (G3′) that when $g \leq \underline{g}_H$, or $g \geq g_H^*$, party H's ideal g_H^* is proposed. The value of g_H^* given in Lemma B.3 is higher than the efficient level under high polarization.

Proposition 4 (iii) follows from inspecting the proposal strategies given in (G1′)-(G4′).

B6. Proof of Lemma B.8 (High-polarization equilibrium characterization)

We proceed by first conjecturing an equilibrium strategy-payoff pair and then verifying that it satisfies guesses (G1′)-(G4′), equilibrium conditions (E1)-(E3), and our assumption on α^i that all proposals made on the equilibrium path are accepted, when $\frac{\theta_H}{\theta_L} > \frac{1-\delta p}{\delta(1-p)}$, $\psi \geq \theta_L^*$, and condition (∗∗) holds.

We conjecture an equilibrium strategy-payoff pair such that for any $i, j \in \{H, L\}$ with $j \neq i$, the acceptance strategy $\alpha^i(g, \mathbf{z})$ satisfies (E1), the proposal strategies are

$$\gamma^L(g) = \begin{cases} g_L^* & \text{for } g \leq g_L^*, \\ g & \text{for } g_L^* \leq g \leq \theta_H + \theta_L, \\ \theta_H + \theta_L & \text{for } \theta_H + \theta_L \leq g, \end{cases}$$

$$\chi_H^L(g) = \begin{cases} 0 & \text{for } g \leq \theta_H + \theta_L, \\ K_H(g) - \theta_H \ln(\theta_H + \theta_L) - \delta[(1-p)V_H(\theta_H + \theta_L) + pW_H(\theta_H + \theta_L)] & \text{for } \theta_H + \theta_L \leq g, \end{cases}$$

$$\gamma^H(g) = \begin{cases} g_H^* & \text{for } g \leq \underline{g}_H, \\ g' \in [\theta_H + \theta_L, g_H^*] & \text{for } \underline{g}_H \leq g \leq \tilde{g}_H, \\ \theta_H + \theta_L & \text{for } \tilde{g}_H \leq g \leq \theta_H + \theta_L, \\ g & \text{for } \theta_H + \theta_L \leq g \leq g_H^*, \\ g_H^* & \text{for } g_H^* \leq g, \end{cases}$$

$$\chi_L^H(g) = \begin{cases} 0 & \text{for } g \leq \tilde{g}_H, \\ K_L(g) - \theta_L \ln(\theta_H + \theta_L) - \delta[(1-p)V_L(\theta_H + \theta_L) + pW_L(\theta_H + \theta_L)] & \text{for } g \in [\tilde{g}_H, \theta_H + \theta_L], \\ 0 & \text{for } g \geq \theta_H + \theta_L, \end{cases}$$

and $\chi_i^i(g) = 1 - \gamma^i(g) - \chi_j^i(g)$, where $g_L^* = \theta_L$, $g_H^* = \theta_H^*$, $\underline{g}_H$ satisfies (B10), $\tilde{g}_H$ satisfies (B15), g' satisfies

$$\theta_L \ln(g') + \delta[(1-p)V_L(g') + pW_L(g')] = K_L(g), \qquad \text{(B18)}$$

and the associated payoff functions are

$$V_L(g) = \begin{cases} V_L^* & \text{for } g \le g_L^*, \\ \frac{1}{1-\delta p}(1 - g + \theta_L \ln(g) + \delta(1-p)W_L^*) & \text{for } g_L^* \le g \le \underline{g}_H, \\ \frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta}\ln(g) & \text{for } \underline{g}_H \le g \le \theta_H + \theta_L, \\ B_L^1 g + C_L^1 \ln(g) + D_L^1 & \text{for } \theta_H + \theta_L \le g \le g_H^*, \\ C_L^2 \ln(g) + D_L^2 & \text{for } g_H^* \le g, \end{cases}$$

$$W_L(g) = \begin{cases} W_L^* & \text{for } g \le \underline{g}_H \text{ and } g \ge g_H^*, \\ \frac{1}{1-\delta p}[\theta_L \ln(g) + \delta(1-p)V_L(g)] & \text{for } \underline{g}_H \le g \le g_H^*, \end{cases}$$

$$V_H(g) = \begin{cases} V_H^* & \text{for } g \le \underline{g}_H, \\ \frac{(1-\delta p)(1-\gamma^H(g))}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{1-\delta}\ln(\gamma^H(g)) & \text{for } \underline{g}_H \le g \le \tilde{g}_H, \\ B_H^1 g + C_H^1 \ln(g) + D_H^1 & \text{for } \tilde{g}_H \le g \le \theta_H + \theta_L, \\ \frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{1-\delta}\ln(g) & \text{for } \theta_H + \theta_L \le g \le g_H^*, \\ V_H^* & \text{for } g_H^* \le g, \end{cases}$$

$$W_H(g) = \begin{cases} W_H^* & \text{for } g \le g_L^*, \\ \frac{1}{1-\delta p}[\theta_H \ln(g) + \delta(1-p)V_H(g)] & \text{for } g_L^* \le g, \end{cases}$$

where $B_i^1 = \frac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}$, $C_i^1 = -\frac{\theta_j}{1-\delta}$, $D_i^1 = \frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{(\theta_H+\theta_L)[\ln(\theta_H+\theta_L)-1]}{1-\delta}$, $C_L^2 = -\frac{\theta_H}{1-\delta p}$, $D_L^2 = \frac{\delta(1-p)}{(1-\delta)(1-\delta p)}(\theta_H - \ln(g_H^*)) + D_L^1$, and

(B19) $$W_L^* = \frac{\delta(1-p)}{(1+\delta-2\delta p)(1-\delta)}(1-g_H^*) + \frac{\theta_L}{1-\delta}\ln(g_H^*),$$

(B20) $$V_L^* = \frac{1}{1-\delta p}[1 - \theta_L + \theta_L \ln(\theta_L) + \delta(1-p)W_L^*],$$

(B21) $$V_H^* = \frac{(1-\delta p)(1-g_H^*)}{(1+\delta-2\delta p)(1-\delta)} + \frac{\theta_H}{1-\delta}\ln(g_H^*),$$

(B22) $$W_H^* = \frac{1}{1-\delta p}[\theta_H \ln(g_L^*) + \delta(1-p)V_H^*].$$

We next verify that this conjecture satisfies (G1′)-(G4′).

For (G1′), since $g_L^* = \theta_L$ and $g_H^* = \theta_H^*$, clearly $g_L^* < \theta_H + \theta_L < g_H^*$ in the high-polarization case, and it only remains to show that $g_i^* \in \arg\max f_i(g)$. In Claim B.4 below, we show that (i) $g_H^* \in \arg\max f_H(g)$, and (ii) $g_L^* \in \arg\max f_L(g)$ when $\psi \ge \theta_L^*$, where ψ is defined in (B10).

Since V_i and W_i are continuous under our conjecture of the equilibrium strategy-payoff pair,

f_i is continuous. It is also piecewise differentiable. Specifically,

$$f_L'(g) = \begin{cases} -1 + \frac{\theta_L}{g} & \text{for } g < g_L^*, \\ \frac{1}{1-\delta p}[-1 + \frac{\theta_L}{g}] & \text{for } g \in (g_L^*, \underline{g}_H), \\ -\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{(1-\delta)g} & \text{for } g \in (\underline{g}_H, \theta_H + \theta_L), \\ -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}(B_L^1 + \frac{C_L^1}{g}) & \text{for } g \in (\theta_H + \theta_L, g_H^*), \\ -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\frac{C_L^2}{g} & \text{for } g \geq g_H^*, \end{cases}$$

$$f_H'(g) = \begin{cases} -1 + \frac{\theta_H}{g} & \text{for } g < g_L^*, \\ -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} & \text{for } g \in (g_L^*, \underline{g}_H), \\ -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}(-\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{(1-\delta)\gamma^H(g)})\frac{d\gamma^H(g)}{dg} & \text{for } g \in (\underline{g}_H, \tilde{g}_H), \\ -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}(B_H^1 + \frac{C_H^1}{g}) & \text{for } g \in (\tilde{g}_H, \theta_H + \theta_L), \\ -\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{(1-\delta)g} & \text{for } g \in (\theta_H + \theta_L, g_H^*), \\ -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g} & \text{for } g > g_H^*. \end{cases}$$

CLAIM B.4: *Under our conjecture of the equilibrium strategy-payoff pair, (i)* $g_H^* \in \arg\max f_H(g)$, *and (ii) if* $\psi \geq \theta_L^*$, *then* $g_L^* \in \arg\max f_L(g)$.

Proof: Part (i): We show $f_H(g)$ is strictly increasing for $g \in (\tilde{g}_H, g_H^*)$, and strictly decreasing for $g > g_H^*$; hence $g_H^* = \arg\max_{g > \tilde{g}_H} f_H(g)$ by continuity of $f_H(g)$. Further, we show $f_H(g) \leq f_H(g_H^*)$ for $g \in (\underline{g}_H, \tilde{g}_H)$, and $f_H(g)$ is strictly increasing for $g < \underline{g}_H$. Hence, $g_H^* \in \arg\max f_H(g)$ by continuity of $f_H(g)$.

- $g < g_L^*$: $f_H'(g)$ is decreasing. At $g_L^* = \theta_L$, $f_H'(g_L^*) > 0$, hence for $g < g_L^*$, $f_H'(g) > 0$.

- $g \in (g_L^*, \underline{g}_H)$: $f_H'(g)$ is decreasing. Since $\underline{g}_H < g_H^*$ and $f_H'(g) = -1 + \frac{g_H^*}{g}$, it follows that $f_H'(g) > 0$ for $g \in (g_L^*, \underline{g}_H)$.

- $g \in (\underline{g}_H, \tilde{g})$: We compare $f_H(g)$ in this range to $f_H(g_H^*)$. First define the functions

$$\begin{aligned} n(x) &= 1 - x + \frac{\theta_H(1+\delta-2\delta p)}{1-\delta p}\ln(x), \text{ and} \\ m(y) &= \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\left[\frac{(1-\delta p)(1-y)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{1-\delta}\ln(y)\right]. \end{aligned}$$

  By these definitions $f_H(g_H^*) = n(g_H^*) + m(g_H^*)$, and $f_H(g) = n(g) + m(\gamma^H(g))$ for $g \in (\underline{g}_H, \tilde{g}_H)$. Further note that $g_H^* = \arg\max n(x)$, and $g_H^* = \arg\max m(y)$, hence $n(g_H^*) \geq n(g)$ and $m(g_H^*) \geq m(\gamma^H(g))$ for all g, so $f_H(g_H^*) > f_H(g)$ for $g \in (\underline{g}_H, \tilde{g})$.

- $g \in (\tilde{g}_H, \theta_H + \theta_L)$: $f_H'(g)$ strictly decreasing. Since $f_H'(\theta_H + \theta_L) = \frac{\theta_H\delta(1-p) - \theta_L(1-\delta p)}{(1-\delta)(1+\delta-2\delta p)(\theta_H+\theta_L)} > 0$, $f_H'(g) > 0$ everywhere in this interval.

- $g \in (\theta_H + \theta_L, g_H^*)$: $f_H'(g)$ strictly decreasing. Since $-\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{(1-\delta)g_H^*} = 0$, it follows that for $g \in (\theta_H + \theta_L, g_H^*)$, $f_H'(g) > 0$.

- $g > g_H^*$: $f_H'(g) = -1 + \frac{g_H^*}{g} < 0$.

Part (ii): We show $f_L(g)$ is strictly increasing for $g < g_L^*$ and strictly decreasing for $g > g_L^*$ and therefore $g_L^* \in \arg\max f_L(g)$.

- $g < g_L^*$: $f_L'(g) > 0$.

- $g \in (g_L^*, \underline{g}_H)$: $f_L'(g)$ is strictly decreasing. Since $f_L'(g) = \frac{1}{1-\delta p}[-1 + \frac{\theta_L}{g}]$, it follows that $f_L'(g) < 0$ for $g \in (g_L^*, \underline{g}_H)$.

- $g \in (\underline{g}_H, \theta_H + \theta_L)$: $f_L'(g)$ is strictly decreasing. Since $\underline{g}_H = \psi$ by Lemma B.5, we have $\underline{g}_H = \psi \geq \theta_L^*$. Since $-\frac{1-\delta p}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{(1-\delta)g} = 0$ if $g = \frac{\theta_L(1+\delta-2\delta p)}{1-\delta p}$, it follows that $f_L'(g) < 0$ for all $g \in (\underline{g}_H, \theta_H + \theta_L)$.

- $g \in (\theta_H+\theta_L, g_H^*)$: The monotonicity of $f_L'(g)$ is determined by $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^1}{1-\delta p}$. If $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^1}{1-\delta p} > 0$, then $f_L'(g)$ is strictly decreasing in g. Since $-1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}(B_L^1 + \frac{C_L^1}{g}) = \frac{\theta_L\delta(1-p)-\theta_H(1-\delta p)}{(1-\delta)(1+\delta-2\delta p)(\theta_H+\theta_L)} \leq 0$ if $g = \theta_H + \theta_L$, it follows that $f_L'(g) < 0$ for $g \in (\theta_H + \theta_L, g_H^*)$. If $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^1}{1-\delta p} \leq 0$, then $f_L'(g)$ is weakly increasing in g. Since $-1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}(B_L^1 + \frac{C_L^1}{g}) = -1 + \frac{\theta_L}{\theta_H} - \frac{\delta(p+\delta-2\delta p)}{(1+\delta-2\delta p)(1-\delta p)} < 0$ when $g = g_H^*$, it follows that $f_L'(g) < 0$ for $g \in (\theta_H + \theta_L, g_H^*)$.

- $g > g_H^*$: The monotonicity of $f_L'(g)$ is determined by $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^2}{1-\delta p}$. If $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^2}{1-\delta p} > 0$, then $f_L'(g)$ is strictly decreasing in g. Since $-1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_L}{g} + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}\frac{C_L^2}{g} = -1 + \frac{\theta_L}{\theta_H} - \frac{\delta(p+\delta-2\delta p)}{(1-\delta p)(1+\delta-2\delta p)} < 0$ if $g = g_H^*$, it follows that $f_L'(g) < 0$ for $g > g_H^*$. If $\frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^2}{1-\delta p} \leq 0$, then $f_L'(g)$ is weakly increasing in g. In this case, $f_L'(g) = -1 + \frac{(1+\delta-2\delta p)\theta_L}{1-\delta p} + \frac{\delta(p+\delta-2\delta p)C_L^2}{1-\delta p} < 0$ when $g = 1$ and therefore $f_L'(g) < 0$ for $g > g_H^*$. ∎

The conjectured equilibrium strategy-payoff pair clearly satisfies (G2′)-(G4′) with the exception of $g_L^* \leq \underline{g}_H < \tilde{g}_H < \theta_H + \theta_L$. When $\psi \geq \theta_L^*$, we have $g_L^* = \theta_L < \theta_L^* \leq \psi = \underline{g}_H$. To verify that $\underline{g}_H < \tilde{g}_H < \theta_H + \theta_L$, we next establish some monotonicity properties of K_L.

CLAIM B.5: *Under our conjecture of the equilibrium strategy-payoff pair, $K_L(g)$ is strictly increasing for $g \in [0, \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)})$ and strictly decreasing for $g \in (\frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}, 1]$.*

Proof: Consider the following cases:

- $g \le g_L^*$: $K_L(g) = \theta_L \ln(g) + \delta[(1-p)V_L^* + pW_L^*]$, which is increasing in g.
- $g \in [g_L^*, \underline{g}_H]$: In this case,

$$K_L(g) = \theta_L \ln(g) + \tfrac{\delta(1-p)}{1-\delta p}(1 - g + \theta_L \ln(g) + \delta(1-p)W_L^*) + \delta p W_L^*.$$

Taking the derivative, we get

$$K_L'(g) = \tfrac{1+\delta-2\delta p}{1-\delta p}\tfrac{\theta_L}{g} - \tfrac{\delta(1-p)}{1-\delta p},$$

and $K_L'(g) > 0$ if and only if $g < \frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L$.

- $g \in [\underline{g}_H, \theta_H + \theta_L]$: In this case,

$$K_L(g) = \tfrac{\theta_L}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}V_L(g) = \tfrac{\theta_L}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}\left[\tfrac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \tfrac{\theta_L}{1-\delta}\ln(g)\right].$$

Taking the derivative, we get

$$K_L'(g) = \tfrac{1}{1-\delta}\left[\tfrac{-\delta(1-p)}{1+\delta-2\delta p} + \tfrac{\theta_L}{g}\right],$$

and $K_L'(g) > 0$ if and only if $g < \frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L$. Note that since $\theta_H + \theta_L > \frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L$ in the high-polarization case, $K_L'(g) < 0$ for $g \in (\frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L, \theta_H + \theta_L)$.

- $g \in [\theta_H + \theta_L, g_H^*]$: In this case,

$$K_L(g) = \tfrac{\theta_L}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}V_L(g) = \tfrac{\theta_L}{1-\delta p}\ln(g) + \tfrac{\delta(1-p)}{1-\delta p}(B_L^1 g + C_L^1 \ln(g) + D_L^1).$$

Taking the derivative, we get

$$K_L'(g) = \tfrac{1}{(1-\delta p)(1-\delta)}\left[\tfrac{(1-\delta)\theta_L - \delta(1-p)\theta_H}{g} + \tfrac{\delta^2(1-p)^2}{1+\delta-2\delta p}\right],$$

which is increasing in g since $(1-\delta)\theta_L - \delta(1-p)\theta_H < 0$ in the high-polarization case. Straightforward calculation shows that $K_L'(g) < 0$ for $g = g_H^*$. Hence, $K_L(g)$ is strictly decreasing for $g \in [\theta_H + \theta_L, g_H^*]$.

- $g \ge g_H^*$: In this case,

$$K_L(g) = \theta_L \ln(g) + \delta[(1-p)V_L(g) + pW_L(g)] = \theta_L \ln(g) + \delta(1-p)(C_L^2 \ln(g) + D_L^2) + \delta p W_L^*.$$

Taking the derivative and substituting for C_L^2, we get

$$K_L'(g) = \tfrac{\theta_L}{g} - \tfrac{\delta(1-p)\theta_H}{(1-\delta p)g},$$

which is negative since $\frac{\theta_H}{\theta_L} > \frac{1-\delta p}{\delta(1-p)}$ in the high-polarization case. Hence, $K_L(g)$ is strictly increasing for $g \in [0, \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)})$ and strictly decreasing for $g \in (\frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}, 1]$. ∎

Recall that in our conjectured equilibrium, $\underline{g}_H$ satisfies $K_L(\underline{g}_H) = K_L(g_H^*)$ and $\tilde{g}_H$ satisfies $K_L(\tilde{g}_H) = K_L(\theta_H + \theta_L)$. Since K_L is continuous, $K_L(g) = -\infty$ when $g = 0$, and $\frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)} < \theta_H + \theta_L < g_H^*$ in the high-polarization case, we have the following corollary of Claim B.5.

COROLLARY B.2: *There exist $\underline{g}_H$ and $\tilde{g}_H$ where $\underline{g}_H < \tilde{g}_H < \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)} < \theta_H + \theta_L < g_H^*$ such that $K_L(\underline{g}_H) = K_L(g_H^*)$ and $K_L(\tilde{g}_H) = K_L(\theta_H + \theta_L)$.*

We next verify that equilibrium conditions (E1)-(E3) are satisfied. Condition (E1) is satisfied by construction. The values V_L^*, W_L^*, V_H^* and W_H^* satisfy

$$\begin{aligned} V_L^* &= 1 - g_L^* + \theta_L \ln(g_L^*) + \delta[pV_L^* + (1-p)W_L^*], \\ W_L^* &= \theta_L \ln(g_H^*) + \delta[(1-p)V_L^* + pW_L^*], \end{aligned}$$

$$\begin{aligned} V_H^* &= 1 - g_H^* + \theta_H \ln(g_H^*) + \delta[pV_H^* + (1-p)W_H(g_H^*)], \\ W_H^* &= \theta_H \ln(g_L^*) + \delta[(1-p)V_H^* + pW_H^*]. \end{aligned}$$

Together with Lemmas B.1, B.2 and B.4, these show that (E3) is satisfied, that is, these payoff functions are consistent with the strategy profile.

Recall $U_i^P(\mathbf{z})$ ($U_i^R(\mathbf{z})$) denotes party i's dynamic payoff when the implemented budget is $\mathbf{z}$ in the current period and party i is the proposer (responder). The next claim establishes that all equilibrium proposals are accepted.

CLAIM B.6: *Under our conjecture of the equilibrium strategy-payoff pair, if condition $(**)$ holds, then $\alpha^j(g, \pi^i(g)) = 1$ for all g and all $i, j \in \{H, L\}$, $j \neq i$.*

Proof: We omit the proof for $j = H$ since it is similar to that for Claim B.3. We use condition $(**)$ in place of condition $(*)$.

Now consider $j = L$. If $g \leq g_L^*$, then $U_L^R(\pi^H(g)) = \theta_L \ln(g_H^*) + \delta[(1-p)V_L^* + pW_L^*] \geq K_L(g) = \theta_L \ln(g_H) + \delta[(1-p)V_L^* + pW_L^*]$ and therefore $\alpha^L(g, \pi^H(g)) = 1$.

If $g \in [g_L^*, \underline{g}_H]$, then $U_L^R(\pi^H(g)) = \theta_L \ln(g_H^*) + \delta[(1-p)V_L^* + pW_L^*] = K_L(g_H^*)$. Since $K_L(\underline{g}_H) = K_L(g_H^*)$ and $K_L(g)$ is increasing on $[g_L^*, \underline{g}_H]$ by Claim B.5, it follows that $U_L^R(\pi^H(g)) \geq K_L(g)$ and therefore $\alpha^L(g, \pi^H(g)) = 1$ for $g \in [g_L^*, \underline{g}_H]$.

If $g \in [\underline{g}_H, g_H^*]$, then $U_L^R(\pi^H(g)) = K_L(g)$ and $\alpha^L(g, \pi^H(g)) = 1$.

If $g \in [g_H^*, 1]$, then $U_L^R(\pi^H(g)) = \theta_L \ln(g_H^*) + \delta[(1-p)V_L^* + pW_L^*] = K_L(g_H^*)$. Since $K_L(g)$ is decreasing on $[g_H^*, 1]$ by Claim B.5 and Corollary B.2, it follows that $U_L^R(\pi^H(g)) \geq K_L(g)$ and therefore $\alpha^L(g, \pi^H(g)) = 1$ for $g \in [g_H^*, 1]$. ∎

The remainder of the proof shows that (E2) is satisfied. The next claim establishes that $K_H(g)$ is increasing, which is useful later in the proof.

CLAIM B.7: *Under our conjecture of the equilibrium strategy-payoff pair, if $\psi \geq \theta_L^*$, then $K_H(g)$ is strictly increasing.*

Proof: Consider the following cases.

- $g \leq g_L^*$: $K_H(g) = \theta_H \ln(g) + \delta[(1-p)V_H^* + pW_H^*]$, which is strictly increasing.
- $g \in [g_L^*, \underline{g}_H]$: $K_H(g) = \theta_H \ln(g) + \delta(1-p)V_H^* + \frac{\delta p}{1-\delta p}[\theta_H \ln(g) + \delta(1-p)V_H^*]$, which is strictly increasing.
- $g \in [\underline{g}_H, \tilde{g}_H]$: $K_H(g) = \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_H(g)$, and $K_H'(g) = \frac{\theta_H}{(1-\delta p)g} + \frac{\delta(1-p)}{1-\delta p}V_H'(g)$. The function $V_H(g)$ is

$$V_H(g) = \frac{(1-\delta p)(1-\gamma^H(g))}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_H}{1-\delta}\ln(\gamma^H(g)),$$

and $\gamma^H(g)$ is given by (B18), which implies

$$\begin{aligned}&\frac{\theta_L}{1-\delta p}\ln(\gamma^H(g)) + \frac{\delta(1-p)}{1-\delta p}\left[\frac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)}\gamma^H(g) - \frac{\theta_H}{1-\delta}\ln(\gamma^H(g)) + D_L^1\right]\\ \text{(B23)}\qquad &= \frac{\theta_L}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}\left[\frac{(1-\delta p)(1-g)}{(1-\delta)(1+\delta-2\delta p)} + \frac{\theta_L}{1-\delta}\ln(g)\right].\end{aligned}$$

Rearranging (B23) gives

$$\ln(\gamma^H(g)) = \frac{1-\delta p}{\theta_L(1-\delta)-\theta_H\delta(1-p)}\left[\theta_L \ln(g) + \frac{\delta(1-p)(1-g)}{1+\delta-2\delta p} - \frac{\delta^2(1-p)^2\gamma^H(g)}{(1-\delta p)(1+\delta-2\delta p)} - \frac{\delta(1-p)(1-\delta)}{1-\delta p}D_L^1\right].$$

Substituting $\ln(\gamma^H(g))$ into $V_H(g)$ and taking the derivative, we have

$$\begin{aligned}V_H'(g) =& \frac{\theta_H\theta_L(1-\delta p)}{(1-\delta)[\theta_L(1-\delta)-\theta_H\delta(1-p)]g} - \frac{\theta_H\delta(1-p)(1-\delta p)}{(1-\delta)[\theta_L(1-\delta)-\theta_H\delta(1-p)](1+\delta-2\delta p)}\\ &- \frac{d\gamma^H(g)}{dg}\frac{\theta_L(1-\delta p)-\theta_H\delta(1-p)}{(1+\delta-2\delta p)[\theta_L(1-\delta)-\theta_H\delta(1-p)]},\end{aligned}$$

and $K_H'(g) = A(g) + B(g)$ where

$$\begin{aligned}A(g) &= \frac{\theta_H}{(1-\delta p)g} + \frac{\theta_H\delta(1-p)}{(1-\delta)[\theta_L(1-\delta)-\theta_H\delta(1-p)]}\left[\frac{\theta_L}{g} - \frac{\delta(1-p)}{1+\delta-2\delta p}\right],\\ B(g) &= -\frac{\delta(1-p)[\theta_L(1-\delta p)-\theta_H\delta(1-p)]}{(1-\delta p)(1+\delta-2\delta p)[\theta_L(1-\delta)-\theta_H\delta(1-p)]}\frac{d\gamma^H(g)}{dg}.\end{aligned}$$

We first show $A(g) > 0$. Suppose the coefficient on $\frac{1}{g}$ is positive. Then $A(g)$ is strictly decreasing and is minimized at $g = \tilde{g}_H$. By Corollary B.2, $\tilde{g}_H < \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}$. Since $A(g) = \frac{\theta_H\delta(1-p)}{\theta_L(1-\delta p)(1+\delta-2\delta p)} > 0$ when $g = \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}$, it follows that $A(g) > 0$ for $g \in [\underline{g}_H, \tilde{g}_H]$ in this case. Now suppose the coefficient on $\frac{1}{g}$ is negative, then $A(g)$ is strictly increasing and is minimized at $g = \underline{g}_H$. We have $\underline{g}_H = \psi \geq \theta_L^*$. When $g = \theta_L^*$, $A(g) = \frac{\theta_H[\theta_H\delta(1-p)-\theta_L(1-\delta p)]}{\theta_L(1+\delta-2\delta p)[\theta_H\delta(1-p)-\theta_L(1-\delta)]}$, which is strictly positive in the high-polarization case. Finally suppose the coefficient on $\frac{1}{g}$ is zero, then $A(g) > 0$. It follows that $A(g) > 0$ for $g \in [\underline{g}_H, \tilde{g}_H]$.

We next show that $B(g) > 0$. Since $\gamma^H(g)$ satisfies (B23), by the implicit function theorem,

$$\frac{d\gamma^H(g)}{dg} = \frac{\gamma^H(g)(1-\delta p)[\theta_L(1+\delta-2\delta p)-g\delta(1-p)]}{g[(1+\delta-2\delta p)(\theta_L(1-\delta)-\theta_H\delta(1-p))+\gamma^H(g)\delta^2(1-p)^2]}. \tag{B24}$$

At $\gamma^H(g) = g^*_H$ the denominator of $\frac{d\gamma^H(g)}{dg}$ is negative. Since the denominator is increasing in $\gamma^H(g)$ and $\gamma^H(g) \leq g^*_H$, the denominator is negative. Since $g \leq \tilde{g}_H < \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}$, the numerator is positive, and therefore $\frac{d\gamma^H(g)}{dg} < 0$. Since this is the high-polarization case and $\frac{d\gamma^H(g)}{dg} < 0$, it follows that $B(g) > 0$.

To summarize, $K'_H(g) = A(g) + B(g) > 0$ for $g \in [\underline{g}_H, \tilde{g}_H]$.

- $g \in [\tilde{g}_H, \theta_H+\theta_L]$: $K_H(g) = \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_H(g)$. Substituting for $V_H(g)$ and taking the derivative, we get

$$K'_H(g) = \frac{(1-\delta)\theta_H - \delta(1-p)\theta_L}{(1-\delta p)(1-\delta)g} + \frac{\delta(1-p)}{1-\delta p}B^1_H. \tag{B25}$$

  If $(1-\delta)\theta_H - \delta(1-p)\theta_L > 0$, then, $K'_H(g) > 0$ since $B^1_H > 0$.

  If $(1-\delta)\theta_H - \delta(1-p)\theta_L < 0$, then $K'_H(g)$ is increasing in g. We have $\tilde{g}_H > \underline{g}_H = \psi \geq \theta^*_L$. Plugging $g = \theta^*_L$ in (B25), we get $K'_H(g) = \frac{\theta_H(1-\delta p)-\theta_L\delta(1-p)}{(1-\delta p)(1+\delta-2\delta p)\theta_L} > 0$, and therefore $K_H(g)$ is strictly increasing for $g \in [\tilde{g}_H, \theta_H + \theta_L]$.

- $g \in [\theta_H+\theta_L, g^*_H]$: $K_H(g) = \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V_H(g)$. Substituting for $V_H(g)$ and taking the derivative, we get

$$K'_H(g) = \frac{\theta_H}{(1-\delta)g} - \frac{\delta(1-p)}{(1-\delta)(1+\delta-2\delta p)},$$

  which is strictly higher than 0 for $g \leq g^*_H$.

- $g > g^*_H$: $K_H(g) = \frac{\theta_H}{1-\delta p}\ln(g) + \frac{\delta(1-p)}{1-\delta p}V^*_H$, which is strictly increasing. ∎

We next show that the proposer has no profitable one-shot deviation. We omit the proof for party L since it is similar to that in the proof of Lemma 7.

We next establish monotonicity properties of $U^P_H(\mathbf{z})$, which is useful for later in the proof. For any status quo g, consider proposals $\mathbf{z}' = (g', x'_H, x'_L)$ such that the responder's acceptance constraint (5) is binding. That is,

$$x'_L = K_L(g) - \theta_L\ln(g') - \delta[(1-p)V_L(g') + pW_L(g')] = K_L(g) - K_L(g'). \tag{B26}$$

Substituting in the proposer's payoff function, we get $U_H^P(\mathbf{z}') = 1 - g' - x'_L + \theta_H \ln(g') + \delta[pV_H(g') + (1-p)W_H(g')]$, which implies

$$\text{(B27)} \qquad \frac{\partial U_H^P}{\partial g'} = -1 + \frac{\theta_H+\theta_L}{g'} + \delta[(1-p)V'_L(g') + pW'_L(g')] + \delta[pV'_H(g') + (1-p)W'_H(g')].$$

Substituting for V'_L, W'_L, V'_H, W'_H, we get a closed-form solution for $\frac{\partial U_H^P}{\partial g'}$ except when $g \in (\underline{g}_H, \tilde{g}_H)$. Specifically, if $g' < g_L^*$, then $\frac{\partial U_H^P}{\partial g'} = \frac{\theta_H+\theta_L}{g'} - 1 > 0$; if $g' \in (g_L^*, \underline{g}_H)$, then $\frac{\partial U_H^P}{\partial g'} = \frac{1+\delta-2\delta p}{1-\delta p}(\frac{\theta_H+\theta_L}{g'} - 1) > 0$; if $g' \in (\tilde{g}_H, \theta_H + \theta_L)$, then $\frac{\partial U_H^P}{\partial g'} = \frac{1+\delta-2\delta p}{1-\delta p}(\frac{\theta_H+\theta_L}{g'} - 1) > 0$; if $g' \in (\theta_H + \theta_L, g_H^*)$, then $\frac{\partial U_H^P}{\partial g'} = \frac{1}{1-\delta p}(\frac{\theta_H+\theta_L}{g'} - 1) < 0$; if $g' > g_H^*$, then $\frac{\partial U_H^P}{\partial g'} = \frac{\theta_H+\theta_L}{g'} - 1 < 0$.

Note that $\frac{\partial U_H^P}{\partial g'} = f'_H(g') + K'_L(g')$. Also, if $g' \in (\underline{g}_H, \tilde{g}_H)$, then $\frac{d\gamma^H(g')}{dg'} = \frac{K'_L(g')}{K'_L(\gamma^H(g'))}$. Hence, for $g' \in (\underline{g}_H, \tilde{g}_H)$,

$$\frac{\partial U_H^P}{\partial g'} = -1 + \frac{1+\delta-2\delta p}{1-\delta p}\frac{\theta_H}{g'} + K'_L(g')C(g')$$

where

$$C(g') = 1 + \frac{\delta(p+\delta-2\delta p)[-(1-\delta p)\gamma^H(g') + (1+\delta-2\delta p)\theta_H]}{[(1-\delta)\theta_L - \delta(1-p)\theta_H](1+\delta-2\delta p) + \gamma^H(g')\delta^2(1-p)^2}.$$

We verify that $C(g') > 0$ in the high-polarization case where $\frac{\theta_H}{\theta_L} > \frac{1-\delta p}{\delta(1-p)}$. Since $K'_L(g') > 0$ for $g' < \tilde{g}_H$ by Claim B.5 and Corollary B.2, it follows that $\frac{\partial U_H^P}{\partial g'} > 0$ for $g' \in (\underline{g}_H, \tilde{g}_H)$.

Below we show that proposer H has no profitable one-shot deviation.

- $g \leq \underline{g}_H$ or $g \geq g_H^*$: In this case, $\gamma^H(g) = g_H^*$ and $\chi_L^H(g) = 0$.

  Since $g_H^* \in \arg\max f_H(g)$, party H has no incentive to deviate from proposing $\gamma^H(g) = g_H^*$ and $\chi_L^H(g) = 0$.

- $\underline{g}_H \leq g \leq \tilde{g}_H$: In this case, $\gamma^H(g) \in [\theta_H + \theta_L, g_H^*]$ and $\chi_L^H(g) = 0$.

  We first show that proposing $\pi^H(g)$ is better than proposing $(\hat{g}, \hat{x}_H, \hat{x}_L)$ with $\hat{g} > \gamma^H(g)$ and then show that it is better than proposing $(\hat{g}, \hat{x}_H, \hat{x}_L)$ with $\hat{g} < \gamma^H(g)$.

  - $\hat{g} > \gamma^H(g)$: Since $\gamma^H(g) > \theta_H + \theta_L > \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}$, by Claim B.5, for $\hat{g} > \gamma^H(g)$, $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$. Since party L's payoff is strictly decreasing in x_L, we only need to consider proposals such that the responder's acceptance constraint (5) is binding. Since $U_H^P(\hat{\mathbf{z}})$ is decreasing in $\hat{g}$ for $\hat{g} > \gamma^H(g) \geq \theta_H + \theta_L$ as shown before, the proposer has no incentive to make any proposal with $\hat{g} > \gamma^H(g)$.
  - $\tilde{g}_H \leq \hat{g} < \gamma^H(g)$: Consider $\hat{\mathbf{z}} = (\hat{g}, 1-\hat{g}, 0)$. Then $U_H^P(\hat{\mathbf{z}}) = f_H(\hat{g})$. As shown in the proof of Claim B.4, $f_H(\hat{g})$ is increasing in $\hat{g}$ for $\tilde{g}_H < \hat{g} < g_H^*$. Since $\pi^H(g) = (\gamma^H(g), 1-\gamma^H(g), 0)$ where $\gamma^H(g) < g_H^*$, it follows that $U_H^P(\pi^H(g)) > U_H^P(\hat{\mathbf{z}})$ for any $\hat{g} < \gamma^H(g) \leq g_H^*$. Since party H's payoff is decreasing in x_L,

$U_H^P(\hat{\mathbf{z}}) \geq U_H^P((\hat{g}, \hat{x}_H, \hat{x}_L))$ for any $(\hat{g}, \hat{x}_H, \hat{x}_L) \in \mathcal{B}$, it follows that $U_H^P(\pi^H(g)) > U_H^P((\hat{g}, \hat{x}_H, \hat{x}_L))$ for any $\hat{g} < \gamma^H(g) \leq g_H^*$. Hence the proposer has no incentive to deviate and make a proposal with $\tilde{g}_H \leq \hat{g} < \gamma^H(g)$.

– $g \leq \hat{g} \leq \tilde{g}_H$. Consider $\hat{\mathbf{z}} = (\hat{g}, 1-\hat{g}, 0)$. Then $U_H^P(\hat{\mathbf{z}}) = f_H(\hat{g})$. Recall that for $g \geq g_L^*$, $f_H(g) = 1 - g + \frac{\theta_H(1+\delta-2\delta p)}{1-\delta p}\ln(g) + \frac{\delta(p+\delta-2\delta p)}{1-\delta p}V_H(g)$. Also, for $\underline{g}_H \leq \hat{g} \leq \tilde{g}$, $V_H(\hat{g}) = V_H(\gamma^H(\hat{g}))$. Hence, $f_H(\gamma^H(\hat{g})) - f_H(\hat{g}) = -\gamma^H(\hat{g}) + \hat{g} + \frac{\theta_H(1+\delta-2\delta p)}{1-\delta p}(\ln(\gamma^H(\hat{g})) - \ln(\hat{g}) > 0$ since $\hat{g} \leq \gamma^H(\hat{g}) \leq \frac{\theta_H(1+\delta-2\delta p)}{1-\delta p}$. Since $\gamma^H(\hat{g}) < \gamma^H(g)$ and $f_H(g)$ is increasing in $(\theta_H + \theta_L, g_H^*)$ as shown in the proof of Claim B.4, it follows that $f_H(\hat{g}) \leq f_H(\gamma^H(\hat{g})) \leq f_H(\gamma^H(g))$ and therefore $U_H^P(\pi^H(g)) \geq U_H^P(\hat{\mathbf{z}})$ for any $\hat{g} \in [g, \tilde{g}_H]$. Hence proposing $\pi^H(g)$ is better than proposing any $(\hat{g}, \hat{x}_H, \hat{x}_L) \in \mathcal{B}$ with $g \leq \hat{g} \leq \tilde{g}_H$.

– $\hat{g} < g$: By Corollary B.2, $g < \tilde{g}_H < \frac{\theta_L(1+\delta-2\delta p)}{\delta(1-p)}$. Hence, for $\hat{g} < g$, $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$ by Claim B.5. Consider $\hat{\mathbf{z}} = (\hat{g}, \hat{x}_H, \hat{x}_L)$ such that (B26) holds. Since $U_H^P(\hat{\mathbf{z}})$ is increasing in $\hat{g}$ for $\hat{g} < g$ as shown before, the proposer has no incentive to deviate and make a proposal with $\hat{g} < g$.

- $\tilde{g}_H \leq g \leq \theta_H + \theta_L$: In this case, $\gamma^H(g) = \theta_H + \theta_L$ and $\chi_L^H(g) \geq 0$.

  Let $h(g) = \max\{g' \in [0,1] : K_L(g') = K_L(g)\}$ and $l(g) = \min\{g' \in [0,1] : K_L(g') = K_L(g)\}$. By Claim B.5, $h(g) \in [\frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L, \theta_H + \theta_L]$ and $l(g) \in [\tilde{g}_H, \frac{1+\delta-2\delta p}{\delta(1-p)}\theta_L]$.

  – $\hat{g} \geq h(g)$: For $\hat{\mathbf{z}} = (\hat{g}, \hat{x}_H, \hat{x}_L)$, Claim B.5 implies that $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$. Consider $\hat{\mathbf{z}}$ such that (B26) holds. As shown before, $U_H^P(\hat{g})$ is increasing for $\hat{g} \in [h(g), \theta_H + \theta_L)$ and decreasing for $\hat{g} > \theta_H + \theta_L$, and therefore the proposer has no incentive to deviate and make any proposal with $\hat{g} \geq h(g)$ and $\hat{g} \neq \theta_H + \theta_L$.

  – $\hat{g} \in [l(g), h(g)]$: Consider $\hat{\mathbf{z}} = (\hat{g}, 1-\hat{g}, 0)$. Since $U_H^P(\hat{\mathbf{z}}) = f_H(\hat{g})$ and $f_H(\hat{\mathbf{z}})$ is increasing for $\hat{g} \in [l(g), h(g)]$, it follows that $U_H^P((h(g), 1-h(g), 0)) > U_H^P(\hat{\mathbf{z}})$ for any $\hat{g} \in (l(g), h(g)$ and therefore the proposer has no incentive to deviate and make a proposal with $\hat{g} \in [l(g), h(g)]$.

  – $\hat{g} < l(g)$: For $\hat{\mathbf{z}} = (\hat{g}, \hat{x}_H, \hat{x}_L)$, Claim B.5 implies that $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$. Consider $\hat{\mathbf{z}}$ such that (B26) holds. As shown before, $U_H^P(\hat{g})$ is increasing for $\hat{g} < l(g)$, and therefore the proposer has no incentive to deviate and make any proposal with $\hat{g} \geq l(g)$.

- $g \in [\theta_H + \theta_L, g_H^*]$: In this case, $\gamma^H(g) = g$ and $\chi_L^H(g) = 0$. Recall that $l(g) = \min\{g' \in [0,1] : K_L(g') = K_L(g)\}$. In this case, $l(g) \in [\underline{g}_H, \tilde{g}_H]$.

  – $\hat{g} > g$: For $\hat{\mathbf{z}} = (\hat{g}, \hat{x}_H, \hat{x}_L)$, Claim B.5 implies that $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$. Consider $\hat{\mathbf{z}}$ such that (B26) holds. As shown before, $U_H^P(\hat{g})$ is decreasing for $\hat{g} > \theta_H + \theta_L$, and therefore the proposer has no incentive to deviate and make any proposal with $\hat{g} \geq g$.

– $\tilde{g}_H \leq \hat{g} < g$: Consider $\hat{\mathbf{z}} = (\hat{g}, 1-\hat{g}, 0)$. Since $U_H^P(\hat{\mathbf{z}}) = f_H(\hat{g})$ and $f_H(\hat{g})$ is increasing if $\tilde{g}_H \leq \hat{g} < g$, it follows that the proposer has no incentive to deviate and make a proposal with $\hat{g} \in [\tilde{g}_H, g)$.

– $l(g) \leq \hat{g} \leq \tilde{g}_H$. Consider $\hat{\mathbf{z}} = (\hat{g}, 1-\hat{g}, 0)$. Note that for $\hat{g} \in [l(g), \tilde{g}_H]$, $f_H(\hat{g}) < f_H(\gamma^H(\hat{g}))$. Also, since $\gamma^H(\hat{g}) < g$ and therefore $f_H(\gamma^H(\hat{g}) < f_H(g)$, it follows that $f_H(\hat{g}) < f_H(g)$. Hence the proposer has no incentive to deviate and make a proposal with $\hat{g} \in [l(g), \tilde{g}_H]$.

– $\hat{g} \leq l(g)$: For $\hat{\mathbf{z}} = (\hat{g}, \hat{x}_H, \hat{x}_L)$, Claim B.5 implies that $\alpha^L(g, (\hat{g}, \hat{x}_H, \hat{x}_L)) = 1$ only if $\hat{x}_L > 0$. Consider $\hat{\mathbf{z}}$ such that (B26) holds. As shown before, $U_H^P(\hat{g})$ is increasing for $\hat{g} \leq l(g)$, and therefore the proposer has no incentive to deviate and make any proposal with $\hat{g} \leq l(g)$.

To summarize, party H has no incentive to deviate from $\pi^H(g)$ for any $g \in [0, 1]$. ∎

B7. High-polarization equilibrium when condition (∗∗) fails:

Figure B2 illustrates the parties' proposal strategies when condition (∗∗) fails. Similar to the low-polarization case, (G1′)-(G4′) are still satisfied in this figure except $\gamma^L(g) > \theta_H + \theta_L$ for very high status quos. As shown in Section V.B, the failure of condition (∗∗) does not affect the set of steady states.

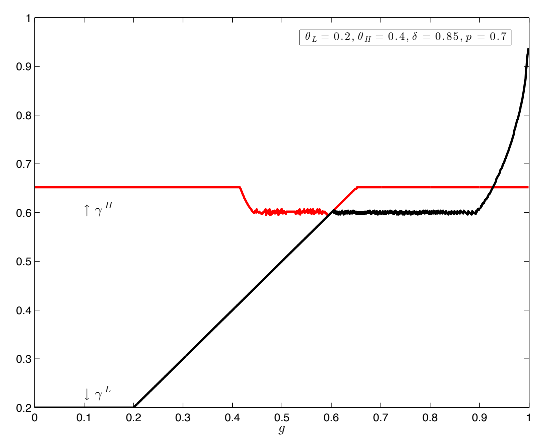
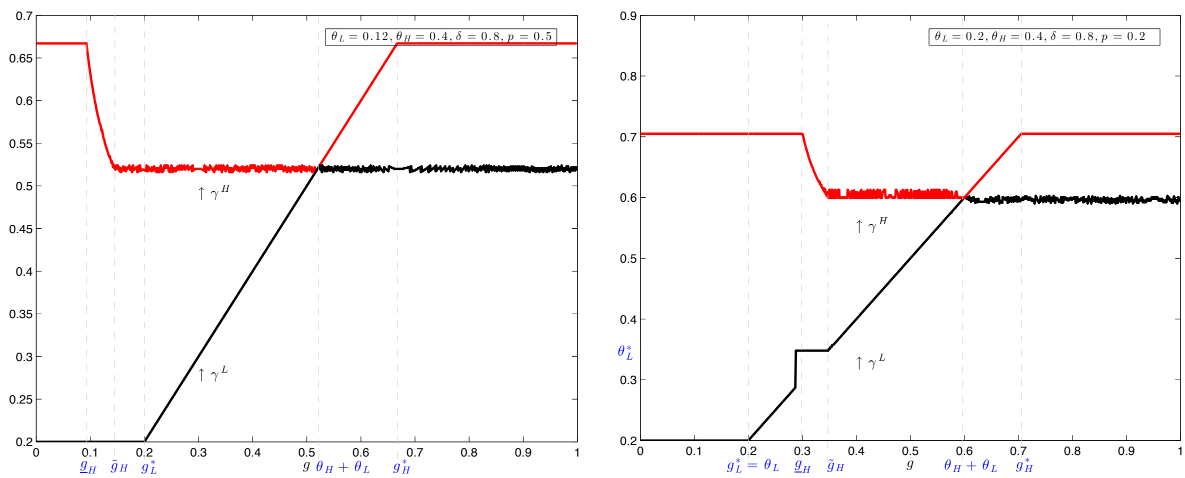


Figure B2. : $\gamma^i(g)$ in high-polarization case when condition (∗∗) does not hold

B8. High-polarization equilibrium when $\psi < \theta_L^*$:

Figure B3 illustrates the proposal strategies when $\psi < \theta_L^*$. In this case, polarization is very high. As the figure shows, two kinds of equilibria arise. In panel (a), the equilibrium strategies still satisfy (G1′)-(G4′) with the exception that $g_L^* > \underline{g}_H$. In this case, party L's dynamic ideal is $g_L^* = \theta_L^* > \theta_L$, an analog to party H's dynamic ideal $g_H^* = \theta_H^*$. To understand the difference between Figure 5 and Figure B3(a), recall that $\underline{g}_H$ is the threshold below which party L's constraint is slack. In Figure 5, $g_L^* = \theta_L < \underline{g}_H$, implying that party L's constraint is slack at its dynamic ideal, but in Figure B3(a), θ_L is greater than $\underline{g}_H$, implying that party L's choice of public good has a dynamic effect because if party H comes to power in the next period, party L's constraint is binding. This dynamic effect results in party L's dynamic ideal g_L^* being higher than its static ideal θ_L. In panel (b), party H's strategy again satisfies the guesses, but party L's strategy violates (G2′). In particular, instead of proposing $g' = g$, now party L proposes a constant level $g' = \theta_L^*$ when the status quo is in a subinterval of $[g_L^*, \theta_H + \theta_L]$ (see the kink in Figure B3(b)). By setting g' at a higher level θ_L^*, party L guarantees a higher bargaining position for itself in the next period. Hence if party H comes to power in the next period, it is forced to set the efficient level of the public good rather than its dynamic ideal (which is much higher).

Although the details of party L's strategy violate certain aspects of (G2′) when $\psi < \theta_L^*$, the efficiency implications and the set of steady states are still the same, as illustrated in Figure B3 and formalized in Section V.B.



(a) $g_L^* = \theta_L^* > \theta_L$

(b) $g_L^* = \theta_L$

Figure B3. : $\gamma^i(g)$ in high-polarization case when $\psi < \theta_L^*$

B9. Proof of Proposition 8

Let $\mathcal{SPEA}$ denote the set of static Pareto efficient allocations.

CLAIM 3: *An infinite repetition of an allocation $\boldsymbol{s} \in \mathcal{SPEA}$ is a dynamic Pareto efficient allocation.*

Proof: Suppose $\mathbf{s} \in \mathcal{SPEA}$. Then there exists $\alpha_L \geq 0$ and $\alpha_H \geq 0$ such that $\mathbf{s}$ is a solution to the following maximization problem:

$$\max_{\mathbf{b} \in \mathcal{B}} \alpha_L u_L(\mathbf{b}) + \alpha_H u_H(\mathbf{b}).$$

It follows that repeating $\mathbf{s}$ in every period is a solution to the following maximization problem:

$$\max_{\{\mathbf{b}^t\}_{t=0}^{\infty} \in \mathcal{B}^\infty} \sum_{t=0}^{\infty} \delta^t \left(\alpha_L u_L(\mathbf{b}^t) + \alpha_H u_H(\mathbf{b}^t) \right),$$

which is equivalent to

$$\max_{\{\mathbf{b}^t\}_{t=0}^{\infty} \in \mathcal{B}^\infty} \alpha_L \sum_{t=0}^{\infty} \delta^t u_L(\mathbf{b}^t) + \alpha_H \sum_{t=0}^{\infty} \delta^t u_H(\mathbf{b}^t).$$

Hence, repeating $\mathbf{s}$ in every period is a dynamic Pareto efficient allocation. ∎

CLAIM 4: *Suppose $\boldsymbol{s} \in \mathcal{SPEA}$. Then, (i) in any equilibrium in the infinite horizon game with all mandatory spending, $V_i(\boldsymbol{s}) = \frac{1}{1-\delta} u_i(\boldsymbol{s})$ and $W_i(\boldsymbol{s}) = \frac{1}{1-\delta} u_i(\boldsymbol{s})$ for $i \in \{H, L\}$, and (ii) there exists an equilibrium in the infinite horizon game with all mandatory spending in which $\pi^i(\boldsymbol{s}) = \boldsymbol{s}$ for $i \in \{H, L\}$.*

Proof: First, note that for any status quo $\mathbf{s} \in \mathcal{B}$, proposing to maintain the status quo will be accepted by the responder. Hence, $V_i(\mathbf{s}) \geq \frac{1}{1-\delta} u_i(\mathbf{s})$ since when party i is the proposer, it can maintain the status quo by proposing $\mathbf{s}$ and when the other party comes to power in the future, it can reject any proposal other than $\mathbf{s}$. Similarly, $W_i(\mathbf{s}) \geq \frac{1}{1-\delta} u_i(\mathbf{s})$ since when party i is the responder, it can maintain the status quo $\mathbf{s}$ by rejecting any proposal not equal to $\mathbf{s}$ and when it comes to power in the future, it can propose to maintain $\mathbf{s}$.

Since $\mathbf{s} \in \mathcal{SPEA}$, an infinite repetition of $\mathbf{s}$ is dynamically Pareto efficient by Claim 3. Hence, $W_j(\mathbf{s}) \geq \frac{1}{1-\delta} u_j(\mathbf{s})$ implies that $V_i(\mathbf{s}) \leq \frac{1}{1-\delta} u_i(\mathbf{s})$ for $i \neq j$. Since $V_i(\mathbf{s}) \geq \frac{1}{1-\delta} u_i(\mathbf{s})$, we have $V_i(\mathbf{s}) = \frac{1}{1-\delta} u_i(\mathbf{s})$ and $W_j(\mathbf{s}) = \frac{1}{1-\delta} u_j(\mathbf{s})$ for $i \neq j$. It also follows that proposing to maintain $\mathbf{s}$ is a best response for party i. Hence, there exists an equilibrium in which $\pi^i(\mathbf{s}) = \mathbf{s}$. ∎

We immediately have the next result, which establishes the "if" part of Proposition 8.

CLAIM 5: *If $\boldsymbol{s}$ is a static Pareto efficient allocation, then there exists an equilibrium in which $\boldsymbol{s}$ is a steady state.*

We next show that only a static Pareto efficient allocation can be a steady state, establishing the "only if" part of Proposition 8.

CLAIM 6: *If $\boldsymbol{s}$ is not a static Pareto efficient allocation, then there is no equilibrium in which $\boldsymbol{s}$ is a steady state.*

Proof: Suppose to the contrary that there exists an equilibrium and a state $\mathbf{s} \notin \mathcal{SPEA}$ such that $\pi^i(\mathbf{s}) = \mathbf{s}$ in this equilibrium. Then, $V_i(\mathbf{s}) = W_i(\mathbf{s}) = K_i(\mathbf{s}) = \frac{1}{1-\delta}u_i(\mathbf{s})$ in this equilibrium for $i \in \{H, L\}$. Since $\mathbf{s} \notin \mathcal{SPEA}$, there exists an $\hat{\mathbf{s}} \in \mathcal{SPEA}$ such that $\hat{\mathbf{s}}$ Pareto dominates $\mathbf{s}$, that is, $u_i(\hat{\mathbf{s}}) > u_i(\mathbf{s})$, $u_j(\hat{\mathbf{s}}) \geq u_j(\mathbf{s})$ for some $i \in \{H, L\}$ and $j \neq i$. Suppose party i proposes $\hat{\mathbf{s}}$ when the status quo is $\mathbf{s}$. If the responder j accepts, its payoff is $u_j(\hat{\mathbf{s}})+\delta[(1-p)V_j(\hat{\mathbf{s}})+pW_j(\hat{\mathbf{s}})]$. Since $\hat{\mathbf{s}} \in \mathcal{SPEA}$, $V_j(\hat{\mathbf{s}}) = W_j(\hat{\mathbf{s}}) = \frac{1}{1-\delta}u_j(\hat{\mathbf{s}})$ in any equilibrium by Claim 4. Hence,

$$u_j(\hat{\mathbf{s}}) + \delta[(1-p)V_j(\hat{\mathbf{s}}) + pW_j(\hat{\mathbf{s}})] = \tfrac{1}{1-\delta}u_j(\hat{\mathbf{s}}) \geq K_j(\mathbf{s}),$$

and therefore responder j accepts $\hat{\mathbf{s}}$ when the status quo is $\mathbf{s}$. It follows that party i's payoff by proposing $\hat{\mathbf{s}}$ when the status quo is $\mathbf{s}$ is

$$u_i(\hat{\mathbf{s}}) + \delta[pV_i(\hat{\mathbf{s}}) + pW_i(\hat{\mathbf{s}})] = \tfrac{1}{1-\delta}u_i(\hat{\mathbf{s}}) > V_i(\mathbf{s}),$$

a contradiction. Hence, there is no equilibrium in which $\mathbf{s} \notin \mathcal{SPEA}$ is a steady state. ∎

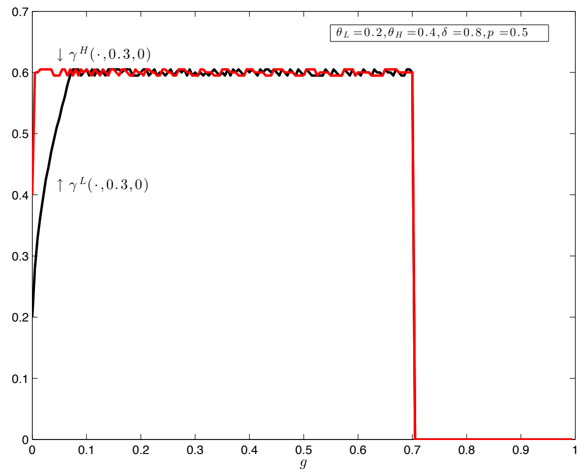


Figure B4. : $\gamma^i(g, 0.3, 0)$

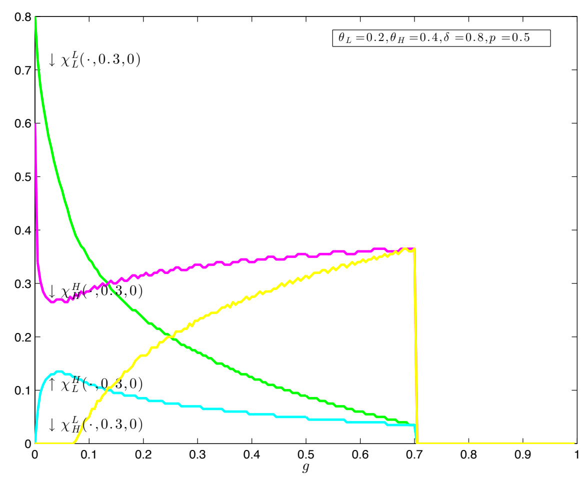


Figure B5. : $\chi_j^i(g, 0.3, 0)$

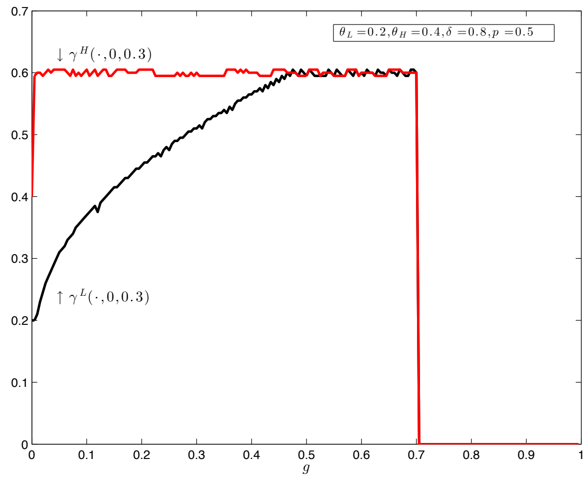


Figure B6. : $\gamma^i(g, 0, 0.3)$

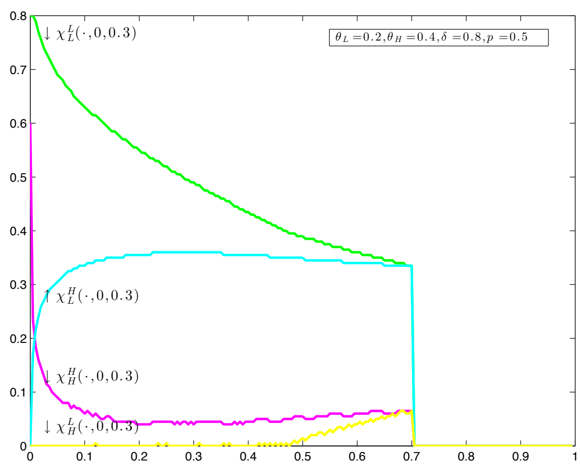


Figure B7. : $\chi_j^i(g, 0, 0.3)$