

Online Appendix (a Threshold Equilibrium): Not Only What But also When: A Theory of Dynamic Voluntary Disclosure.

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June 2013

This Appendix is organized as follows. First, in Section A.1, we discuss a variant of a static disclosure model that provides a numerical result and analytical insights we later use in the proof of Proposition 1. This variant of the static model may also be of independent interest. Then, in Section A.2 we provide a proof of the Proposition. The proof starts with noticing properties of the equilibrium prices if the agent follows any threshold strategy. Given these properties, we show that the best response of the agent is indeed to follow a threshold strategy, establishing existence of a threshold equilibrium with the properties we discussed. In the same section we also establish the second claim in Proposition 1. Finally, Section A.3 contains omitted proofs of some lemmas describing the sensitivity of equilibrium prices to the disclosed signals if the agent follows a threshold strategy.

A.1 A Variant of a Static Model

Consider the following static disclosure setting, similar to Dye (1985) and Jung and Kwon (1988). With probability p the agent learns the firm's value, which is the realization of a random variable $S \sim N(\mu, \sigma^2)$.¹ If the agent learns the realization of S he may choose to disclose it. We are interested in investors' beliefs about the firm's value given no disclosure for an arbitrary threshold disclosure policy. That is, what is the expectation of S given that the agent discloses s if and only if $s \geq z$, for exogenously determined z . Unlike Dye (1985) and Jung and Kwon (1988), we are not constraining z to be consistent with optimal disclosure strategy by the agent, i.e., z is not part of an equilibrium. We will refer to this setting as the "*Dye setting with an exogenous disclosure threshold.*"

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¹The reason we are considering general μ is that in our dynamic setting investors will update their beliefs about the undisclosed signal, y , based on the value of the disclosed signal, x .

Denote by $h^{stat}(\mu, z)$ investors' expectation of S given that no disclosure was made and given that the disclosure threshold is z . Figure A.1 plots $h^{stat}(\mu, z)$ for $S \sim N(0, 1)$ and $p = 0.5$.

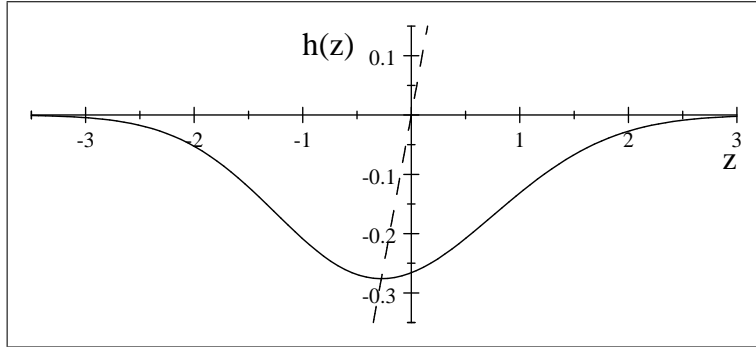


Figure A.1: Price Given No-Disclosure in a Dye Setting with Exogenous Disclosure Threshold z

For $z \rightarrow \infty$ none of the agents discloses, and hence, following no disclosure investors do not revise their beliefs relative to the prior. For $z \rightarrow (-\infty)$ all agents who obtain a signal disclose it, and therefore, following no disclosure investors infer that the agent is uninformed, so investors posterior beliefs equal the prior distribution (as for $z \rightarrow \infty$). As the exogenous disclosure threshold, z , increases from $-\infty$, upon observing no disclosure investors know that the agent is either uninformed or that the agent is informed and his type is lower than z . Therefore, for any finite disclosure threshold, z , investors' expectation of S following no disclosure is lower than the prior mean (zero). The following lemma provides a further characterization of investors' expectation about S given no disclosure, $h^{stat}(\mu, z)$.

Lemma A.1 *Consider the Dye setting with an exogenous disclosure threshold. Then:*

1. $h^{stat}(\mu + \Delta, z + \Delta) = h^{stat}(\mu, z) + \Delta$ for any constant Δ ; this implies that

$$\frac{\partial}{\partial \mu} h^{stat}(\mu, z) + \frac{\partial}{\partial z} h^{stat}(\mu, z) = 1.$$

2. $z^* = \arg \min_z h^{stat}(\mu, z)$ if and only if $z^* = h^{stat}(\mu, z^*)$. This implies that the equilibrium disclosure threshold in the standard Dye (1985) and Jung and Kwon (1988) equilibrium minimizes $h^{stat}(\mu, z)$.

The second point follows from Lemma 2 (the Generalized Minimum Principle). Note that for all $z < h^{stat}(\mu, z)$ the price given no disclosure, $h^{stat}(\mu, z)$, is decreasing in z (and for $z > h^{stat}(\mu, z)$ it is increasing in z).

Direct analysis of the $h^{stat}(\mu, z)$ shows that:

Claim A.1 (Numerical Result) *For $p < 0.95$ the absolute value of the slope of $h^{stat}(\mu, z)$ with respect to z is uniformly bounded by 1.*

We use this claim extensively in the proof below since it allow us to bound how future prices (in particular, $P_2(x, 1)$ and $P_2(x, 2)$) change with x and in turn that allows us to establish existence of a threshold strategy equilibrium. This is where we use the assumption $p < 0.77$ in the proof of Proposition 1. Note the difference in the bound in the proposition ($p < 0.77$) and in the claim ($p < 0.95$). The reason is that in the dynamic setup in period 2 the agent is informed about Y with probability $p + p(1 - p)$, which needs to be less than 0.95 for us to apply this claim.

For the analysis of our dynamic model it will prove useful to consider an even richer variant of this model, allowing a random threshold policy. In particular, first, nature chooses publicly μ , the unconditional mean of S . Then, with probability λ_i , $i \in \{1, \dots, K\}$, where $\sum_{i=1}^K \lambda_i = p$, the agent discloses s if and only if $s \geq z_i(\mu)$.

The reason we are considering a random disclosure policy is as follows. In our dynamic setting, when by $t = 2$ the agent disclosed a single signal investors do not know whether the agent learned a second signal, and if so, whether he learned it at $t = 1$ or at $t = 2$. Since the agent follows different disclosure thresholds at the two possible dates, investors in equilibrium must assign a probability distribution over different disclosure thresholds. Moreover, in the dynamic model the disclosure thresholds for Y change with x and the disclosed x affects investors' unconditional expectation of Y . Therefore to apply these generic results to our dynamic model we write z as a function of the unconditional mean, μ .

Let us denote by $h^{stat}(\mu, \{z_i(\mu)\})$ the conditional expectation of S given no disclosure and given that the disclosure thresholds are $\{z_i(\mu)\}$ (assuming that $\{z_i(\mu)\}$ are differentiable).

Lemma A.2 *For $p \leq 0.95$ suppose that $z_i(\mu) < h^{stat}(\mu, \{z_i(\mu)\})$ and $z'_i(\mu) \in [0, c]$ for all i . Then $\frac{d}{d\mu} h^{stat}(\mu, \{z_i(\mu)\}) \in (\min\{1, 2 - c\}, 2)$.*

Before we formally prove Lemma A.2, we analyze the particular case in which the disclosure strategy is nonrandom, i.e., $K = 1$. This provides the basic intuition for Lemma A.2.

We start by providing the two simplest examples, for the cases where $z'(\mu) = 1$ and $z'_i(\mu) = 0$. These examples are useful in demonstrating the basic logic and how it can be analyzed using Figure A.1. These two examples also provide most of the intuition for the case with no restriction on $z'_i(\mu)$, which is presented in Example 3. Note that Example 3 also provides the upper and lower bounds for the more general case in Lemma A.2.

Examples (all the examples assume $K = 1$):

1. If $z'(\mu) = 1$ then $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) = 1$.

Using point 1 in Lemma A.1 we have $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) = \frac{\partial}{\partial \mu} h^{stat}(\mu, z) + z'(\mu) * \frac{\partial}{\partial z} h^{stat}(\mu, z) =$

1. The intuition can be demonstrated using Figure A.1. A unit increase in μ (keeping z constant) shifts the entire graph both upwards and to the right by one unit. However, since also z increases by one unit, the overall effect is an increase in $h^{stat}(\mu, z(\mu))$ by one unit.

2. If $z'(\mu) = 0$ and $z(\mu) = z^*$, then $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) \in (1, 2)$.

From Lemma A.1 we know that $\frac{\partial}{\partial \mu} h^{stat}(\mu, z^*) + \frac{\partial}{\partial z^*} h^{stat}(\mu, z^*) = 1$ and therefore $\frac{\partial}{\partial \mu} h^{stat}(\mu, z^*) =$

$1 - \frac{\partial}{\partial z^*} h^{stat}(\mu, z^*)$. From Claim A.1 we also know that $\frac{\partial}{\partial z^*} h^{stat}(\mu, z^*) \in (-1, 0)$ since $z^* \leq h^{stat}(\mu, z^*)$. Therefore, $\frac{\partial}{\partial \mu} h^{stat}(\mu, z^*) \in (1, 2)$. The intuition can be demonstrated using Figure A.1. The effect of a unit increase in μ can be presented as a sum of two effects: (i) a unit increase in the disclosure threshold, z , as well as a shift of the entire graph both to the right and upwards by one unit, and (ii) a unit decrease in the disclosure threshold, z , (as $z'(\mu) = 0$). The first effect is similar to Example 1 above and therefore increases $h^{stat}(\mu, z(\mu))$ by one. The second effect increases $h^{stat}(\mu, z(\mu))$ by the absolute value of the slope of $h^{stat}(\mu, z)$, which is between zero and one.

3. In case $z'(\mu) = c$, we have $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) \in (\min\{1, 2 - c\}, \max\{1, 2 - c\})$.

The previous examples are nested in this more general case. Following a similar logic, we conclude that $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) = \frac{\partial}{\partial \mu} h^{stat}(\mu, z) + c \frac{\partial}{\partial z} h^{stat}(\mu, z) = 1 + (c - 1) \frac{\partial}{\partial z} h^{stat}(\mu, z(\mu))$. Recall that $\frac{\partial}{\partial z} h^{stat}(\mu, z(\mu)) \in (-1, 0)$ for $p < 0.95$.

We next provide the a formal proof of Lemma A.2.

Proof of Lemma A.2

By applying Bayes rule, $h^{stat}(\mu, \{z_i(\mu)\})$ is given by:

$$h^{stat}(\mu, \{z_i(\mu)\}) = \frac{(1-p)\mu + \sum_{i=1}^K \lambda_i \int_{-\infty}^{z_i(\mu)} y \phi(y|\mu) dy}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(\mu)|\mu)}.$$

Taking the derivative of $h^{stat}(\mu, \{z_i(\mu)\})$ with respect to μ and applying some algebraic manipulation yields:

$$\frac{d}{d\mu} h^{stat}(\mu, \{z_i(\mu)\}) = 1 + \frac{\sum_{i=1}^K \lambda_i (z'_i(\mu) - 1) \phi(z_i(\mu)|\mu) (z_i(\mu) - h^{stat}(\mu, \{z_i(\mu)\}))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(\mu)|\mu)}.$$

We start by proving the supremum of this derivative.

Given that $z'_i(\mu) \geq 0$ and $z_i(\mu) \leq h^{stat}(\mu, \{z_i(\mu)\})$ for all $i \in \{1, \dots, K\}$ we have

$$\begin{aligned} \frac{d}{d\mu} h^{stat}(\mu, \{z_i(\mu)\}) &\leq 1 + \frac{\sum_{i=1}^K \lambda_i \phi(z_i(\mu)|\mu) (h^{stat}(\mu, \{z_i(\mu)\}) - z_i(\mu))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(\mu)|\mu)} \\ &\leq 1 + \max_{\substack{z_i \leq h(x) \\ i \in \{1, \dots, K\}}} \frac{\sum_{i=1}^K \lambda_i \phi(z_i(\mu)|\mu) (h^{stat}(\mu, \{z_i(\mu)\}) - z_i(\mu))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(\mu)|\mu)}. \end{aligned} \tag{1}$$

Due to symmetry, for all $i \in \{1, \dots, K\}$ the maximum is achieved at the same $z_i(\mu) = \hat{z}(\mu)$. To see

this, note that the FOC of the maximization with respect to $z_i(\mu)$ is

$$0 = (\phi'(z_i(\mu) | \mu) (h^{stat}(\mu, \{z_i(\mu)\}) - z_i(\mu)) - \phi(z_i(\mu) | \mu)) \left((1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(\mu) | \mu) \right) - \left(\sum_{i=1}^K \lambda_i \phi(z_i(\mu) | \mu) (h^{stat}(\mu, \{z_i(\mu)\}) - z_i(\mu)) \right) \phi(z_i(\mu) | \mu).$$

Since $\phi'(z_i(\mu) | \mu) = -\alpha(z_i(\mu) - \mu) \phi(z_i(\mu) | \mu)$ (for some constant $\alpha > 0$), this simplifies to

$$-\alpha(z_i(\mu) - \mu) (h^{stat}(\mu, \{z_i(\mu)\}) - z_i(\mu)) = \frac{\sum_{i=1}^K \lambda_i \phi(z_i(\mu) | \mu) (h^{stat}(\mu, \{z_i(\mu)\}) - z_i(\mu))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(\mu) | \mu)} + 1.$$

In the range $z_i(\mu) \leq h^{stat}(\mu, \{z_i(\mu)\}) \leq \mu$, the LHS is decreasing in $z_i(\mu)$.² The RHS is the same for all i . Therefore, the unique solution to this system of FOC is for all $z_i(\mu)$ to be equal (and note that the maximum is achieved at an interior point since at $z_i(\mu) = h^{stat}(\mu, \{z_i(\mu)\})$ the LHS is zero and the RHS is positive; and as $z_i(\mu)$ goes to $-\infty$ the LHS goes to $+\infty$ while the RHS is bounded).

Returning to the bound in (1), that the maximum is achieved for some $\hat{z}(\mu)$ constant for all i , implies that Example 3 (discussed above) can be used to provide the upper bound: $\frac{d}{d\mu} h^{stat}(\mu, \{z_i(\mu)\}) \leq \max\{1, 2 - \min_i \{z'_i(\mu)\}\}$. The lower bound can be achieved in a similar way by observing that if we want to minimize the slope we will again choose the same $z_i(\mu)$ for all i , and therefore by Example 3 $\frac{d}{d\mu} h^{stat}(\mu, \{z_i(\mu)\}) \geq \min\{1, 2 - \max_i \{z'_i(\mu)\}\}$.³ Computing uniform bounds over all slopes $z'_i(\mu) \in [0, c]$ yields the result.

QED Lemma A.2

A.2 Existence of a Threshold Equilibrium

We now turn the proof of existence of a threshold equilibrium. The proof of Proposition 1 is complicated and technical, so we start with a road-map.

Road-map of Proof of Proposition 1

First, we assume that the manager follows some threshold strategy and establish bounds on the slopes of equilibrium prices under the assumption that $p < 0.77$ (Claim A.2 below). We then show that if prices have these properties then the manager's best response is indeed to follow a threshold strategy. This requires looking at all possible private histories of the agent and verifying that claim for each one of them. By appropriately choosing off-equilibrium beliefs, we then establish the existence of a threshold equilibrium. Finally, in the last step of the proof we show that there exists an x' such that for $x > x'$ later disclosure receives a strictly better interpretation, i.e.,

²Since $z_i(x) \leq h(x, \{z_i(\cdot)\})$ also $h(x, \{z_i(\cdot)\}) \leq E[x|y] = \beta_1 x$.

³For a complete analysis of this case see proof of Lemma A.7 below.

$P_2(x, 2) > P_2(x, 1)$. To keep the flow of the reasoning we delegate some of the most algebra-heavy proofs to Section A.3.

Proof of Proposition 1

To establish existence of a threshold equilibrium we need to look at many possible private histories at $t = 1$ and $t = 2$. We make the following observations about all equilibria:

1) Once an agent reveals one of the signals, he follows a myopic disclosure strategy (i.e. reveals the second signal if and only if it increases the current price), so his disclosure policy is a threshold policy (see Lemma 3).

2) At $t = 2$, if the agent has not revealed any of the signals, he reveals at least one if $P_2(\emptyset) \leq P_2(x, 2)$. For this to be a threshold strategy we need that $P_2(x, 2)$ is increasing (as in Pae (2005)). We establish this property below for the equilibria we construct.

3) In a threshold equilibrium we must have that at $t = 1$, $P_1(\emptyset) < P_1(x, 1)$ for any $x \geq x^*$. Otherwise an agent that learned only the signal X at $t = 1$ would strictly prefer to postpone disclosure since there is a positive probability that he will learn the second signal at $t = 2$ and reveal only the second signal (if $P_2(y, 2) > \max\{P(x, y), P_2(x, 2), P_2(\emptyset)\}$). In addition, recall that $P_2(x, 2) \geq P_2(x, 1)$ so there may be another benefit to waiting, which applies to both the agent that learned on X at $t = 1$ and for an agent that learned both signals at $t = 1$ and will disclose only one signal by $t = 2$.

4) The most difficult analysis is for $t = 1$ since the agent incentives to disclose depend not only on the current prices but also on how his current disclosure affects continuation payoffs. Therefore, most of our proof considers different possible private histories of the agent at $t = 1$.

It proves convenient to introduce a new definition:

Definition A.1 *Denote investors' expectation of the value of the signal y , as of time t , given that the manager disclosed only x at time t_x , by $h_t(x, t_x)$. The notation is borrowed from the notation of prices, $P_t(x, t_x)$.*

With this notation, the equilibrium prices that play a central role in our proof are:

$$\begin{aligned} P_1(x, 1) &= \beta_2(x + h_1(x, 1)), \\ P_2(x, 1) &= \beta_2(x + h_2(x, 1)), \\ P_2(x, 2) &= \beta_2(x + h_2(x, 2)). \end{aligned}$$

The following Claim derives upper and lower bounds to the slopes of these prices:

Claim A.2 *Suppose that investors believe that the manager follows a threshold reporting strategy*

as in Proposition 1. Then, for $p \leq 0.77$ and $x > x^*$:

$$\begin{aligned} \frac{\partial}{\partial x} h_1(x, 1) & \begin{cases} = \beta_1 & \text{if } h_1(x, 1) < x \\ \in (2\beta_1 - 1, \beta_1) & \text{if } h_1(x, 1) > x \end{cases}, \\ \frac{\partial}{\partial x} h_2(x, 2) & \begin{cases} = \beta_1 & \text{if } h_2(x, 2) < x^* \\ \in (2\beta_1 - 1, 2\beta_1) & \text{if } h_2(x, 2) > x^* \end{cases}, \\ \frac{\partial}{\partial x} h_2(x, 1) & \begin{cases} = \beta_1 & \text{if } h_2(x, 1) < x \\ \in (2\beta_1 - 1, \beta_1) & \text{if } h_2(x, 1) > x \end{cases}. \end{aligned} \quad (2)$$

This Claim established part (iii) of Proposition 1. In particular, the bound on $\frac{\partial}{\partial x} h_2(x, 2)$ implies that $P_2(x, 2)$ increases in x , so the agent indeed best responds with a threshold strategy at time $t = 2$. So from now on we focus on $t = 1$.

In proving the existence of a threshold equilibrium, we first consider partially informed agents that learn a single signal, x , at $t = 1$ ($\tau_x = 1, \tau_y \neq 1$) and then we consider fully informed agents that learn both signals at $t = 1$. For each of these cases we show that: (i) for sufficiently high (low) realizations of x the agent discloses (does not disclose) x at $t = 1$; and (ii) On the equilibrium path, the difference between the agent's expected payoff if he discloses only x at $t = 1$ and if he does not disclose at $t = 1$ is increasing in x , implying that the agent's best response is indeed a threshold strategy.

Partially Informed Agents ($\tau_x = 1, \tau_y \neq 1$)

First consider an agent that knows only x at $t = 1$. For sufficiently low realizations of x the agent is always better off not disclosing it at $t = 1$, as he can “hide” behind uninformed agents. We next establish that for an agent that learns a single signal, x , at $t = 1$ his incentives to disclose it are monotone in x and hence a threshold strategy is a best response (the proof of the lemma is in the next section).

Lemma A.3 *Consider an agent that learns a single signal, x , at $t = 1$. If $\beta_1 \geq \frac{1}{2}$ or if $h_2(x^*, 1) \leq x^*$ then if investors believe that the agent follows a threshold strategy, the incentives to disclose x at $t = 1$ are strictly increasing in x . That is,*

$$\frac{\partial}{\partial x} (E[U|\tau_x = 1, \tau_y \neq 1, t_x = 1] - E[U|\tau_x = 1, \tau_y \neq 1, t_x \neq 1]) > 0,$$

and there exists x high enough that the agent is better off revealing it than not.

Fully Informed Agents ($\tau_x = \tau_y = 1$)

We next discuss an agent that learns both signals at $t = 1$ (such that $x > y$).

Using Theorem 1, we divide these private histories into three cases:

- 1) $y \geq h_2(x, 2)$.
- 2) $y \in (h_2(x, 1), h_2(x, 2))$
- 3) $y \leq h_2(x, 1)$

Types $y \geq h_2(x, 2)$ disclose y in period 2 no matter if they disclose x at $t = 1$ or not. Therefore, such types will disclose x at $t = 1$ if

$$\max \{P_1(x, 1), P(x, y)\} \geq P_1(\emptyset).$$

and since Claim A.2 implies that the left-hand side is increasing in x , these types will follow a threshold strategy.

Now consider the case $y \leq h_1(x, 1)$ so that y is sufficiently low that it will not be disclosed at $t = 2$ if it was not disclosed at $t = 1$ but the agent disclosed x . There are two sub-cases: either after not disclosing x at $t = 1$ the agent will remain silent at $t = 2$ or he will disclose x . The first sub-case is easier since the payoff from non-disclosing x is a constant and hence the incentives to disclose are increasing in x if and only if $P_1(x, 1) + P_2(x, 1)$ are increasing and that follows from Claim A.2. The next lemma covers the second sub-case.

Lemma A.4 *Consider an agent that learned both signals at $t = 1$ and the realization of $y \leq x$ is such that $y \leq h_2(x, 1)$ (so that it will not be disclosed at $t = 2$) and $\beta_2[x + h_2(x, 2)] \geq P_2(\emptyset)$. Then:*

(i) *For sufficiently high realizations of x the agent prefers to disclose x at $t = 1$ over not disclosing x at $t = 1$.*

(ii) $\frac{\partial}{\partial x}(E[U|\tau_x = 1, \tau_y = 1, t_x = 1] - E[U|\tau_x = 1, \tau_y = 1, t_x \neq 1]) > 0$.

Proof.

(i) We need to show that for sufficiently high x :

$$\beta_2[x + h_1(x, 1)] + \beta_2[x + h_2(x, 1)] > P_1(\emptyset) + \beta_2[x + h_2(x, 2)].$$

Rearranging yields

$$\beta_2[x + h_2(x, 1)] - P_1(\emptyset) > \beta_2[h_2(x, 2) - h_1(x, 1)].$$

By Claim A.2 the LHS of the above inequality, $\beta_2[x + h_2(x, 1)] - P_1(\emptyset)$, goes to infinity as x goes to infinity. Therefore, it is sufficient to show that $h_2(x, 2) - h_1(x, 1)$ is bounded from above. Both $h_2(x, 2)$ and $h_1(x, 1)$ are lower than $\beta_2 x$. From the Generalized Minimum Principle (Lemma 2) we know that $h_1(x, 1)$ is higher than the price given no disclosure in a Dye (1985), Jung and Kwon (1988) setting where $y \sim N(\beta_1 x, \text{Var}(y|x))$. The price given no disclosure in such a setting is $\beta_1 x - \text{Const}$, so $h_1(x, 1) > \beta_1 x - \text{Const}$. Hence, given that $h_2(x, 2) < \beta_1 x$ we have $h_2(x, 2) - h_1(x, 1) < \text{Const}$.

(ii) We need to show that

$$\frac{\partial}{\partial x} (\beta_2 [x + h_1(x, 1)] + \beta_2 [x + h_2(x, 1)] - P_1(\emptyset) - \beta_2 [x + h_2(x, 2)]) > 0,$$

which is identical to condition 2 in the proof of Lemma A.3. ■

Finally, for the sub-case $y \in (h_2(x, 1), h_2(x, 2))$ the agent will reveal y in period 2 if he reveals x at $t = 1$, but will not reveal it if he does not reveal x at $t = 1$. This agent will reveal x today if

$$\beta_2 [x + h_1(x, 1)] + \beta_2 [x + y] > P_1(\emptyset) + \beta_2 [x + h_2(x, 2)].$$

and these incentives are monotone in x for the same reasons as in the previous lemma.

Fixed point and off-equilibrium beliefs

That finishes the analysis of all possible private histories. To summarize, we have proven that (assuming $p < 0.77$) if $\beta_2 \geq \frac{1}{2}$ or if or if $h_2(x^*, 1) \leq x^*$, then the best response of the agent is to indeed follow a threshold strategy. We now need to find a fixed-point for the threshold. That is, we need to find x^* such that if the market believes that in period 1 the agent uses threshold x^* then he best responds using that exact threshold. We also need to specify off-equilibrium beliefs and it turns out that these two tasks are connected.

In a model with only one signal (static or dynamic), the only off-path history is when the agent reveals a signal below the equilibrium threshold but that does not matter for beliefs since at that point there is no information asymmetry. In contrast, in a model with two signals, when the agent reveals only one of them and it is below x^* , we need to specify the market's beliefs about the probability that he has learned the other signal and if so, what is y . In particular, we can set the beliefs to be arbitrarily negative about y and hence the price $P_t(x, t_x)$ to be arbitrarily low off-path, making sure that the agent does not have incentives to reveal such x .

Therefore, any x^* such that for all $x \geq x^*$ the agent prefers (weakly or strictly) to reveal x (and possibly also y) when he is partially informed (knows only x) or fully informed (knows both x and y , in which case the incentives have to hold for all $y \leq x$) can be used to complete a construction of our equilibrium. (Note: a model with two-dimensional signals has multiple equilibria supported by appropriate off-path beliefs).

To see that such x^* exists note that as investors belief x^* goes to infinity then the price upon nondisclosure, $P_1(\emptyset)$ converges to 0 (since there is no inference from nondisclosure in the limit) while for any $x > x^*$ prices $P_1(x, 1)$ and $P_2(x, 1)$ get arbitrarily large (and recall that we have proven above that $h_2(x, 2) - h_1(x, 1) < Const$). So for sufficiently large x^* after all private histories in period 1 the agent prefers to reveal x if it is above x^* to not revealing anything.

That finishes the proof that there exists an equilibrium in threshold strategies.

Finally, we establish in the following lemma the last part of Proposition 1.

Lemma A.5 *There exists an $x' \geq x^*$ such that $P_2(x, 2) > P_2(x, 1)$ for any $x \geq x'$.*

Proof. In Theorem 1 we have shown that $P_2(x, 2) \geq P_2(x, 1)$ for any x , which implies in the setting of Section 4 that $h_2(x, 2) \geq h_2(x, 1)$.

As established in Section 3, given disclosure of the signal x the manager behaves myopically in the sense that he discloses the signal y (when he learned y) if and only if it increases the price relative to the price when y is not disclosed. This holds for both $t = 1$ and $t = 2$. We can now introduce the equilibrium inference on the sets $\mathcal{B}_1^1, \mathcal{B}_1^2, \mathcal{B}_2^1$ and \mathcal{B}_2^2 that were defined in Section 3. In particular, we adjust the set \mathcal{B}_i^j by taking into account also the equilibrium disclosure strategy when defining the potential disclosers and denote it by \mathcal{D}_i^j . The sets \mathcal{D}_i^j for $i, j = 1, 2$ are given by:

$$\begin{aligned}\mathcal{D}_1^1 &= \{(y, \tau_y) | \tau_y = 1, t_x = 1 \text{ and } y \leq \min\{x, h_1(x, 1), h_2(x, 1)\}\} \\ \mathcal{D}_1^2 &= \{(y, \tau_y) | \tau_y = 2, t_x = 1 \text{ and } y \leq h_2(x, 2)\}\end{aligned}$$

$$\begin{aligned}\mathcal{D}_2^1 &= \{(y, \tau_y) | \tau_y = 1, t_x = 2 \text{ and } y \leq \min\{x^*, h_2(x, 2)\}\} \\ \mathcal{D}_2^2 &= \{(y, \tau_y) | \tau_y = 2, t_x = 2 \text{ and } y \leq \min\{x, h_2(x, 2)\}\}\end{aligned}$$

Note that $h_1(x, 1) > h_2(x, 1)$ so \mathcal{D}_1^1 can be written as $\mathcal{D}_1^1 = \{(y, \tau_y) | \tau_y = 1, t_x = 1 \text{ and } y \leq \min\{x, h_2(x, 1)\}\}$.

We next show that $h_2(x, 2) > h_2(x, 1)$ for all x such that $h_2(x, 2) > x^*$. From Section 3 we know that $h_2(x, 2) \geq h_2(x, 1)$ so we only need to preclude $h_2(x, 2) = h_2(x, 1)$. Assume by contradiction that $h_2(x, 2) = h_2(x, 1)$. Since $x > x^*$ we have $\mathcal{D}_2^1 \subset \mathcal{D}_1^1$ and $\mathcal{D}_2^2 \subseteq \mathcal{D}_1^2$. Moreover, any $y \in (x^*, h_2(x, 2))$ is strictly lower than $h_2(x, 2)$ which equals $E[y | y \in \mathcal{S}_{\mathcal{A}, \mathcal{D}_2}]$. From part (i) of the Generalized Minimum Principle (Lemma 2) we have $h_2(x, 2) > h_2(x, 1)$ which leads to a contradiction. Therefore, for all values of x such that $h_2(x, 2) > x^*$ we have $h_2(x, 2) > h_2(x, 1)$.

The last thing to be shown is that there exists an x' such that $h_2(x, 2) > x^*$ for any $x \geq x'$. This is immediate given that $\frac{\partial}{\partial x} h_2(x, 2) = \beta_1 (> 0)$ for value of x such that $h_2(x, 2) < x^*$ (see Claim A.2). Note that x' can be, but is not necessarily, greater than x^* . ■

QED Proposition 1.

A.3 Omitted Proofs.

In this section we provide proofs for the lemmas and claims in the previous section.

A.3.1 Proof of Claim A.2

Claim A.2 above is:

Suppose that investors believe that the manager follows a threshold reporting strategy as in

Proposition 1. Then, for $p \leq 0.77$:

$$\begin{aligned} \frac{\partial}{\partial x} h_1(x, 1) & \begin{cases} = \beta_1 & \text{if } h_1(x, 1) < x \\ \in (2\beta_1 - 1, \beta_1) & \text{if } h_1(x, 1) > x \end{cases}, \\ \frac{\partial}{\partial x} h_2(x, 2) & \begin{cases} = \beta_1 & \text{if } h_2(x, 2) < x^* \\ \in (2\beta_1 - 1, 2\beta_1) & \text{if } h_2(x, 2) > x^* \end{cases}, \\ \frac{\partial}{\partial x} h_2(x, 1) & \begin{cases} = \beta_1 & \text{if } h_2(x, 1) < x \\ \in (2\beta_1 - 1, \beta_1) & \text{if } h_2(x, 1) > x \end{cases}. \end{aligned} \quad (3)$$

Proof of Claim A.2

In the proof we use a terminology "non-binding" and "binding" case to distinguish between $h_t(x, 1) \leq x$ and $h_t(x, 1) > x$. These cases are qualitatively different because in general investors infer that if $\tau_y = 1$ then $y \leq x$ and $y < h_t(x, 1)$. In the "non-binding" the second inequality implies the first. In the "binding" case, $y \leq x$ implies the second inequality.

We start the proof with $\frac{\partial}{\partial x} h_1(x, 1)$.

Lemma A.6 For $p \leq 0.95$, $\frac{\partial}{\partial x} h_1(x, 1) \begin{cases} = \beta_1 & \text{if } h_1(x, 1) < x \\ \in (2\beta_1 - 1, \beta_1) & \text{if } h_1(x, 1) > x \end{cases}$

Proof. As shown in Section 3, for any x that is disclosed at $t = 1$ such that $h_1(x, 1) < x$ (the non binding case), if $\tau_y = 1$ the agent is myopic with respect to the disclosure of y and discloses it whenever $y \geq h_1(x, 1)$. This case is captured by Example 1 in Section A.1: an increase in the mean of the distribution results in an identical increase in both the equilibrium beliefs and the equilibrium disclosure threshold. The quantitative difference in our setting is that a unit increase in x increases investors' beliefs about y by β_1 (rather than by 1), and therefore also increases both the beliefs about y and the threshold for disclosure of y by β_1 . As a result, for $h_1(x, 1) < x$ we have $\frac{\partial}{\partial x} h_1(x, 1) = \beta_1$.⁴

In the binding case, i.e., for all x such that $h_1(x, 1) > x$ (if such $x > x^*$ exists) we know that if $\tau_y = 1$ then $y < x$ (otherwise, the manager would have disclosed y). An increase in x increases the beliefs about y at a rate of β_1 , while the increase in the constraint/disclosure threshold ($y < x$) increases the beliefs about y at a rate of 1. Therefore, this is a special case of Example 3 in Section A.1, where we increase the mean by β_1 and $z'(\mu) \equiv c = \frac{1}{\beta_1}$. From Example 3 we know that an increase in the beliefs about y given a unit increase in x (which is equivalent to an increase of β_1

⁴Since both the beliefs about Y and the disclosure threshold increase at the same rate, the probability that the agent learned y at $t = 1$ but did not disclose it, conditional on him disclosing x at $t = 1$, is independent of x .

in the value of μ in Example 3) is given by $\beta_1 (1 + (c - 1) \frac{\partial}{\partial z} h^{stat}(\mu, z))$. Substituting $c = \frac{1}{\beta_1}$ and rearranging terms yields

$$\frac{\partial}{\partial x} h_1(x, 1) = \beta_1 + (1 - \beta_1) \frac{\partial}{\partial z} h^{stat}(\mu, z).$$

Since $\frac{\partial}{\partial z} h^{stat}(\mu, z) \in (-1, 0)$ (recall Claim A.1), we have $\frac{\partial}{\partial x} h_1(x, 1) \in (2\beta_1 - 1, \beta_1)$. ■

Analyzing the effect of x on $h_2(x, 2)$ and $h_2(x, 1)$ is more involved and more technical. The reason these cases are more complicated is that when pricing the firm at $t = 2$ investors do not know whether the manager learned y at $t = 1$ or at $t = 2$ (in the case where the agent did in fact learn y). Investors' inferences about y depend on when the agent learned it, and therefore the analysis of $h_2(x, 2)$ and $h_2(x, 1)$ requires stochastic disclosure thresholds. This is where we use Lemma A.2.

We next analyze $h_2(x, 2)$.

When an agent discloses $x > x^*$ at $t = 2$ investors know that $\tau_x = 2$ (otherwise the agent would have disclosed x at $t = 1$). Investors' beliefs about the manager's other signal at $t = 2$ are set as a weighted average of three scenarios: $\tau_y = 1$, $\tau_y = 2$ and $\tau_y > 2$. We start by describing the disclosure thresholds conditional on each of these three scenarios.

(i) If $\tau_y > 2$ the agent cannot disclose y and therefore the disclosure threshold is not relevant. In the pricing of the firm conditional on $\tau_y > 2$ investors use $E[y|x]$ which equals $\beta_1 x$.

(ii) If $\tau_y = 2$ investors know that $y < h_2(x, 2)$ and also that $y < x$. We need to distinguish between the binding case and the non-binding case. In the non-binding case, where $h_2(x, 2) \leq x$, investors know that $y < h_2(x, 2)$, so conditional on $\tau_y = 2$ investors set their beliefs as if the manager follows a disclosure threshold of $h_2(x, 2)$. In the binding case, where $h_2(x, 2) > x$, investors know that $y < x$, so it is equivalent to a disclosure threshold of x .

(iii) If $\tau_y = 1$ investors know that $y < x^*$ (where $x^* \leq x$) and also $y < h_2(x, 2)$. Here again we should distinguish between a non-binding case, in which $h_2(x, 2) < x^*$ (if such case exists), and a binding case in which $h_2(x, 2) > x^*$. In the non-binding case the disclosure threshold is $h_2(x, 2)$. In the binding case the disclosure threshold is x^* , which is independent of x .

The next lemma provides an upper and lower bound for $\frac{\partial}{\partial x} h_2(x, 2)$ and since the proof uses generic disclosure thresholds for each of the three scenarios above, it applies also to $\frac{\partial}{\partial x} h_2(x, 1)$.

Lemma A.7 *For any $p < 0.77$*

$$\frac{\partial}{\partial x} h_2(x, 2), \frac{\partial}{\partial x} h_2(x, 1) \in (2\beta_1 - 1, 2\beta_1).$$

Proof of Lemma A.7.

In this proof we use a slightly different notation, as part of the proof is more general than our setting. Note that the first part of this proof is quite similar to the proof of Lemma A.2.

Suppose that x and y have joint normal distribution and the agent is informed about y with probability p and uninformed with probability $1 - p$.⁵ Conditional on being informed the agent's disclosure strategy is assumed to be as follows: with probability λ_i , $i \in \{1, \dots, K\}$, he discloses if his type is above $z_i(x)$, where the various $z_i(x)$ are determined exogenously such that $z_i(x) \leq h(x, \{z_i(x)\})$ for all i (which always holds in our setting). Note that $\sum_{i=1}^K \lambda_i = p$. Let's denote the conditional expectation of y given x and given the disclosure thresholds, $z_i(x)$, by $h(x, \{z_i(x)\})$.

By applying Bayes rule, $h(x, \{z_i(x)\})$ is given by:

$$h(x, \{z_i(x)\}) = \frac{(1-p)E[y|x] + \sum_{i=1}^K \lambda_i \int_{-\infty}^{z_i(x)} y \phi(y|x) dy}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x)|x)}.$$

Taking the derivative of $h(x, \{z_i(x)\})$ with respect to x and applying some algebraic manipulation (recall that $\frac{\partial E[y|x]}{\partial x} = \beta_1$) yields:

$$\frac{d}{dx} h(x, \{z_i(x)\}) = \beta_1 + \frac{\sum_{i=1}^K \lambda_i (z'_i(x) - \beta_1) \phi(z_i(x)|x) (z_i(x) - h(x, \{z_i(x)\}))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x)|x)}. \quad (4)$$

We start by proving the supremum of $\frac{d}{dx} h(x, \{z_i(x)\})$.

Given that $z'_i(x) \geq 0$ and $(z_i(x) - h(x, \{z_i(x)\})) \leq 0$ for all $i \in \{1, \dots, K\}$ we have

$$\begin{aligned} \frac{d}{dx} h(x, \{z_i(x)\}) &\leq \beta_1 + \frac{\beta_1 \sum_{i=1}^K \lambda_i \phi(z_i(x)|x) (h(x, \{z_i(x)\}) - z_i(x))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x)|x)} \\ &\leq \beta_1 + \max_{\substack{z_i \leq h(x) \\ i \in \{1, \dots, K\}}} \frac{\beta_1 \sum_{i=1}^K \lambda_i \phi(z_i|x) (h(x, \{z_i(x)\}) - z_i)}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i|x)}. \end{aligned}$$

Due to symmetry, for all $i \in \{1, \dots, K\}$ the maximum is achieved at some $z_i(x) = \hat{z}(x)$, which is the same for all i . To see this, note that the FOC of the maximization with respect to $z_i(x)$ is

$$\begin{aligned} 0 &= (\phi'(z_i(x)|x) (h(x, \{z_i(x)\}) - z_i(x)) - \phi(z_i(x)|x)) \left((1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x)|x) \right) \\ &\quad - \left(\sum_{i=1}^K \lambda_i \phi(z_i(x)|x) (h(x, \{z_i(x)\}) - z_i(x)) \right) \phi(z_i(x)|x). \end{aligned}$$

Since $\phi'(z_i(x)|x) = -\alpha(z_i(x) - \beta_1 x) \phi(z_i(x)|x)$ (for some constant $\alpha > 0$), this simplifies to

$$-\alpha(z_i(x) - \beta_1 x) (h(x, \{z_i(x)\}) - z_i(x)) = \frac{\sum_{i=1}^K \lambda_i \phi(z_i(x)|x) (h(x, \{z_i(x)\}) - z_i(x))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x)|x)} + 1.$$

In the range $z_i(x) \leq h(x, \{z_i(x)\}) \leq \beta_1 x$, the LHS is decreasing in $z_i(x)$.⁶ The RHS is the

⁵We apologize for the abuse of notation: the p in this proof corresponds to $p + p(1-p)$ in our model since this is the probability that the agent is informed about signal Y in the beginning of period 2.

⁶Since $z_i(x) \leq h(x, \{z_i(\cdot)\})$ also $h(x, \{z_i(\cdot)\}) \leq E[x|y] = \beta_1 x$.

same for all i . Therefore, the unique solution to this system of FOC is for all $z_i(x)$ to be equal (and note that the maximum is achieved at an interior point since at $z_i(x) = h(x, \{z_i(x)\})$ the LHS is zero and the RHS is positive; and as $z_i(x)$ goes to $-\infty$ the LHS goes to $+\infty$ while the RHS is bounded).

Let $\hat{z}(x)$ be the maximizing value. Then

$$\begin{aligned} \frac{d}{dx} h(x, \{z_i(x)\}) &\leq \beta_1 + \frac{\beta_1 \sum_{i=1}^K \lambda_i \phi(\hat{z}(x) | x) (h(x, \{z_i(x)\}) - \hat{z}(x))}{(1-p) + p\Phi(\hat{z}(x) | x)} \\ &= \beta_1 + \frac{p\beta_1 \phi(\hat{z}(x) | x) (h(x, \{z_i(x)\}) - \hat{z}(x))}{(1-p) + p\Phi(\hat{z}(x) | x)}. \end{aligned}$$

The right hand side of the above inequality is identical to the slope in a Dye setting with exogenous disclosure threshold with probability of being uninformed $(1-p)$ and a disclosure threshold $\hat{z}(x)$, constant in x (see the discussion in Section A.1). In such a setting, we can think of the effect of a marginal increase in x as the sum of two effects. The first effect is a shift by β_1 in both the distribution and the disclosure threshold. This will increase $h(x)$ by β_1 . The second effect is a decrease in the disclosure threshold by β_1 (as the disclosure threshold does not change in x). Since $\hat{z}(x) < \beta_1 x$ we are in the decreasing part of the beliefs about y given no disclosure (to the left of the minimum beliefs). Therefore, the decrease in the disclosure threshold increases the beliefs about y by the change in the disclosure threshold times the slope of the beliefs about y given no disclosure. Since for $p < 0.95$ the slope of the beliefs about y given no disclosure is greater than -1 , the latter effect increases the beliefs about y by less than β_1 . The overall effect is therefore smaller than $2\beta_1$.

Next we prove the infimum of $\frac{d}{dx} h(x, \{z_i(x)\})$.

Equation (4) captures a general case with any number of potential disclosure strategies. In our particular case $K = 1$ where $i = 1$ represents the case of $\tau_y = 1$ and $i = 2$ represents the case of $\tau_y = 2$. So, in our setting equation (4) can be written as

$$\begin{aligned} \frac{d}{dx} h(x, \{z_i(x)\}) &= \beta_1 + \frac{\lambda_1 (z'_1(x) - \beta_1) \phi(z_1(x) | x) (z_1(x) - h(x, \{z_i(x)\}))}{(1-p) + \sum_{i=1}^2 \lambda_i \Phi(z_i(x) | x)} \\ &\quad + \frac{\lambda_2 (z'_2(x) - \beta_1) \phi(z_2(x) | x) (z_2(x) - h(x, \{z_i(x)\}))}{(1-p) + \sum_{i=1}^2 \lambda_i \Phi(z_i(x) | x)}. \end{aligned}$$

When calculating $h_2(x, 2)$ and $h_1(x, 1)$ in our setting, the disclosure threshold, $z_i(x)$, in any possible scenario (the binding and non-binding case for both $\tau_y = 1$ and $\tau_y = 2$) takes one of the following three values: $h_i(x, \cdot)$, x or x^* . Note that whenever $z_i(x) = h(x, \{z_i(x)\})$ we have

$$\frac{(z'_i(x) - \beta_1) \phi(z_i(x) | x) (z_i(x) - h(x, \{z_i(x)\}))}{(1-p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x) | x)} = 0.$$

For the remaining two cases ($z_i(x) = x$ and $z_i(x) = x^*$), for all $i \in \{1, 2\}$ we have $z'_i(x) \leq 1$

and $(z_i(x) - h(x, \{z_i(x)\})) \leq 0$. This implies

$$\frac{d}{dx} h(x, \{z_i(x)\}) \geq \beta_1 - \frac{(1 - \beta_1) \sum_{i=1}^K \lambda_i \phi(z_i(x) | x) (h(x, \{z_i(x)\}) - z_i(x))}{(1 - p) + \sum_{i=1}^K \lambda_i \Phi(z_i(x) | x)}.$$

Using the same symmetry argument for the first order condition as before, $\frac{d}{dx} h(x, \{z_i(x)\})$ is minimized for some $z^{\min}(x)$, and hence,

$$\frac{d}{dx} h(x, \{z_i(x)\}) \geq \beta_1 + \frac{p(1 - \beta_1) \phi(z^{\min}(x) | x) (h(x, \{z_i(x)\}) - z^{\min}(x))}{(1 - p) + p\Phi(z^{\min}(x) | x)}.$$

The right hand side of the above inequality is identical to the slope in a Dye setting with exogenous disclosure threshold in which: the probability of being uninformed is $(1 - p)$, the threshold is $z^{\min}(x)$, and $\frac{\partial}{\partial x} z^{\min}(x) = 1$ (see the discussion in Section A.1). In such a setting, we can think of the effect of a marginal increase in x as the sum of two effects. The first is a shift by β_1 in both the distribution and the disclosure threshold. This will increase beliefs about y by β_1 . The second effect is an increase in the disclosure threshold by $(1 - \beta_1)$ (as the disclosure threshold increases by 1). Since $z^{\min}(x) < \beta_1 x$ we are in the decreasing part of the beliefs about y given no disclosure (to the left of the minimum beliefs). Therefore, the increase in the disclosure threshold decreases the beliefs about y by the change in the disclosure threshold, $(1 - \beta_1)$, times the slope of the beliefs about y given no disclosure. Since for $p < 0.95$ the slope of the beliefs about y given no disclosure is greater than -1 , the latter effect decreases the beliefs about y by less than $(1 - \beta_1)$. The overall effect is therefore greater than $\beta_1 - (1 - \beta_1) = 2\beta_1 - 1$.

The reasoning we have presented is independent of the actual thresholds, so the bounds apply to $h_2(x, 1)$ as well. ■

This covers the range $h_2(x, 2) \geq x^*$.

For the case $h_2(x, 2) < x^*$ (if such case exists) we claim that $\frac{\partial}{\partial x} h_2(x, 2) = \beta_1$.

To see this, note that $h_2(x, 2)$ is a weighted average of the beliefs about y over the three scenarios $\tau_y = 1$, $\tau_y = 2$ and $\tau_y > 2$. That is, we can write

$$h_2(x, 2) = \lambda_1 g_1 + \lambda_2 g_2 + (1 - \lambda_1 - \lambda_2) g_3,$$

where $\lambda_i = \Pr(\tau_y = i | ND_y)$ and $g_i = E[y | \tau_y = i, ND_y]$ for $i = 1, 2, 3$ (where $i = 3$ represents the case of $\tau_y > 2$). ND_y stands for No-Disclosure of y (where x was disclosed at $t = 2$). Since $h_2(x, 2) < x^*$ the disclosure threshold for both $\tau_y = 2$ and $\tau_y = 1$ is $h_2(x, 2)$.

Assume, by contradiction, that $\frac{\partial}{\partial x} h_2(x, 2) > \beta_1$. Then, an increase in x increases $h_2(x, 2)$ by more than the increase in the expectation of y (which is β_1) and therefore, the probability of obtaining a signal below the disclosure threshold increases for both the first and the second period. This implies that both λ_1 and λ_2 increase. In addition, note that the increase in g_1 and in g_2 is lower than $\frac{\partial}{\partial x} h_2(x, 2)$ and the increase in g_3 is β_1 - which is also lower than $\frac{\partial}{\partial x} h_2(x, 2)$. The fact that both g_1 and g_2 are lower than g_3 leads to a contradiction, since an increase in x puts more

weight on the lower values (λ_1 and λ_2 increase) and in addition all the values g_1, g_2, g_3 increase at a rate weakly lower than the assumed increase in $h_2(x, 2)$. A symmetric argument can be made when assuming by contradiction that $\frac{\partial}{\partial x} h_2(x, 2) < \beta_1$. The case of $\frac{\partial}{\partial x} h_2(x, 2) = \beta_1$ does not lead to a contradiction, as an increase in x does not affect the probabilities λ_1, λ_2 and the derivatives of g_1 and g_2 and g_3 are all equal to β_1 .

Finally, we analyze $h_2(x, 1)$.

Recall that Lemma A.7 applies also to $h_2(x, 1)$. However, for $h_2(x, 1)$ we can show tighter bounds.

1) If $h_2(x, 1) < x$ (the non-binding case) then when pricing the firm at $t = 2$ investors know that if the agent learned y (at either $t = 1$ or $t = 2$) then $y < h_2(x, 1)$. If the agent did not learn y then investors use in their pricing $E[Y|x] = \beta_1 x$. So, the beliefs about y are a weighted average of $E[Y|y < h_2(x, 1)]$ and $E[Y|x] = \beta_1 x$. This is similar to a Dye (1985) and Jung and Kwon (1988) setting, and therefore, in equilibrium we have $\frac{\partial}{\partial x} h_2(x, 1) = \beta_1$.

2) Next, we show that for x such that $h_2(x, 1) > x$ (if such case exists) $\frac{\partial}{\partial x} h_2(x, 1) \in (2\beta_1 - 1, \beta_1)$.

The argument is similar to the one we made in the proof of Lemma A.6 (that $\frac{\partial}{\partial x} h_1(x, 1) \in (2\beta_1 - 1, \beta_1)$, for x such that $h_1(x, 1) > x$). First note that for $h_2(x, 1) > x$ investors' beliefs about y conditional on that the agent has learned y are independent of whether he learned y at $t = 1$ or at $t = 2$. Moreover, given that $\tau_y \leq 2$ investors know that $y < x$. So, from investors' perspective, it doesn't matter if the agent learned y at $t = 1$ or at $t = 2$. Their pricing, $h_2(x, 1)$, will reflect a weighted average between $E[Y|y < x]$ and $E[Y|\tau_y > 2, x] = \beta_1 x$. From here on the proof is qualitatively the same as in the proof for $\frac{\partial}{\partial x} h_1(x, 1) \in (2\beta_1 - 1, \beta_1)$, where the only quantitative difference is the probability that the agent learned y .

QED Claim A.2

A.3.2 Lemma A.3

Proof of Lemma A.3. For simplicity of exposition, we partition the support of x into two cases: realizations of x for which $\beta_2(x + h_2(x, 2)) \geq P_2(\emptyset)$ and for which $\beta_2(x + h_2(x, 2)) < P_2(\emptyset)$.⁷

Case I - $\beta_2(x + h_2(x, 2)) \geq P_2(\emptyset)$ (i.e. an agent that does not learn the second signal will prefer to disclose x at $t = 2$)

Rewriting $E[U|\tau_x = 1, \tau_y \neq 1, t_x = 1, x] - E[U|\tau_x = 1, \tau_y \neq 1, t_x \neq 1]$ yields:

$$\beta_2[x + h_1(x, 1) + h_2(x, 1) - h_2(x, 2)] - P_1(\emptyset) + p\beta_2 \left[\int_{h_2(x, 1)}^{\infty} (y - h_2(x, 1)) f(y|x) dy - \int_{h_2(x, 2)}^{\infty} (y - h_2(x, 2)) f(y|x) dy - \int_{y^H(x)}^{\infty} (h_2(y, 2) - x) f(y|x) dy \right].$$

⁷Note that on the equilibrium path we are always in case I, i.e., $\beta_2(x + h_2(x, 2)) \geq P_2(\emptyset)$.

The derivative of this expression with respect to x has the same sign as

$$D = 1 + \frac{\partial}{\partial x} (h_1(x, 1) + h_2(x, 1) - h_2(x, 2)) + p[A + B + C], \quad (5)$$

where

$$\begin{aligned} A &= \frac{\partial}{\partial x} \int_{h_2(x,1)}^{\infty} (y - h_2(x, 1)) f(y|x) dy \\ B &= -\frac{\partial}{\partial x} \int_{h_2(x,2)}^{\infty} (y - h_2(x, 2)) f(y|x) dy \\ C &= -\frac{\partial}{\partial x} \int_{y^H(x)}^{\infty} (h_2(y, 2) - x) f(y|x) dy. \end{aligned}$$

To evaluate this derivative we use the following, easy to obtain, equations:

$$\begin{aligned} \frac{\partial}{\partial x} f(y|x) &= -\beta_1 \frac{\partial}{\partial y} f(y|x), \\ \frac{\partial}{\partial x} (F(y(x)|x)) &= f(y(x)|x) \left(\frac{\partial}{\partial x} y(x) - \beta_1 \right). \end{aligned}$$

Next, we analyze the three terms A , B , and C . Note that the derivative with respect to the limits of integrals for A , B and C is zero.

$$A = -\frac{\partial h_2(x, 1)}{\partial x} (1 - F(h_2(x, 1)|x)) - \beta_1 \int_{h_2(x,1)}^{\infty} (y - h_2(x, 1)) \frac{\partial}{\partial y} f(y|x) dy.$$

Integrating by parts (w.r.t. y) the term $\int_{h_2(x,1)}^{\infty} (y - h_2(x, 1)) \frac{\partial}{\partial y} f(y|x) dy$ yields:

$$\begin{aligned} &\int_{h_2(x,1)}^{\infty} (y - h_2(x, 1)) \frac{\partial}{\partial y} f(y|x) dy \\ &= -(h_2(x, 1) - h_2(x, 1)) f(h_2(x, 1)|x) - \int_{h_2(x,1)}^{\infty} f(y|x) dy = -(1 - F(h_2(x, 1)|x)). \end{aligned}$$

Plugging it back to A we get

$$A = -\left(\frac{\partial h_2(x, 1)}{\partial x} - \beta_1 \right) (1 - F(h_2(x, 1)|x)).$$

Next, we calculate B :

$$\begin{aligned} B &= \int_{h_2(x,2)}^{\infty} \frac{h_2(x, 2)}{\partial x} f(y|x) dy + \beta_1 \int_{h_2(x,2)}^{\infty} (y - h_2(x, 2)) \frac{\partial}{\partial y} f(y|x) dy \\ &= \left(\frac{\partial h_2(x, 2)}{\partial x} - \beta_1 \right) (1 - F(h_2(x, 2)|x)). \end{aligned}$$

Finally,

$$\begin{aligned}
C &= (1 - F(y^H(x)|x)) + \beta_1 \int_{y^H(x)}^{\infty} (h_2(y, 2) - x) \frac{\partial}{\partial y} f(y|x) dy \\
&= (1 - F(y^H(x)|x)) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h_2(y, 2)}{\partial y} f(y|x) dy.
\end{aligned}$$

Substituting A , B and C back to (5) and re-arranging terms yields:

$$\begin{aligned}
D &= 1 + \frac{\partial}{\partial x} (h_1(x, 1) + h_2(x, 1) - h_2(x, 2)) \\
&\quad - p \left[\begin{aligned} &\left(\frac{\partial h_2(x, 1)}{\partial x} - \beta_1 \right) (1 - F(h_2(x, 1)|x)) + \left(\frac{\partial h_2(x, 2)}{\partial x} - \beta_1 \right) (1 - F(h_2(x, 2)|x)) + \\ &\quad (1 - F(y^H(x)|x)) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h_2(y, 2)}{\partial y} f(y|x) dy \end{aligned} \right] \\
&= (1 - p) \left(1 + \frac{\partial}{\partial x} (h_1(x, 1) + h_2(x, 1) - h_2(x, 2)) \right) \\
&\quad + p \left[\begin{aligned} &1 + \frac{\partial h_1(x, 1)}{\partial x} + \frac{\partial h_2(x, 1)}{\partial x} F(h_2(x, 1)|x) - F(h_2(x, 1)|x) \beta_1 - \frac{\partial h_2(x, 2)}{\partial x} F(h_2(x, 2)|x) \\ &\quad + F(h_2(x, 2)|x) \beta_1 + (1 - F(y^H(x)|x)) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h_2(y, 2)}{\partial y} f(y|x) dy \end{aligned} \right]
\end{aligned}$$

Additional rearranging yields:

$$\begin{aligned}
D &= (1 - p) \left(1 + \frac{\partial}{\partial x} (h_1(x, 1) + h_2(x, 1) - h_2(x, 2)) \right) \\
&\quad + p \left[1 + \frac{\partial h_1(x, 1)}{\partial x} + \left(\frac{\partial h_2(x, 1)}{\partial x} - \beta_1 \right) F(h_2(x, 1)|x) - \left(\frac{\partial h_2(x, 2)}{\partial x} - \beta_1 \right) F(h_2(x, 2)|x) \right] \\
&\quad + p \beta_1 \int_{y^H(x)}^{\infty} \frac{1}{\beta_1} - \frac{\partial h_2(y, 2)}{\partial y} f(y|x) dy.
\end{aligned}$$

Since $\frac{\partial h_2(x,1)}{\partial x} \leq \beta_1$ (see Claim A.2) and $F(h_2(x,2)|x) \geq F(h_2(x,1)|x)$ we have

$$\begin{aligned}
D &\geq (1-p) \left(1 + \frac{\partial}{\partial x} (h_1(x,1) + h_2(x,1) - h_2(x,2)) \right) \\
&\quad + p \left[1 + \frac{\partial h_1(x,1)}{\partial x} + \left(\frac{\partial h_2(x,1)}{\partial x} - \frac{\partial h_2(x,2)}{\partial x} \right) F(h_2(x,2)|x) \right] \\
&\quad + p\beta_1 \int_{y^H(x)}^{\infty} \frac{1}{\beta_1} - \frac{\partial h_2(y,2)}{\partial y} f(y|x) dy \\
&= (1-p(1-F(h_2(x,2)|x))) \left(1 + \frac{\partial}{\partial x} (h_1(x,1) + h_2(x,1) - h_2(x,2)) \right) + \\
&\quad + p(1-F(h_2(x,2)|x)) \left(1 + \frac{\partial h_1(x,1)}{\partial x} \right) + p\beta_1 \int_{y^H(x)}^{\infty} \frac{1}{\beta_1} - \frac{\partial h_2(y,2)}{\partial y} f(y|x) dy \\
&= (1-p(1-F(h_2(x,2)|x))) \left(1 + \frac{\partial}{\partial x} (h_1(x,1) + h_2(x,1) - h_2(x,2)) \right) + \\
&\quad + p \int_{h_2(x,2)}^{y^H(x)} \left(1 + \frac{\partial h_1(x,1)}{\partial x} \right) f(y|x) dy + p \int_{y^H(x)}^{\infty} 2 + \frac{\partial h_1(x,1)}{\partial x} - \beta_1 \frac{\partial h_2(y,2)}{\partial y} f(y|x) dy \\
&\geq (1-p(1-F(h_2(x,2)|x))) \left(1 + \frac{\partial}{\partial x} (h_1(x,1) + h_2(x,1) - h_2(x,2)) \right) \\
&\quad + p \int_{y^H(x)}^{\infty} 2 + \frac{\partial h_1(x,1)}{\partial x} - \beta_1 \frac{\partial h_2(y,2)}{\partial y} f(y|x) dy
\end{aligned}$$

So, the following two conditions are sufficient for the proof of Case I.

For all x :

1. $\frac{\partial}{\partial x} h_1(x,1) + \frac{\partial}{\partial x} h_2(x,1) \geq \frac{\partial}{\partial x} h_2(x,2) - 1$.
2. $\frac{\partial h_2(y,2)}{\partial y} \leq \left(2 + \frac{\partial h_1(x,1)}{\partial x} \right) \frac{1}{\beta_1}$ for any $y > x$.

Case II - $\beta_2(x + h_2(x,2)) < P_2(\emptyset)$ (i.e. an agent that does not learn the second signal will prefer to not disclose x at $t = 2$)

The analysis of Case I was for generic bounds of the integrals $h_2(x,1)$ and $y^H(x)$. The difference between Case I and Case II is that the price at $t = 2$ given no disclosure of y (which occurs when the agent does not obtain a signal y or obtains a low realization of y) is $P_2(\emptyset)$ in Case II and $\beta_2(x + h_2(x,2))$ in Case I. Therefore, the expected payoff of the agent in Case II is less sensitive to x than in Case I. As a result, the fact that for values of x for which $\beta_2(x + h_2(x,2)) \geq P_2(\emptyset)$ (in Case I) $\frac{\partial}{\partial x} (E[U|\tau_x = 1, \tau_y \neq 1, t_x = 1] - E[U|\tau_x = 1, \tau_y \neq 1, t_x \neq 1]) > 0$ implies that it also holds for $\beta_2(x + h_2(x,2)) < P_2(\emptyset)$.

To summarize the analysis of Partially Informed Agents, conditions 1 and 2 above are sufficient for both Case I and Case II. Claim A.2 established that condition 2 above holds.

So, it is only left to show that also condition 1 holds. For any $\beta_1 > \frac{1}{2}$, it is immediate to see that condition 1 holds since the LHS of condition 1 is greater than $2(2\beta_1 - 1) > 0$ and the RHS

is less than $2\beta_1 - 1$ (again by Claim A.2) For the case $\beta_1 < \frac{1}{2}$ we use the assumption that x^* satisfies $h_2(x^*, 1) \leq x^*$. For such x^* we know from Claim A.2 that $\frac{\partial}{\partial x} h_2(x, 1) = \beta_1$ for all $x \geq x^*$. Substituting this into condition 1 above yields $\frac{\partial}{\partial x} h_1(x, 1) + \beta_1 \geq \frac{\partial}{\partial x} h_2(x, 2) - 1$ which given Claim A.2 is always satisfied. ■