Ambiguous Business Cycles: Online Appendix

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This paper studies a New Keynesian business cycle model with agents who are averse to ambiguity (Knightian uncertainty). Shocks to confidence about future TFP are modeled as changes in ambiguity. To assess the size of those shocks, our estimation uses not only data on standard macro variables, but also incorporates the dispersion of survey forecasts about growth as a measure of confidence. Our main result is that TFP and confidence shocks together can explain roughly two thirds of business cycle frequency movements in the major macro aggregates. Confidence shocks account for about 70% of this variation.
JEL: D81, D84, E20, E32, E52

I. Solution method

Here we describe how to extend the solution method to allow for higher order approximations of the equilibrium dynamics. The general principle is the same as applied for the linear case: First, we solve the model as a rational expectations model in which the worst case scenario expectations are correct on average. This is due to the fact that agents act *as if* the worst-case belief characterizes the true data generating process. If a perturbation approach is desired, in this step of the solution we can use standard methods to solve for the equibrium dynamics up to desired levels of approximations. For example, we may be interested in second or third order approximations. The second step is to take these decision rules but feed in the econometrician's data generating process. In our model the difference between the latter and the worst-case belief is only a change to the conditional mean of an exogenous process. Thus this second step requires only to shock the economy with an average innovation that reflects this difference in means.

We now proceed to present some details on this algorithm while referring to the existing literature on details about the specifics of the higher-order approximations under expected utility. For a recent review of this literature see Fernández-Villaverde and Rubio-Ramírez (2010), whose notation we follow in describing the general class of dynamic stochastic models that we consider:

$$E_t^* \mathbf{f}(\mathbf{y_{t+1}}, \mathbf{y_t}, \mathbf{x_{t+1}}, \mathbf{x_t}, \mathbf{s_{t+1}}, \mathbf{s_t}) = 0 \tag{1}$$

where E_t^* denotes the expectations operator conditional on time t information,

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$\mathbf{f}(\cdot)$ is a vector of functions that characterize the equilibrium conditions, $\mathbf{y_t}$ is the vector of non-predetermined variables of size k, $\mathbf{x_t}$ is the vector of endogenous predetermined variables of size m, and $\mathbf{s_t}$ is the vector of exogenous predetermined variables. These latter variables follow a Markov process of the form

$$\mathbf{s_t} = \mathbf{P}\mathbf{s_{t-1}} + \mathbf{\Lambda}\epsilon_\mathbf{t} \tag{2}$$

where $\mathbf{\Lambda}$ is a perturbation parameter and $\epsilon_\mathbf{t} \overset{iid}{\sim} N(0, \mathbf{\Sigma})$.

The solution to the system (1) is to characterize the evolution of the predetermined variables, $\mathbf{x_{t+1}} = \mathbf{h}(\mathbf{x_t}, \mathbf{s_{t-1}}, \epsilon_\mathbf{t}, \mathbf{\Lambda})$ and that of the non-predetermined ones, $\mathbf{y_t} = \mathbf{g}(\mathbf{x_t}, \mathbf{s_{t-1}}, \epsilon_\mathbf{t}, \mathbf{\Lambda})$. The perturbation approach is to seek approximations, of various orders, to the two functions $\mathbf{h}(\cdot)$ and $\mathbf{g}(\cdot)$. The usual approach is to take this approximation around $(\overline{\mathbf{x}}^\mathbf{0}, \overline{\mathbf{s}}^\mathbf{0}, \mathbf{0}, \mathbf{0})$, where $\mathbf{x^0}$ and $\overline{\mathbf{s}}^\mathbf{0}$ are the steady states of the state variables when $\mathbf{\Lambda}$ is set to $\mathbf{0}$.

To solve our model with ambiguity, we have to form these higher-order approximations under the worst-case belief E_t^*. In our case, this means that we need to make sure that the $\mathbf{P}$ matrix used in (2) reflects E_t^*. In the specific model used in our paper, where there is time-varying ambiguity a_t about the conditional mean of $z_t := \log Z_t$, this amounts to

$$\hat{\mathbf{s}}_\mathbf{t}^\mathbf{0} := \begin{bmatrix} \widehat{\widetilde{\mathbf{s}}}_\mathbf{t} \\ \widehat{z}_t \\ \widehat{a}_t \end{bmatrix} = \mathbf{P} \begin{bmatrix} \widehat{\widetilde{\mathbf{s}}}_\mathbf{t-1} \\ \widehat{z}_{t-1} \\ \widehat{a}_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_\mathbf{t} \\ u_t \\ \varepsilon_t^a \end{bmatrix}$$

where $\hat{\mathbf{s}}_\mathbf{t}^\mathbf{0}$ are deviations of $\mathbf{s_t}$ from the worst-case steady state $\overline{\mathbf{s}}^\mathbf{0}$ and $\widetilde{\mathbf{s}}_\mathbf{t}$ denotes the other exogenous shocks of size n. To reflect the time t worst case belief about $\widehat{z}_{t+1}$, the matrix $\mathbf{P}$ is given by:

$$\mathbf{P} = \begin{bmatrix} \rho_{n\times n} & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_a \end{bmatrix}$$

Based on $\overline{\mathbf{s}}^\mathbf{0}$, we find the steady states $\overline{\mathbf{x}}^\mathbf{0}$ and the corresponding values for $\overline{\mathbf{y}}^\mathbf{0}$ using the equilibrium conditions $\mathbf{f}(\cdot)$ evaluated at $(\overline{\mathbf{y}}^\mathbf{0}, \overline{\mathbf{x}}^\mathbf{0}, \overline{\mathbf{s}}^\mathbf{0})$. Standard techniques can then be applied to obtain higher-order approximations around $(\overline{\mathbf{x}}^\mathbf{0}, \overline{\mathbf{s}}^\mathbf{0}, \mathbf{0}, \mathbf{0})$ for the functions $\mathbf{h}(\cdot)$ and $\mathbf{g}(\cdot)$.

Denote these approximations by $\widetilde{x}_{i,t+1}^* = h_i(\mathbf{x_t}, \mathbf{s_{t-1}}, \epsilon_\mathbf{t}, \mathbf{\Lambda})$ for $i = \{1, ..., m\}$ and $\widetilde{y}_{j,t}^* = g_j(\mathbf{x_t}, \mathbf{s_{t-1}}, \epsilon_\mathbf{t}, \mathbf{\Lambda})$ for $j = \{1, ..., k\}$. These functions control the evolution, under the worst-case dynamics for the exogenous variables, of the state variables indexed by i and control variables indexed by j. To describe these dynamics under the econometrician law of motion, we then have to feed a modified law of motion for $\mathbf{s_t}$. In particular, we have to account for the fact that, from the perspective of the agent's worst case beliefs at $t-1$, the average innovation of the technology shock at time t is not equal to 0, but rather equal to a_{t-1} :

$$z_t = E^*_{t-1} z_t + u_t + a_{t-1}.$$

This means that to describe the dynamics of the economy under the econometrician's law of motion we can take the approximated functions $\widetilde{\mathbf{x}}^*_{\mathbf{t+1}}$ and $\widetilde{\mathbf{y}}^*_{\mathbf{t}}$ and add the corresponding response to a TFP innovation equal to a_{t-1}. In particular, denote by $\widetilde{x}^u_i$ and $\widetilde{y}^u_j$ the coefficient controlling the response to the innovation u_t of some state variable x_i, and control variable y_j, respectively. The dynamics under the econometrician's law of motion are thus given by $\widetilde{\mathbf{x}}^*_{\mathbf{t+1}}$ and $\widetilde{\mathbf{y}}^*_{\mathbf{t}}$, such that:

$$\widetilde{x}_{i,t+1} = \widetilde{x}^*_{i,t+1} + \widetilde{x}^u_i a_{t-1} \tag{3}$$

$$\widetilde{y}_{j,t} = \widetilde{y}^*_{j,t} + \widetilde{y}^u_j a_{t-1} \tag{4}$$

Notice that the coefficients $\widetilde{x}^u_i$ and $\widetilde{y}^u_j$ are obtained in the first-order approximation of the function $\mathbf{h}(\cdot)$ and $\mathbf{g}(\cdot)$. Moreover, because of the recursive nature of finding the higher order approximations, these coefficients are the same independent of the order of approximation. Formulas (3) and (4) imply that the dynamics of $\widetilde{\mathbf{x}}_{\mathbf{t+1}}$ and $\widetilde{\mathbf{y}}_{\mathbf{t}}$ are different from $\widetilde{\mathbf{x}}^*_{\mathbf{t+1}}$ and $\widetilde{\mathbf{y}}^*_{\mathbf{t}}$ by the same amount, independent of the order used in the approximation of $\widetilde{\mathbf{x}}^*_{\mathbf{t+1}}$ and $\widetilde{\mathbf{y}}^*_{\mathbf{t}}$.

Figure 1 presents impulse responses, under the econometrician's DGP, to an increase in ambiguity for the estimated model described in the quantitative section of our paper. The figure shows that dynamics computed under first, second and third-order approximations produce very similar results. To interpret these results, formulas (3) and (4) show that $\widetilde{\mathbf{x}}_{\mathbf{t+1}}$ and $\widetilde{\mathbf{y}}_{\mathbf{t}}$ change across different orders of approximation only due to changes in the approximation of the decision rules $\widetilde{\mathbf{x}}^*_{\mathbf{t+1}}$ and $\widetilde{\mathbf{y}}^*_{\mathbf{t}}$ which were computed under expected utility.

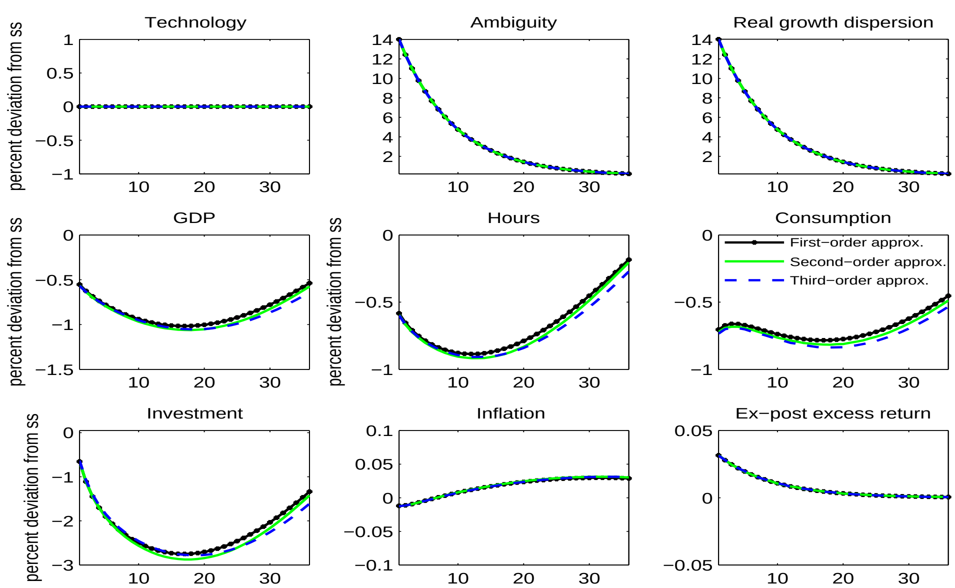
II. Convergence properties

Table 1 reports the Brooks-Gelman-Rubin (Gelman and Rubin (1992) and Brooks and Gelman (1998)) Potential Scale Reduction Factor (PSRF) for each parameter. Posterior distribution is obtained from 4 chains of 600,000 draws generated using a Random walk Metropolis algorithm with an acceptance rate of 29%. We discard the initial 100,000 draws for each chain and retain one out of every 4 subsequent draws. The PSRF values are very close to 1 and thus well below the benchmark value of 1.1 used as an upper bound for convergence.

Table 1—Potential Scale Reduction Factor

Parameter	PSRF	Parameter	PSRF	Parameter	PSRF
α	1.0001	a_π	1.0001	$100\sigma_{R,\epsilon}$	1.0005
$100(\beta^{-1}-1)$	1.0003	a_y	1.0002	$100\sigma_{\pi,\epsilon}$	1.0001
$100(\gamma-1)$	1.001	ρ_R	1.0001	$100\sigma_{I,\epsilon}$	1.0001
$100(\bar{\pi}-1)$	1.0009	n	1.0055	$100\sigma_{C,\epsilon}$	1.0001
ξ_p	1.0001	ρ_z	1.0001	$100\sigma_{H,\epsilon}$	1.0002
ξ_w	1.0002	ρ_a	1.0001	$100\sigma_{D,\epsilon}$	1.0002
κ	1.0001	σ_z	1.0002		
σ_L	1.0009	σ_n	1.0003		

Figure 1. Impulse response: an increase in ambiguity



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