Auctions, Actions, and the Failure of Information Aggregation
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Online Appendix

Lemma O.1. *For any* $s > s_R^\kappa$, $\lim_z \rho_z^-(s) = \rho(s)\frac{\kappa-(1-F(s|R)))}{F(s|R)}\frac{F(s|L)}{\kappa-(1-F(s|L)))}$. *There is a unique signal* s^p *which is a limit point of the cutoffs* $\{s_z^p\}_{z\geq 1}$. *Moreover either* $s^p = s_R^\kappa$ *or* s^p *is the unique signal with the property that satisfies for* $\bar{\rho}(s) := \rho(s)\frac{\kappa-(1-F(s|R)))}{F(s|R)}\frac{F(s|L)}{\kappa-(1-F(s|L)))}$, $u(\bar{\rho}(s)) = u(\rho(s))$.

PROOF:

Pick any convergent sequence of signals $\{s_z^p\}_{z\geq 1}$ with a limit $s^p > s_R^\kappa$. Let *objects taken* denote the random variable which is equal to the minimum of κz and the number of bidders with signals higher than s_z^p. We first note that, $\rho_z^-(s_z^p)$ can be more conveniently expressed by the following equality:

$$\rho_z^-(s_z^p) = \rho(s_z^p)\frac{E\left[\frac{\kappa z-(\text{objects taken})}{z-(\text{objects taken})}|R\right]}{E\left[\frac{\kappa z-(\text{objects taken})}{z-(\text{objects taken})}|L\right]}.$$

Our first observation is the following: Conditional on state ω, $\frac{\text{objects taken}}{z} \to 1 - F(s^p|\omega)$ in probability as $z \to \infty$. Therefore, as $z \to \infty$,

$$E\left[\frac{\kappa z-(\text{objects taken})}{z-(\text{objects taken})}|\omega\right] \to \frac{\kappa-(1-F(s^p|\omega))}{F(s^p|\omega)},$$

and hence,

$$\frac{E\left[\frac{\kappa z-(\text{objects taken})}{z-(\text{objects taken})}|R\right]}{E\left[\frac{\kappa z-(\text{objects taken})}{z-(\text{objects taken})}|L\right]} \to \frac{\frac{\kappa-(1-F(s^p|R))}{F(s^p|R)}}{\frac{\kappa-(1-F(s^p|L))}{F(s^p|L)}}.$$

Therefore as $z \to \infty$, $\rho_z^-(s_z^p) \to \rho(s^p)\frac{\frac{\kappa-(1-F(s^p|R))}{F(s^p|R)}}{\frac{\kappa-(1-F(s^p|L))}{F(s^p|L)}}$. This proves the first claim of the lemma, by taking $\{s_z^p\}_{z\geq 1}$ to be the constant sequence whose elements are equal to $s > s_R^\kappa$. Now let the sequence $\{s_z^p\}_{z\geq 1}$ be the sequence of cutoffs, and let s^p be a limit point of the sequence, and renumber the new sequence so that its limit is s^p. We have already shown that $s^p \geq s_R^\kappa$ in claim 1 in the proof of Lemma **??**. So now assume that $s^p > s_R^\kappa$. Since $s^p > s_R^\kappa$, as $z \to \infty$, $\rho_z^+(s_z^p) \to \rho(s^p)$.

Since each s_z^p has the feature that $u(\rho_z^-(s_z^p)) = u(\rho_z^+(s_z^p))$, and since for each $s > s_R^\kappa$, $\rho_z^-(s) \to \rho(s)\frac{\frac{\kappa-(1-F(s|R))}{F(s|R)}}{\frac{\kappa-(1-F(s|L))}{F(s|L)}}$, we have that for $s \in (s_R^\kappa, 1)$,

$$\Delta(s) := u\left(\rho(s)\left(\frac{\kappa-(1-F(s|R)))}{F(s|R)}\frac{F(s|L)}{\kappa-(1-F(s|L)))}\right)\right) - u\left(\rho(s)\right) \overset{(\geq)}{\leq} 0 \text{ for } s \overset{(\leq)}{\geq} s^p.$$

The term $\rho(s)\left(\frac{\kappa-(1-F(s|R)))}{F(s|R)}\frac{F(s|L)}{\kappa-(1-F(s|L)))}\right)$ is strictly increasing and is always strictly less than $\rho(s)$ in the interval $[s_R^\kappa, 1)$, and is strictly negative when s is close to 1, therefore, there should be at most one signal s^p that can be a limit point in the range $(s_R^\kappa, 1)$.

Now suppose that s_R^κ is a limit point. We will show that no signal $s > s_R^\kappa$ can be a limit point. If s_R^κ is a limit point of the sequence, then it should be that $\lim_z u(\rho_z^-(s_z^p)) = u(0)$, and $\limsup_z u(\rho_z^+(s_z^p)) \le u(\rho(s_R^\kappa))$. Since $u(\rho_z^-(s_z^p)) = u(\rho_z^+(s_z^p))$, it has to be that $u(0) \le u(\rho(s_R^\kappa))$. But then, for every $s > s_R^\kappa$, $\Delta(s) < 0$.

Hence we have shown that if s_R^κ is a limit point, then it is the unique limit point, and if it is not, and if an $s > s_R^\kappa$ is a limit point, then it is unique. This completes the argument that the sequence has a unique limit point.

Lemma O.2. *If $f(0|L) \neq f(0|R)$, then $\rho_z^-(s) > \rho_z^+(s)$ for any $s \in (0,1)$. Moreover both of these functions are strictly increasing in s.*

PROOF:

The first claim in this lemma is identical to the argument in Pesendorfer and Swinkels (1997, Lemma 7, page 1272), and is called loser's curse. The claim that $\rho_z^+(s)$ and $\rho_z^-(s)$ are strictly increasing is standard and follows from the MLRP assumption. The proof can be found in the technical appendix of Milgrom and Weber (1982).

Lemma O.3. *In an increasing equilibrium bidding function b, if there is an atom at bid b^p, then $\Pr(\omega = L|s_1 = s, p) > \Pr(\omega = L|s_1 = s, p = b^p, 1$ wins with $b^p)$ for any $p < b^p$. Also $\Pr(\omega = L|s_1 = s, p) < \Pr(\omega = L|s_1 = s, p = b^p, 1$ loses with $b^p)$ for any $p > b^p$.*

PROOF:

The two claims are proven in a very similar way, so we will only prove the first one, i.e., $\Pr(\omega = L|s_1 = s, p) > \Pr(\omega = L|s_1 = s, p = b^p, 1 \text{ wins with } b^p)$ for any $p < b^p$. This inequality is very intuitive, but we could not find it in any source, so we are proving it directly using the standard combinatorial techniques.

Let the interval of types who are bidding at the atom bid be (s', s''). Then $\Pr(\omega = L|s_1 = s, p) > f(\omega = L|s_1 = s)\Pr(\omega = L|Y_{z-1}^k = s')$. The term

$\Pr(\omega = L|s_1 = s, p = b^p, 1 \text{ wins with } b^p)$ is calculated using the following steps:

$$1 - F_t(s', s''|\omega) := \frac{F(s''|\omega) - F(s'|\omega)}{F(s''|\omega)}$$

$$C_j^{n-1-i}(\omega) := \binom{z-1-i}{j}(1 - F_t(s', s''|\omega))^j \left(F_t(s', s''|\omega)\right)^{z-1-i-j}$$

$$D^i(\omega) := \binom{z-1}{i}(1 - F(s''|\omega))^i (F(s''|\omega))^{z-1-i} \sum_{z-1-i \geq j \geq k-i} C_j^{z-1-i}(\omega) \frac{k-i}{j+1}$$

$$\Pr(s_1 = s, p = b^p, 1 \text{ wins with } b^p|\omega) = f(s|\omega) \sum_{0 \leq i \leq k-1} D^i(\omega)$$

$$\Pr(\omega = L|s_1 = s, p = b^p, 1 \text{ wins with } b^p) = \frac{f(s|L) \sum_{0 \leq i \leq k-1} D^i(L)}{f(s|R) \sum_{0 \leq i \leq k-1} D^i(R)}$$

Explanation: The probability that 1 wins with b^p, the price is b^p conditional on ω can be calculated as the sum of the probabilities of winning in each of the following events, $w^{i,j}$ where $i \leq k-1$ bidders bid above s'', and $k-i \leq j \leq z-1-i$ bidders bid the pooling bid. The probability of winning conditional on event $w^{i,j}$ is $\frac{k-i}{j+1}$, since there are $k-i$ objects remaining for the $j+1$ bidders bidding the pooling bid. The above expressions calculate the probability of each event $w^{i,j}$ in each state and calculate the total winning probability in each state. Similarly the term $f(\omega = L|s_1 = s) \Pr(\omega = L|Y_{z-1}^k = s')$ is calculated using the following steps:

$$\Pr(Y_{z-1}^k = s'|\omega) = \tbinom{z-1}{1} f(s'|\omega) \textstyle\sum_{0 \leq i \leq k-1} \tbinom{z-2}{i} (1 - F(s''|\omega))^i (F(s''|\omega))^{z-2-i} C_{k-i-1}^{z-2-i}(\omega)$$

$$f(\omega = L|s_1 = s) \Pr(\omega = L|Y_{z-1}^k = s') = \tfrac{f(s|L) \Pr(Y_{z-1}^k = s'|L)}{f(s|R) \Pr(Y_{z-1}^k = s'|R)}.$$

We will now show the following:

$$\frac{f(s|L) \Pr(Y_{z-1}^k = s'|L)}{f(s|R) \Pr(Y_{z-1}^k = s'|R)} > \frac{f(s|L) \sum_{0 \leq i \leq k-1} D^i(L)}{f(s|R) \sum_{0 \leq i \leq k-1} D^i(R)},$$

or equivalently the following,

$$\frac{\binom{z-1}{1} f(s'|L) \sum_{0 \leq i \leq k-1} \binom{z-2}{i} (1 - F(s''|L))^i (F(s''|L))^{z-2-i} C_{k-i-1}^{z-2-i}(L)}{\binom{z-1}{1} f(s'|R) \sum_{0 \leq i \leq k-1} \binom{z-2}{i} (1 - F(s''|R))^i (F(s''|R))^{z-2-i} C_{k-i-1}^{z-2-i}(R)} > \frac{\sum_{0 \leq i \leq k-1} D^i(L)}{\sum_{0 \leq i \leq k-1} D^i(R)}.$$

Let, $E^i(\omega) := \binom{z-1}{1} f(s'|\omega) \binom{z-2}{i} (1 - F(s''|\omega))^i (F(s''|\omega))^{z-2-i} C_{k-i-1}^{z-2-i}(\omega)$. We first obtain the following identity by direct algebra:

$$\frac{D^i(\omega)}{E^i(\omega)} = \frac{(1-F_t(s',s''|\omega))F(s''|\omega)}{f(s'|\omega)} \sum_{k-i\leq j\leq z-i-1} \frac{(k-i)!(z-k-1)!}{(j+1)!(z-j-i-1)!} \left(\frac{1-F_t(s',s''|\omega)}{F_t(s',s''|\omega)}\right)^{j+i-k}.$$

A simplification of the above identity via a change of variables by letting $u := j-k+i$ delivers the following:

$$D^i(\omega) = E^i(\omega)\frac{(1-F_t(s',s''|\omega))F(s''|\omega)}{f(s'|\omega)} \sum_{0\leq u\leq z-k-1} \frac{(k-i)!(z-k-1)!}{(k-i+u+1)!(z-k-u-1)!} \left(\frac{1-F_t(s',s''|\omega)}{F_t(s',s''|\omega)}\right)^{u}.$$

The following are consequences of MLRP: $\frac{(1-F_t(s',s''|L))F(s''|L)}{f(s'|L)} < \frac{(1-F_t(s',s''|R))F(s''|R)}{f(s'|R)}$, and, for any positive integer u, $\left(\frac{1-F_t(s',s''|L)}{F_t(s',s''|L)}\right)^u < \left(\frac{1-F_t(s',s''|R)}{F_t(s',s''|R)}\right)^u$. Also, for any fixed $u \in \{0,...,z-k-1\}$, the term $\frac{(k-i)!(z-k-1)!}{(k-i+u+1)!(z-k-u-1)!}$ is strictly increasing in i.

Our final observation is that $\frac{E^i(L)}{E^i(R)}$ is strictly decreasing in i. This observation also follows from the MLRP assumption by doing some algebraic manipulations. Putting these observations together yields the desired result.

REFERENCES

Milgrom, Paul R., and Robert J. Weber. 1982. "A theory of auctions and competitive bidding." *Econometrica*, 50(5): 1089–1122.

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