Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality

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Online Appendix

INDIVIDUAL'S PROBLEM: DETAILS:

For any period t, history s^{t-1}, and quality $j \in \{1, \ldots, J\}$, let $k_{i,j,t}(s^{t-1})$ denote individual i's beginning-of-period t holdings of quality j trees. For any period t, history s^t, quality $j \in \{1, \ldots, J\}$, and set $P \subset \mathbb{R}_+$, let $q_{i,j,t}(P; s^t)$ denote his net purchase in period t of quality j trees at a price $p \in P$. The individual chooses a history-contingent sequence for consumption $c_{i,t}(s^t)$ and measures of tree holdings $k_{i,j,t+1}(s^t)$ and net tree purchases $q_{i,j,t}(P; s^t)$ to maximize his expected lifetime utility

$$\sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{\tau=0}^{t-1} \pi_{s_\tau} \beta_{s_\tau} \right) \pi_{s_t} c_{i,t}(s^t).$$

This states that the individual maximizes the expected discounted value of consumption, given the stochastic process for the discount factor. The individual faces a standard budget constraint,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) = c_{i,t}(s^t) + \int_0^\infty p \left(\sum_{j=1}^{J} q_{i,j,t}(\{p\}; s^t) \right) dp,$$

for all t and s^t. The left hand side is the fruit produced by the trees he owns at the start of period t. The right hand side is consumption plus the net purchase of trees at nonnegative prices p. He also faces a law of motion for his tree holdings,

$$k_{i,j,t+1}(s^t) = k_{i,j,t}(s^{t-1}) + q_{i,j,t}(\mathbb{R}_+; s^t),$$

for all $j \in \{1, \ldots, J\}$. This states that the increase in his tree holdings is given by his net purchase of that quality tree. Finally, the individual faces a set of constraints that depends on whether his discount factor is high or low.

If the individual has a high discount factor, $s_t = h$, he is a buyer, which implies $q_{i,j,t}(P; s^t)$ is nonnegative for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. In addition, he must have enough fruit to purchase trees,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) \geq \int_0^\infty \max\{\Theta(p), 1\} p \left(\sum_{j=1}^{J} q_{i,j,t}(\{p\}; s^t) \right) dp.$$

If the individual wishes to purchase q trees at a price p and $\Theta(p) > 1$, he will be rationed and so must bring $\Theta(p)pq$ fruit to the market to make this purchase.

This constrains his ability to buy trees in markets with excess demand. Together with the budget constraint, this also ensures consumption is nonnegative. Finally, he can only purchase quality j trees at a price p if individuals are selling them at that price, that is

$$q_{i,j,t}(P;s^t) = \int_P \gamma_j(p) \left(\sum_{j'=1}^{J} q_{i,j',t}(\{p\};s^t) \right) dp$$

for all $j \in \{1,\ldots,J\}$ and $P \subset \mathbb{R}_+$. The left hand side is the quantity of quality j trees purchased at a price $p \in P$. The integrand on the right hand side is the product of quantity of trees purchased at price p and the share of those trees that are of quality j.

If the individual has a low discount factor, $s_t = l$, he is a seller, which implies $q_{i,j,t}(P;s^t)$ is nonpositive for all $j \in \{1,\ldots,J\}$ and $P \subset \mathbb{R}_+$. In addition, he may not try to sell more trees than he owns:

$$k_{i,j,t}(s^{t-1}) \geq -\int_0^\infty \max\{\Theta(p)^{-1},1\} q_{i,j,t}(\{p\};s^t) dp,$$

for all $j \in \{1,\ldots,J\}$. Each tree only sells with probability $\min\{\Theta(p),1\}$ at price p, so if $\Theta(p) < 1$, an individual must bring $\Theta(p)^{-1}$ trees to the market to sell one of them. Sellers are not restricted from selling trees in the wrong market. Instead, in equilibrium they will be induced not to do so.

Let $\bar{V}^*(\{k_j\})$ be the supremum of the individuals' expected lifetime utility over feasible policies, given initial tree holding vector $\{k_j\}$. We prove in Proposition 1 that the function $\bar{V}^*$ satisfies the following functional equation:

$$\bar{V}(\{k_j\}) = \pi_h V_h(\{k_j\}) + \pi_l V_l(\{k_j\}), \tag{1}$$

where

(2)

$$V_h(\{k_j\}) = \max_{\{q_j, k_j'\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_h \bar{V}(\{k_j'\}) \right)$$

subject to $k_j' = k_j + q_j(\mathbb{R}_+)$ for all $j \in \{1, \ldots, J\}$

$$\sum_{j=1}^{J} \delta_j k_j \geq \int_0^\infty \max\{\Theta(p), 1\} p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp,$$

$$q_j(P) = \int_P \gamma_j(p) \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+$$

$$q_j(P) \geq 0 \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+,$$

and

$$V_l(\{k_j\}) = \max_{\{q_j, k_j'\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_l \bar{V}(\{k_j'\}) \right) \quad (3)$$

subject to $k_j' = k_j + q_j(\mathbb{R}_+)$ for all $j \in \{1, \ldots, J\}$

$$k_j \geq -\int_0^\infty \max\{\Theta(p)^{-1}, 1\} q_j(\{p\}) dp \text{ for all } j \in \{1, \ldots, J\},$$

$$q_j(P) \leq 0 \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+,$$

PROOF OF PROPOSITION 1:

Let $\bar{\Theta}(p) \equiv \max\{\Theta(p), 1\}$ and $\underline{\Theta}(p) = \min\{\Theta(p), 1\}$. Fix Θ and Γ and take any positive-valued numbers $\{v_{s,j}\}$ and λ that solve the Bellman equations (1), (3), and (4) for $s = l, h$. Let p_h be an optimal price for buying trees,

$$p_h \in \arg\max_p \left(\bar{\Theta}(p)^{-1} \left(\frac{\beta_h \sum_{j=1}^{J} \gamma_j(p) \bar{v}_j}{p} - 1 \right) \right).$$

Similarly let $p_{l,j}$ be an optimal price for selling quality j trees,

$$p_{l,j} = \arg\max_p \underline{\Theta}(p) \big(p - \beta_l \bar{v}_j \big)$$

for all δ. We seek to prove that $\bar{V}^*(\{k_j\}) \equiv \sum_{j=1}^{J} \bar{v}_j k_j$ where $\bar{v}_j = \pi_h v_{h,j} + \pi_l v_{l,j}$.

If $\lambda = 1$, equations (1) and (3) imply

$$\bar{v}_j = \pi_h\big(\delta_j + \beta_h \bar{v}_j\big) + \pi_l\big(\delta_j + \underline{\Theta}(p_{l,j})p_{l,j} + (1-\underline{\Theta}(p_{l,j}))\beta_l \bar{v}_j\big).$$

for all δ. Equivalently,

$$\bar{v}_j = \frac{\delta_j + \pi_l \underline{\Theta}(p_{l,j})p_{l,j}}{1 - \pi_h\beta_h - \pi_l\beta_l(1-\underline{\Theta}(p_{l,j}))} > 0.$$

Alternatively, if $\lambda > 1$, the same equations imply

$$\bar{v}_j = \pi_h\left(\delta_j\left(\big(1-\bar{\Theta}(p_h)^{-1}\big) + \bar{\Theta}(p_h)^{-1}\frac{\beta_h\sum_{j'=1}^J \gamma_{j'}(p_h)\bar{v}_{j'}}{p_h}\right) + \beta_h\bar{v}_j\right) + \pi_l\big(\delta_j + \underline{\Theta}(p_{l,j})p_{l,j} + (1-\underline{\Theta}(p_{l,j}))\beta_l\bar{v}_j\big)$$

for all δ. Since $v_{l,j}$ and $v_{h,j}$ are positive by assumption so is $\bar{v}_j$, and equivalently we can write

$$\bar{v}_j\left(1 - \pi_h\beta_h - \pi_l\beta_l(1-\underline{\Theta}(p_{l,j})) - \pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j\sum_{j'=1}^J\gamma_{j'}(p_h)\bar{v}_{j'}}{p_h\bar{v}_j}\right) = \pi_h\delta_j\left(1-\bar{\Theta}(p_h)^{-1}\right) + \pi_l\big(\delta_j + \underline{\Theta}(p_{l,j})p_{l,j}\big).$$

The right hand side of this expression is positive for all j. Once again since $\bar{v}_j > 0$, with $\lambda > 1$, this holds if and only if

$$(4) \qquad 1 - \pi_h\beta_h - \pi_l\beta_l(1-\underline{\Theta}(p_{l,j})) > \pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j\sum_{j'=1}^J\gamma_{j'}(p_h)\bar{v}_{j'}}{p_h\bar{v}_j}.$$

If this restriction fails at any prices p_h and $p_{l,j}$, it is possible for an individual to obtain unbounded expected utility by buying and selling trees at the appropriate prices. We are interested in cases in which it is satisfied.

Next, let $\bar{V}(\{k_j\}) = \sum_{j=1}^J \bar{v}_j k_j$ and $V_s(\{k_j\}) \equiv \sum_{j=1}^J v_{s,j}k_j$ for $s = l, h$. It is easy to prove that $\bar{V}$ and $\bar{V}_s$ solve equations (1), (2), and (3) and that the same policy is optimal.

Finally, we adapt Theorem 4.3 from Ivan Werning (2009), which states the following: suppose $\bar{V}(k)$ for all k satisfies the recursive equations (1), (2), and (3) and there exists a plan that is optimal given this value function which gives rise to a sequence of tree holdings $\{k^*_{i,j,t}(s^{t-1})\}$ satisfying

$$(5) \qquad \lim_{t\to\infty}\sum_{s^t}\left(\prod_{\tau=0}^{t-1}\pi_{s_\tau}\beta_{s_\tau}\right)\bar{V}(\{k^*_{i,j,t}(s^{t-1})\}) = 0.$$

Then, $\bar{V}^* = \bar{V}$.

If $\lambda = 1$, an optimal plan is to sell quality j trees at price $p_{l,j}$ when impatient and not to purchase trees when patient. This gives rise to a non-increasing sequence for tree holdings. Given the linearity of $\bar{V}$, condition (5) holds trivially.

If $\lambda > 1$, it is still optimal to sell quality j trees at price $p_{l,j}$ when impatient, but patient individuals purchase trees at price p_h and do not consume. Thus

$$k'_{h,j} = k_j + \bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^{J}\delta_{j'}k_{j'}}{p_h}$$
$$k'_{l,j} = \big(1 - \underline{\Theta}(p_{l,j})\big)k_j.$$

Using linearity of the value function, the expected discounted value next period of an individual with tree holdings $\{k_j\}$ this period is

$$\begin{aligned}
&\sum_{j=1}^{J}\bar{v}_j\big(\pi_h\beta_h k'_{h,j} + \pi_l\beta_l k'_{l,j}\big)\\
&= \sum_{j=1}^{J}\bar{v}_j\left(\pi_h\beta_h\left(k_j + \bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^{J}\delta_{j'}k_{j'}}{p_h}\right) + \pi_l\beta_l\big(1-\underline{\Theta}(p_{l,j})\big)k_j\right)\\
&= \sum_{j=1}^{J}\bar{v}_j k_j\left(\pi_h\beta_h + \pi_l\beta_l\big(1-\underline{\Theta}(p_{l,j})\big) + \pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j\sum_{j'=1}^{J}\gamma_{j'}(p_h)\bar{v}_{j'}}{p_h\bar{v}_j}\right),
\end{aligned}$$

where the second equality simply rearranges terms in the summation. Equation (4) implies that each term of this sum is strictly smaller than $\bar{v}_j k_j$. This implies that there exists an $\eta < 1$ such that

$$\eta > \frac{\sum_{j=1}^{J}\bar{v}_j\big(\pi_h\beta_h k'_{h,j} + \pi_l\beta_l k_{l,j}\big)}{\sum_{j=1}^{J}\bar{v}_j k_j} = \frac{\pi_h\beta_h\bar{V}(\{k'_{h,j}\}) + \pi_l\beta_l\bar{V}(\{k'_{l,j}\})}{\bar{V}(\{k_j\})},$$

and so condition (5) holds.

PROOF OF LEMMA 1:

Consider problem (P_1). Given that there is no $j' < 1$, the only constraint is (5). If such a constraint were slack, we could increase p and hence raise the value of the objective function, which ensures the constraint binds. Eliminating the price by substituting the binding constraint into the objective function gives

$$v_{l,1} = \delta_1 + \max_{\theta}\left(\min\{\theta,1\}\frac{\beta_h\min\{\theta^{-1},1\}}{\lambda - 1 + \min\{\theta^{-1},1\}} + (1-\min\{\theta,1\})\beta_l\right)\bar{v}_1.$$

If $\lambda = 1$, any $\theta_1 \geq 1$ attains the maximum. If $\lambda = \beta_h/\beta_l$, any $\theta_1 \in [0,1]$ attains the maximum. For intermediate values of λ, the unique maximizer is $\theta_1 = 1$. Substituting back into the original problem gives $v_{l,1} = \delta_1 + p_1$ and $p_1 = \beta_h \bar{v}_1/\lambda$, establishing the result for $j = 1$.

For $j \geq 2$ we proceed by induction. Assume for all $j' \in \{2, \ldots, j-1\}$, we have established the characterization of $p_{j'}$, $\theta_{j'}$, $v_{l,j'}$ and $\bar{v}_{j'}$ in the statement of the lemma. We first prove that $\bar{v}_j > \bar{v}_{j-1}$. To do this, consider the policy (θ_{j-1}, p_{j-1}). If this solved problem (P_j), combining the objective function and the definition of $\bar{v}_j$ gives

$$\begin{aligned}\bar{v}_j &= \frac{\delta_j(\pi_h\lambda + \pi_l) + \pi_l \min\{\theta_{j-1}, 1\} p_{j-1}}{1 - \pi_h\beta_h - \pi_l\beta_l(1 - \min\{\theta_{j-1}, 1\})} \\ &> \frac{\delta_{j-1}(\pi_h\lambda + \pi_l) + \pi_l \min\{\theta_{j-1}, 1\} p_{j-1}}{1 - \pi_h\beta_h - \pi_l\beta_l(1 - \min\{\theta_{j-1}, 1\})} = \bar{v}_{j-1}.\end{aligned}$$

The inequality uses the fact that the denominator is positive together with $\delta_j > \delta_{j-1}$; and the last equality comes from the objective function and the definition of $\bar{v}_{j-1}$ in problem (P_{j-1}). Since the proposed policy satisfies all of the constraints in problem (P_{j-1}) and $\bar{v}_j > \bar{v}_{j-1}$, it also satisfies all the constraints in problem (P_j). The optimal policy must deliver a weakly higher value, proving $\bar{v}_j > \bar{v}_{j-1}$.

Next we prove that at any solution to problem (P_j) the constraint (5) is binding. If there were an optimal policy (θ, p) such that it was slack, consider a small increase in p to $p' > p$ and a reduction in θ to $\theta' < \theta$ so that $\min\{\theta, 1\}(p - \beta_l \bar{v}_{j-1}) = \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j-1})$ while constraint (5) is still satisfied. Now suppose for some $j' \neq j-1$, $\min\{\theta, 1\}(p - \beta_l \bar{v}_{j'}) < \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j'})$. Subtracting the inequality from the preceding equation gives

$$\big(\min\{\theta, 1\} - \min\{\theta', 1\}\big)(\bar{v}_{j'} - \bar{v}_{j-1}) > 0.$$

Given that $\theta' < \theta$, the above inequality yields $\bar{v}_{j'} > \bar{v}_{j-1}$ and hence $j' \geq j$. This implies that the change in policy does not tighten the constraints (6) for $j' < j$, while it raises the value of the objective function in problem (P_j), a contradiction. Therefore constraint (5) must bind at the optimum.

We now show that the binding constraint (5) implies that $\theta_j \leq 1$ for all $j \geq 2$. By contradiction, assume that the solution to problem (P_j) is some (θ, p) with $\theta > 1$. In this case, the objective function reduces to $v_{l,j} = \delta_j + p$, while the constraint (6) for $j' = 1$ imposes $v_{l,1} \geq \delta_1 + p$. Since we have shown that $v_{l,1} = \delta_1 + p_1$, this implies $p \leq p_1$. Moreover, $\bar{v}_j > \bar{v}_1$ implies $\beta_h \bar{v}_j/\lambda > \beta_h \bar{v}_1/\lambda = p_1$ and hence $\beta_h \bar{v}_j/p > \lambda$. Now a change to the policy $(1, p)$ relaxes the constraint (5) without affecting any other piece of the problem (P_j) and is therefore weakly optimal. But this cannot be optimal because (5) is slack, a contradiction. This proves that $\theta_j \leq 1$ for all $j \geq 2$ and hence, using the binding constraint (5), $p_j = \beta_h \bar{v}_j/\lambda$.

Next, we prove that if $\lambda < \beta_h/\beta_l$, the constraint (6) is binding at $j' = j-1$. We break our proof into two parts. First, consider $j = 2$ and, to find a contradiction, assume that there is a solution (θ, p) to problem (P_2) such that constraint (6) is slack for $j' = 1$. Then problem (P_2) is equivalent to problem (P_1) except for the value of the dividend $\delta_2 > \delta_1$. Following the same argument used for problem (P_1), we can show that $\theta_2 \geq 1$ and so constraint (6) reduces to $v_{l,1} \geq \delta_1 + p_2$. But since $p_1 = \beta_h \bar{v}_1/\lambda < p_2 = \beta_h \bar{v}_2/\lambda$, this contradicts $v_{l,1} = \delta_1 + p_1$. Constraint (6) must bind when $j = 2$.

Next consider $j > 2$ and again assume by contradiction that there is a solution (θ, p) to problem (P_j) such that constraint (6) is slack for $j' = j - 1$. Then problem (P_j) is equivalent to problem (P_{j-1}) except in the value of the dividend δ. Since constraint (6) is binding in the solution to problem (P_{j-1}) and $\theta_{j-1} \leq 1$, we have

$$v_{l,j-2} = \delta_{j-2} + \theta_{j-1} p_{j-1} + (1 - \theta_{j-1}) \beta_l \bar{v}_{j-2} = \delta_{j-2} + \theta p + (1 - \theta) \beta_l \bar{v}_{j-2},$$

and hence

$$\theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-2}) = \theta(p - \beta_l \bar{v}_{j-2}). \tag{6}$$

Since $p = \beta_h \bar{v}_j/\lambda$ and $p_{j-1} = \beta_h \bar{v}_{j-1}/\lambda$, $p - \beta_l \bar{v}_{j-2} > p_{j-1} - \beta_l \bar{v}_{j-2} > 0$ and so $\theta_{j-1} > \theta > 0$. But now combine equation (6) with $\theta_{j-1} > \theta$ and $\bar{v}_{j-1} > \bar{v}_{j-2}$ to get

$$\theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-1}) < \theta(p - \beta_l \bar{v}_{j-1}).$$

This implies that constraint (6) for $j' = j - 1$ is violated, a contradiction. This proves that constraint (6) must bind whenever $\lambda < \beta_j/\beta_l$ and establishes all the equations in the statement of the lemma.

Alternatively, suppose $\lambda = \beta_h/\beta_l$. Since $p_j = \beta_h \bar{v}_j/\lambda = \beta_l \bar{v}_j$, the objective function in problem (P_j) reduces to $v_{l,j} = \delta_j + \beta_l \bar{v}_j$, while constraint (6) imposes

$$v_{l,j'} = \delta_{j'} + \beta_l \bar{v}_{j'} \geq \delta_{j'} + \beta_l \big(\theta \bar{v}_j + (1 - \theta) \bar{v}_{j'}\big)$$

for all $j' < j$. Since $\bar{v}_j > \bar{v}_{j'}$, this implies $\theta = 0$ in the solution to the problem. It is easy to verify that this is implied by the equations in the statement of the lemma.

Finally, we need to prove that there is a unique value of $\bar{v}_j > \bar{v}_{j-1}$ that solves the four equations in the statement of the lemma. Combining them we obtain

$$(1 - \pi_h \beta_h - \pi_l \beta_l) \bar{v}_j = \delta_j (\pi_l + \lambda \pi_h) + \pi_l \min\{\theta_{j-1}, 1\} \frac{\big(\beta_h - \beta_l \lambda\big)^2 \bar{v}_{j-1} \bar{v}_j}{\big(\beta_h \bar{v}_j - \beta_l \lambda \bar{v}_{j-1}\big) \lambda}. \tag{7}$$

If $\lambda = \beta_h/\beta_l$, the last term is zero and so this pins down $\bar{v}_j$ uniquely. Otherwise we prove that there is a unique solution to equation (7) with $\bar{v}_j > \bar{v}_{j-1}$. In particular, the left hand side is a linearly increasing function of $\bar{v}_j$, while the right hand side is an increasing, concave function, and so there are at most two solutions to the equation. As $\bar{v}_j \to \infty$, the left hand side exceeds the right hand side, and so we simply need to prove that as $\bar{v}_j \to \bar{v}_{j-1}$, the right hand side exceeds the left hand side.

First assume $j = 2$ so $\theta_{j-1} = \theta_1 \geq 1$. Then we seek to prove that

$$(1 - \pi_h \beta_h - \pi_l \beta_l)\bar{v}_1 < \delta_2(\pi_l + \lambda \pi_h) + \pi_l \frac{(\beta_h - \beta_l \lambda)\bar{v}_1}{\lambda}.$$

Since $\bar{v}_1 = (\delta_1 \lambda(\pi_l + \lambda\pi_h))/(\lambda - \beta_h(\pi_l + \lambda\pi_h))$ and $\delta_1 < \delta_2$, we can confirm this directly. Next take $j \geq 3$. In this case, in the limit with $\bar{v}_j \to \bar{v}_{j-1}$, the right hand side of (7) converges to

$$\delta_j(\pi_l + \lambda\pi_h) + \pi_l \theta_{j-1} \frac{(\beta_h - \beta_l\lambda)\bar{v}_{j-1}}{\lambda} > \delta_{j-1}(\pi_l + \lambda\pi_h) + \pi_l \min\{\theta_{j-2}, 1\} \frac{(\beta_h - \beta_l\lambda)^2 \bar{v}_{j-2}\bar{v}_{j-1}}{(\beta_h \bar{v}_{j-1} - \beta_l \lambda \bar{v}_{j-2})\lambda},$$

where the inequality uses the indifference condition

$$\min\{\theta_{j-2}, 1\}(p_{j-2} - \beta_l \bar{v}_{j-2}) = \theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-2})$$

and the assumption $\delta_{j-1} < \delta_j$. The right hand side of the inequality is the same as the right hand side of equation (7) for quality $j-1$. The desired inequality then follows by comparing the left hand side of the inequality to the left hand side of equation (7) for quality $j-1$. This completes the proof.

PROOF OF PROPOSITION 2:

We first prove that the solution to problem (P) describes a partial equilibrium and then prove that there is no other equilibrium.

Existence

As in the statement of the proposition, we look for a partial equilibrium where $\mathbb{P} = \{p_j\}$, $\Theta(p_j) = \theta_j$, $\gamma_j(p_j) = 1$, $dF(p_j) = K_j / \sum_{j'} K_{j'}$, and $v_{s,j}$ solves problem (P_j). Also for notational convenience define $p_{J+1} = \infty$. To complete the characterization, we define Θ and Γ on their full support $\mathbb{R}_+$. For $p < p_1$, $\Theta(p) = \infty$ and $\Gamma(p)$ can be chosen arbitrarily, for example $\gamma_1(p) = 1$. For $j \in \{1, \ldots, J\}$ and $p \in (p_j, p_{j+1})$, $\gamma_j(p) = 1$ and $\Theta(p)$ satisfies sellers' indifference condition $v_{l,j} = \delta_j + \min\{\Theta(p), 1\}p + \big(1 - \min\{\Theta(p), 1\}\big)\beta_l \bar{v}_j$; equivalently, $\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) = \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$. To prove that this is a partial equilibrium, we need to verify that the five equilibrium conditions hold.

To show that the third and fourth equilibrium conditions—Buyers' Optimality and Active Markets—are satisfied, it is enough to prove that the prices $\{p_j\}$ solve the optimization problem in equation (4). Lemma 1 implies that $p_j = \beta_h \bar{v}_j / \lambda$

for all λ and j; and $\Theta(p_j) \leq 1$ if $\lambda > 1$. Together these conditions imply that any price p_j achieves the maximum in this optimization problem. For any price $p \in (p_j, p_{j-1})$, $\gamma_j(p) = 1$ by construction, and so the right hand side of equation (4) is smaller than when evaluate at p_j. Moreover, for any $p < p_1$, $\Theta(p) = \infty$ and so the right hand side is $1 \leq \lambda$.

Next we prove that $\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$ for all j and p, with equality if $p \in [p_j, p_{j+1})$. The first and second equilibrium conditions—Sellers's Optimality and Equilibrium Beliefs— follow immediately from this. The equality holds by construction. Let us now focus on the inequalities.

First take any $j' \in \{2, \dots, J\}$, $j < j'$, and $p \in [p_{j'}, p_{j'+1})$. By the construction of Θ,

$$\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_{j'}) = \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_{j'}).$$

Then $p_{j'} \leq p$ implies that $\min\{\Theta(p_{j'}), 1\} \geq \min\{\Theta(p), 1\}$. Since $j < j'$, Lemma 1 implies that $\bar{v}_{j'} > \bar{v}_j$ and so

$$\min\{\Theta(p_{j'}), 1\}(\bar{v}_{j'} - \bar{v}_j) \geq \min\{\Theta(p), 1\}(\bar{v}_{j'} - \bar{v}_j).$$

Adding this to the previous equation gives

$$\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j).$$

Also condition (6) in problem $(P_{j'})$ implies

$$\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_j).$$

Combining the last two inequalities gives

$$\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$$

for all $p \in [p_{j'}, p_{j'+1})$ and $j < j'$.

Similarly, take any $j' \in \{1, \dots, J-1\}$, $j > j'$, and $p \in [p_{j'}, p_{j'+1})$. The construction of Θ implies

$$\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_{j'}) = \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_{j'}),$$

while Lemma 1 together with $\Theta(p_j) = \theta_j$ implies

$$\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l \bar{v}_{j'}) = \min\{\Theta(p_{j'+1}), 1\}(p_{j'+1} - \beta_l \bar{v}_{j'}).$$

The two equalities together imply

$$\min\{\Theta(p_{j'+1}), 1\}(p_{j'+1} - \beta_l \bar{v}_{j'}) = \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_{j'}).$$

Then $p_{j'+1} > p$ implies $\min\{\Theta(p_{j'+1}), 1\} \leq \min\{\Theta(p), 1\}$. Since $j > j'$, Lemma 1 implies that $\bar{v}_j > \bar{v}_{j'}$ and so

$$\min\{\Theta(p_{j'+1}), 1\}(\bar{v}_{j'} - \bar{v}_j) \geq \min\{\Theta(p), 1\}(\bar{v}_{j'} - \bar{v}_j).$$

Adding this to the previous equation gives

$$\min\{\Theta(p_{j'+1}), 1\}(p_{j'+1} - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j).$$

Also, since $(\Theta(p_{j'+1}), p_{j'+1})$ is a feasible policy in problem (P_j),

$$\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p_{j'+1}), 1\}(p_{j'+1} - \beta_l \bar{v}_j).$$

Combining inequalities gives

$$\min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$$

for all $p \in [p_{j'}, p_{j'+1})$ and $j > j'$.

Finally, consider $p < p_1$. Since $\Theta(p) = \infty$, $\min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j) = p - \beta_l \bar{v}_j < p_1 - \beta_l \bar{v}_j \leq \min\{\Theta(p_1), 1\}(p_1 - \beta_l \bar{v}_j)$, where the first inequality uses $p < p_1$ and the second uses the fact that $\Theta(p_1) < 1$ only if $\lambda = \beta_h/\beta_l$; but in this case, $p_1 = \beta_l \bar{v}_1 \leq \beta_l \bar{v}_j$. Since we have already proved that $\min\{\Theta(p_1), 1\}(p_1 - \beta_l \bar{v}_j) \leq \min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j)$, this establishes the inequality for $p < p_1$.

The last piece of the definition of equilibrium is Consistency of Supplies with Beliefs. This holds by the construction of the distribution function F in the statement of the Proposition.

Uniqueness

Now take any partial equilibrium $\{\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, F\}$. We first claim that $\bar{v}$ is increasing in j. Take $j > j'$ and let $p_{j'}$ denote the price offered by j'. Type j Sellers' Optimality implies

$$v_{l,j} \geq \delta_j + \min\{\Theta(p_{j'}), 1\} p_{j'} + (1 - \min\{\Theta(p_{j'}), 1\}) \beta_l \bar{v}_j,$$

and so combining with quality j Buyers' Optimality, equation (3), and solving for $\bar{v}_j$ gives

$$\bar{v}_j \geq \frac{\delta_j(\pi_l + \pi_h \lambda) + \pi_l \min\{\Theta(p_{j'}), 1\} p_{j'}}{\pi_l(1 - \min\{\Theta(p_{j'}), 1\})\beta_l + \pi_h \beta_h} > \frac{\delta_{j'}(\pi_l + \pi_h \lambda) + \pi_l \min\{\Theta(p_{j'}), 1\} p_{j'}}{\pi_l(1 - \min\{\Theta(p_{j'}), 1\})\beta_l + \pi_h \beta_h} = \bar{v}_{j'},$$

where the second inequality uses $\delta_j > \delta_{j'}$ and the equality solves the same equations for $\bar{v}_{j'}$.

Consistency of Supplies with Beliefs implies that for each $j \in \{1, \ldots, J\}$, there exists a price $p_j \in \mathbb{P}$ with $\gamma_j(p_j) > 0$.

Now in the remainder of the proof, assume also that $\theta_j \equiv \Theta(p_j) > 0$. First we prove that the constraint $\lambda \leq \min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_j/p_j + (1 - \min\{\theta_j^{-1}, 1\})$ is satisfied. Second we prove that the constraint $v_{l,j'} \geq \delta_{j'} + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j'})$ is satisfied for all $j' < j$. Third we prove that the pair (θ_j, p_j) delivers value $v_{l,j}$ to sellers of quality j trees. Fourth we prove that (θ_j, p_j) solves (P_j).

Step 1. To derive a contradiction, assume $\lambda > \min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_j/p_j + 1 - \min\{\theta_j^{-1}, 1\}$. Active Markets implies that the expected value of a unit of fruit to a buyer who pays p_j must equal λ and so there must be a j' with $\gamma_{j'}(p_j) > 0$ and $\lambda < \min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_{j'}/p_j + 1 - \min\{\theta_j^{-1}, 1\}$. If $\theta_j = \infty$, $\min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_{j'}/p_j + 1 - \min\{\theta_j^{-1}, 1\} = 1 \leq \lambda$, which is impossible; therefore $\theta_j < \infty$. Then Equilibrium Beliefs implies p_j is an optimal price for quality j' sellers and so for all p' and $\theta' \equiv \Theta(p')$, $\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'}) \geq \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j'})$. Since $\theta_j > 0$, $\min\{\theta_j, 1\}(p' - \beta_l \bar{v}_{j'}) > \min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'})$ for all $p' > p_j$, and so the two inequalities imply $\min\{\theta_j, 1\} > \min\{\theta', 1\}$.

Now take any $j'' < j'$, so $\bar{v}_{j''} < \bar{v}_{j'}$. Then since $\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'}) \geq \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j'})$, $\min\{\theta_j, 1\} > \min\{\theta', 1\}$, and $\bar{v}_{j''} < \bar{v}_{j'}$,

$$\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j''}) > \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j''}).$$

Type j'' Sellers' Optimality condition implies $\bar{v}_{j''} \geq \delta_{j''} + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j''}$ and so the previous inequality gives $\bar{v}_{j''} > \delta_{j''} + \min\{\theta', 1\}p' + (1 - \min\{\theta', 1\})\beta_l \bar{v}_{j''}$. Rational beliefs implies $\gamma_{j''}(p') = 0$. That is, any $p' > p_j$ attracts only quality j' sellers or higher and so delivers value at least equal to $\min\{\theta'^{-1}, 1\}\beta_h \bar{v}_{j'}/p' + (1 - \min\{\theta'^{-1}, 1\})$ to buyers. For p' sufficiently close to p_j, this exceeds λ, contradicting buyers' optimality.

Step 2. Sellers' Optimality implies $v_{l,j'} \geq \delta_{j'} + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j'})$ for all j', p_j, and $\theta_j = \Theta(p_j)$.

Step 3. Equilibrium Beliefs implies $v_{l,j} = \delta_j + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j'})$ for all j, p_j, and $\theta_j = \Theta(p_j) < \infty$ with $\gamma_j(p_j) > 0$.

Step 4. Suppose there is a policy (θ, p) that satisfies the constraints of problem (P_j) and delivers a higher payoff. That is,

$$v_{l,j} < \delta_j + \min\{\theta, 1\}p + (1 - \min\{\theta, 1\})\beta_l \bar{v}_j$$
$$\lambda \leq \min\{\theta^{-1}, 1\}\beta_h \bar{v}_j/p + 1 - \min\{\theta^{-1}, 1\}$$
$$v_{l,j'} \geq \delta_{j'} + \min\{\theta, 1\}p + (1 - \min\{\theta, 1\})\beta_l \bar{v}_{j'} \text{ for all } j' < j.$$

If these inequalities hold with $\theta > 1$, then the same set of inequalities holds with $\theta = 1$, and so we may assume $\theta \leq 1$ without loss of generality. Choose $p' < p$

such that

$$v_{l,j} < \delta_j + \theta p' + (1-\theta)\beta_l \bar{v}_j \tag{8}$$

$$\lambda < \beta_h \bar{v}_j / p' \tag{9}$$

$$v_{l,j'} > \delta_{j'} + \theta p' + (1-\theta)\beta_l \bar{v}_{j'} \text{ for all } j' < j. \tag{10}$$

The previous inequalities imply that this is always feasible by setting p' close enough to p. Now sellers' optimality implies $v_{l,j} \geq \delta_j + \min\{\Theta(p'), 1\}p' + (1 - \min\{\Theta(p'), 1\})\beta_l \bar{v}_j)$, which, together with inequality (8), implies $\Theta(p') < \theta$. This together with inequality (10) implies that

$$v_{l,j'} > \delta_{j'} + \Theta(p')p' + (1-\Theta(p'))\beta_l \bar{v}_{j'} \text{ for all } j' < j,$$

and so, due to Equilibrium Beliefs, $\gamma_{j'}(p') = 0$ for all $j' < j$. But then, using inequality (9), we obtain

$$\begin{aligned}\lambda < \frac{\beta_h \bar{v}_j}{p'} &\leq \frac{\beta_h \sum_{j'=1}^{J} \gamma_{j'}(p')\bar{v}_{j'}}{p'} \\ &= \min\{\Theta(p')^{-1}, 1\} \frac{\beta_h \sum_{j'=1}^{J} \gamma_{j'}(p')\bar{v}_{j'}}{p'} + \left(1 - \min\{\Theta(p')^{-1}, 1\}\right),\end{aligned}$$

where the second inequality uses monotonicity of $\bar{v}_j$ and $\gamma_{j'}(p') = 0$ for $j' < j$; and the last equation uses $\Theta(p') < \theta \leq 1$. This contradicts Buyers' Optimality condition and completes the proof.

PROOF OF PROPOSITION 3:

To prove that there exists a unique competitive equilibrium, it is enough to prove that there exists a unique $\lambda \in [1, \beta_h/\beta_l]$ such that the partial equilibrium associated to that λ clears the fruit market.

For given $\lambda \in [1, \beta_h/\beta_l]$, let $x_j(\lambda) \equiv \theta_j(\lambda)p_j(\lambda)$, where $\theta_j(\lambda)$ and $p_j(\lambda)$ are the partial equilibrium sale probability and price for quality j trees. For all $j > 1$ and given $x_{j-1}(\lambda)$, define

$$f_j(x_j, \lambda) \equiv x_j \left[1 - \frac{\beta_l}{\beta_h}\lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)}\right] - x_{j-1}(\lambda)\left[1 - \frac{\beta_l}{\beta_h}\lambda\right],$$

where, with some abuse of notation,

$$p_j(x_j, \lambda) = \frac{\delta_j \beta_h(\pi_l + \lambda\pi_h) + x_j \pi_l[\beta_h - \beta_l\lambda]}{\lambda(1 - \pi_h\beta_h - \pi_l\beta_l)}. \tag{11}$$

Given $\lambda \in [1, \beta_h/\beta_l]$, Proposition 2 and Lemma 1 ensure that $p_j(x_j(\lambda), \lambda)$ is the equilibrium price for quality j trees with $x_j(\lambda)$ being implicitly defined by

$f_j(x_j, \lambda) = 0$ for all $j > 1$. Moreover, for $\lambda \in (1, \beta_h/\beta_l)$

$$x_1(\lambda) = p_1(x_1(\lambda), \lambda) = \frac{\delta_1 \beta_h(\pi_l + \lambda\pi_h)}{\lambda - \beta_h(\pi_l + \lambda\pi_h)}. \tag{12}$$

Lemma 1 also implies that $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$ for all $j > 1$. From $f_j(x_j, \lambda) = 0$ for all $j > 1$ immediately follows that $x_j(\lambda) < x_{j-1}(\lambda)$ for all $j > 1$.

Next, define $M(\lambda)$ as

$$M(\lambda) \equiv \sum_{j=1}^{J} [\pi_h \delta_j - \pi_l x_j(\lambda)] K_j.$$

Market clearing requires $M(\lambda) = 0$. Now we show that $x'_j(\lambda) < 0$ and hence $M'(\lambda) > 0$ for all $\lambda \in (1, \beta_h/\beta_l)$. For $j = 1$ we can directly calculate

$$x'_1(\lambda) = -\frac{\delta_1 \beta_h \pi_l}{[\lambda - \beta_h(\pi_l + \lambda\pi_h)]^2} < 0.$$

For all $j > 1$, given $x'_{j-1}(\lambda) < 0$ we can proceed recursively as follows. Applying the implicit function theorem to $f_j(x_j, \lambda) = 0$, we obtain

$$x'_j(\lambda) = -\frac{\partial f_j(x_j, \lambda)/\partial \lambda}{\partial f_j(x_j, \lambda)/\partial x_j}.$$

First, we can calculate

$$\frac{\partial f_j(x_j, \lambda)}{\partial x_j} = 1 - \frac{\beta_l}{\beta_h}\lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)} + x_j \frac{\beta_l}{\beta_h}\lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)^2} \frac{\partial p_j(x_j, \lambda)}{\partial x_j}.$$

It is easy to show that $\frac{\partial f_j(x_j,\lambda)}{\partial x_j} > 0$ given that $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$ and

$$\frac{\partial p_j(x_j, \lambda)}{\partial x_j} = \frac{\pi_l[\beta_h - \beta_l \lambda]}{\lambda(1 - \bar{\beta})} > 0.$$

Second, we can calculate

$$\begin{aligned}
\frac{\partial f_j(x_j, \lambda)}{\partial \lambda} =& \frac{\beta_l}{\beta_h}\left[x_{j-1}(\lambda) - x_j \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)}\right] - \left(1 - \frac{\beta_l}{\beta_h}\lambda\right) x'_{j-1}(\lambda) \\
&- \frac{\beta_l}{\beta_h}\lambda \frac{x_j}{p_j(x_j, \lambda)} \frac{\partial p_{j-1}(x_{j-1}(\lambda), \lambda)}{\partial x_{j-1}(\lambda)} x'_{j-1}(\lambda) \\
&- \frac{\beta_l}{\beta_h}\lambda \frac{x_j}{p_j(x_j, \lambda)} \left[\frac{\partial p_{j-1}(x_{j-1}(\lambda), \lambda)}{\partial \lambda} - \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)} \frac{\partial p_j(x_j, \lambda)}{\partial \lambda}\right]
\end{aligned}$$

where the first term is positive because $x_j(\lambda) < x_{j-1}(\lambda)$ and $p_j(x_j(\lambda), \lambda) >$

$p_{j-1}(x_{j-1}(\lambda), \lambda)$, the second term is positive because $\lambda \in (1, \beta_h/\beta_l)$ and $x'_{j-1}(\lambda) < 0$, and the third term is positive because of the last inequality together with $\partial p_j(x_j, \lambda)/\partial x_j > 0$. Finally, to show that the last term is also positive we need to show that the term in square bracket is negative where

$$\frac{\partial p_j(x_j, \lambda)}{\partial \lambda} = -\frac{\beta_h \pi_l (\delta_j + x_j)}{\lambda^2 (1 - \bar{\beta})}.$$

Using expression (11) for $p_j(x_j, \lambda)$ and $f_j(x_j, \lambda) = 0$ for all j, after some algebra, one can show that this is always the case given that $\lambda \in (1, \beta_h/\beta_l)$. This implies that $x'_j(\lambda) < 0$ for all j and hence $M'(\lambda) > 0$.

Finally, define

$$\underline{\pi} \equiv \frac{\sum_{j=1}^{J} \delta_j K_j}{\sum_{j=1}^{J} [\delta_j + x_j(1)] K_j} \text{ and } \overline{\pi} \equiv \frac{\sum_{j=1}^{J} \delta_j K_j}{\sum_{j=1}^{J} [\delta_j + x_j(\beta_h/\beta_l)] K_j},$$

where $x_1(\lambda)$ is given in equation (12) and $x_j(\lambda)$ solves $f_j(x_j, \lambda) = 0$ for all $j > 1$. It is easy to see that $\underline{\pi} < \overline{\pi}$ given that $x'_j(\lambda) < 0$. Moreover, $M(1) < 0$ iff $\pi_l > \underline{\pi}$ and $M(\beta_h/\beta_l) > 0$ iff $\pi_l < \overline{\pi}$. Given that $M'(\lambda) > 0$, it follows that if $\pi_l \in (\underline{\pi}, \overline{\pi})$, there exists a unique equilibrium with $\lambda \in (1, \beta_h/\beta_l)$. If instead $\pi_l \leq \underline{\pi}$, then both $M(1)$ and $M(\beta_h/\beta_l)$ are larger than zero, while if $\pi_l \geq \overline{\pi}$, they are both smaller than zero. Lemma 1 implies that $x_1(1) \geq p_1(1)$ and $x_1(\beta_h/\beta_l) \leq p_1(\beta_h/\beta_l)$. This implies that if $\pi_l \leq \underline{\pi}$, there exists a unique equilibrium with $\lambda = 1$, where $x_1(1)$ is pinned down by market clearing. If instead $\pi_l \geq \overline{\pi}$, then there exists a unique equilibrium with $\lambda = \beta_h/\beta_l$, where $x_1(\beta_h/\beta_l)$ is pinned down by market clearing. This completes the proof.

PROOF OF PROPOSITION 4:

It is straightforward to verify that if P_1 and Θ_1 satisfy equations (12) and (13) with $\underline{p}_1 = P_1(\underline{\delta})$, then the specified functions P_2 and Θ_2 satisfy the same pair of equations with $\underline{p}_2 = P_2(\kappa\underline{\delta})$. The remaining results follow directly from the definition of liquidity, volume, asking price, and transaction price.

PROOF OF PROPOSITION 5:

By construction, $\underline{\delta}_2 = \underline{\delta}_1 \underline{\varepsilon}$. Now equation (12) implies $\underline{p}_2 = \underline{p}_1 \underline{\varepsilon}$ since $\Theta_2(\underline{p}_2) = \Theta_1(\underline{p}_1) = 1$. Then equation (13) implies $\Theta_1(p) = \Theta_2(p\underline{\varepsilon})$ for all $p > 0$. Finally, equation (12) implies $\Theta_1(P_1(\delta)) = \Theta_2(P_2(\delta\underline{\varepsilon}))$ and $P_2(\delta\underline{\varepsilon}) = P_1(\delta)\underline{\varepsilon}$ for all $\delta > 0$. We use these relationships throughout our proof, which we break into pieces corresponding to the four claims.

Liquidity

The liquidity of type 2 assets is

$$L_2 = \pi_l \int_{\underline{\delta}_1}^{\bar{\delta}_1} \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta_1 \varepsilon)) h(\varepsilon) g_1(\delta_1) d\varepsilon d\delta_1$$
$$< \pi_l \int_{\underline{\delta}_1}^{\bar{\delta}_1} \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta_1 \underline{\varepsilon})) h(\varepsilon) g_1(\delta_1) d\varepsilon d\delta_1 = L_1.$$

The inequality uses the fact that P_2 is increasing and Θ_2 is decreasing. The second equality uses $\Theta_2(P_2(\delta_1 \underline{\varepsilon})) = \Theta_1(P_1(\delta_1))$ and integrates over ε.

Volume

We have proved that $L_2 < L_1$. Below we prove $T_2 < T_1$. The definition of V_a then implies $V_2 < V_1$.

Asking Price

The average asking price of type 2 assets is

$$A_2 = \int_{\underline{\delta}_1}^{\bar{\delta}_1} \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} P_2(\delta_1 \varepsilon) h(\varepsilon) g_1(\delta_1) d\varepsilon d\delta_1 < \int_{\underline{\delta}_1}^{\bar{\delta}_1} \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} P_2(\delta_1 \underline{\varepsilon}) \frac{\varepsilon}{\underline{\varepsilon}} h(\varepsilon) g_1(\delta_1) d\varepsilon d\delta_1 = A_1.$$

The inequality uses the fact that price-dividend ratio is lower for higher quality assets, that is $P_2(\delta)/\delta$ is decreasing in δ since $\Theta(p)$ is decreasing (equation 12). The equality uses $P_1(\delta_1) = P_2(\delta_1 \underline{\varepsilon})/\underline{\varepsilon}$ and integrates over ε.

Transaction Price

We break this into two pieces. Consider first the average transaction price for all type 2 assets with dividend $\delta_2 = \delta\varepsilon$, conditional on the value of δ. This is

$$T_2(\delta) \equiv \frac{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) P_2(\delta\varepsilon) h(\varepsilon) d\varepsilon}{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) h(\varepsilon) d\varepsilon}.$$

We again use the fact that the price-dividend ratio is decreasing in δ and hence $P_2(\delta\varepsilon)/\varepsilon < P_2(\delta\underline{\varepsilon})/\underline{\varepsilon} = P_1(\delta)$ for all $\varepsilon > \underline{\varepsilon}$ to get

$$T_2(\delta) < P_1(\delta) \frac{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) \varepsilon h(\varepsilon) d\varepsilon}{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon)) h(\varepsilon) d\varepsilon}.$$

Now, because the function $\Theta_2(P_2(\delta\varepsilon))$ is decreasing in ε, the likelihood ratio $\frac{\Theta_2(P_2(\delta\varepsilon))h(\varepsilon)}{h(\varepsilon)}$ is monotone decreasing. It follows that the generalized density

$\Theta_2(P_2(\delta\varepsilon))h(\varepsilon)$ is first order stochastically dominated by the density $h(\varepsilon)$. This implies that the preceding expression is smaller than

$$P_1(\delta)\frac{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \varepsilon h(\varepsilon)d\varepsilon}{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} h(\varepsilon)d\varepsilon} = P_1(\delta),$$

since the expected value of ε is 1. This proves $T_2(\delta) < P_1(\delta)$.

Now express the average transaction price for type 2 assets as a weighted average of the average transaction price conditional on δ:

$$(13) \quad T_2 = \frac{\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon\right) T_2(\delta)g_1(\delta)d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon\right) g_1(\delta)d\delta}$$
$$< \frac{\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon\right) P_1(\delta)g_1(\delta)d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon\right) g_1(\delta)d\delta},$$

where the inequality uses $T_2(\delta) < P_1(\delta)$. Consider how the difficulty of selling a type $\delta\varepsilon$ asset changes with δ and ε. Using the functional form for Θ in equation (13),

$$\frac{\partial \log \Theta_2(P_2(\delta\varepsilon))}{\partial\delta} = -\frac{\beta_h}{\beta_h - \lambda\beta_l}\frac{\varepsilon P_2'(\delta\varepsilon)}{P_2(\delta\varepsilon)}.$$

We prove that this is decreasing in ε, i.e. that $\Theta_2(P_2(\delta\varepsilon))$ is log-submodular in δ and ε. This is true if and only if the elasticity of P_2 is increasing. Implicitly differentiate equation (12) and use the functional form for $\Theta(p)$ in equation (13) to get

$$\frac{\delta P_a'(\delta)}{P_a(\delta)} = \left[1 + \frac{\pi_l}{\pi_l + \lambda\pi_h}\frac{P_a(\delta)}{\delta}\Theta_a(P_a(\delta))\right]^{-1}$$

This elasticity is increasing in δ since both $P_a(\delta)/\delta$ and $\Theta_a(P_a(\delta))$ are decreasing. This establishes log-submodularity of $\Theta_2(P_2(\delta\varepsilon))$. Equivalently, for any $\delta_1 < \delta_2$ and $\underline{\varepsilon} < \varepsilon$,

$$\frac{\Theta_2(P_2(\delta_1\varepsilon))}{\Theta_2(P_2(\delta_1\underline{\varepsilon}))} > \frac{\Theta_2(P_2(\delta_2\varepsilon))}{\Theta_2(P_2(\delta_2\underline{\varepsilon}))}$$

Weighting by $h(\varepsilon)$ and integrating over $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$, this implies

$$\frac{\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon}{\Theta_2(P_2(\delta\underline{\varepsilon}))}$$

is decreasing in δ. Once again, since the relevant likelihood ratio is monotone, the generalized density $\left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon\right)g_1(\delta)$ is first order stochastically dominated by the generalized density $\Theta_2(P_2(\delta\underline{\varepsilon}))g_1(\delta)$. Since $P_1(\delta)$ is increasing, this implies

$$\frac{\int_{\underline{\delta}}^{\bar{\delta}}\left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}}\Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon\right)P_1(\delta)g_1(\delta)d\delta}{\int_{\underline{\delta}}^{\bar{\delta}}\left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}}\Theta_2(P_2(\delta\varepsilon))h(\varepsilon)d\varepsilon\right)g_1(\delta)d\delta} < \frac{\int_{\underline{\delta}}^{\bar{\delta}}\Theta_2(P_2(\delta\underline{\varepsilon}))P_1(\delta)g_1(\delta)d\delta}{\int_{\underline{\delta}}^{\bar{\delta}}\Theta_2(P_2(\delta\underline{\varepsilon}))g_1(\delta)d\delta}.$$

The left hand side is bigger than T_2 by inequality (13), while $\Theta_2(P_2(\delta\underline{\varepsilon})) = \Theta_1(P_1(\delta))$ implies that the right hand side is equal to T_1.

PROOF OF PROPOSITION 6:

Since the two assets have the same support, the pricing function $P(\delta)$ and the buyer-seller ratio $\Theta(p)$ are the same as well. Implicitly differentiate $P(\delta)$ using equation (12) and the functional form for $\Theta(p)$ in equation (13) to prove the price function is convex:

$$P''(\delta) = \frac{\pi_l\beta_l\beta_h\Theta(P(\delta))P'(\delta)^2}{(\beta_h-\lambda\beta_l)\big(1-\pi_h\beta_h-\pi_l\beta_l(1-\Theta(P(\delta)))\big)P(\delta)} > 0.$$

Similarly, differentiate $\Theta(P(\delta))$ and simplify using the same equations:

$$\frac{d^2\Theta(P(\delta))}{d\delta^2} = \frac{P''(\delta)}{P(\delta)\pi_l\beta_l}\left(\frac{\beta_h(1-\pi_h\beta_h-\pi_l\beta_l)}{\beta_h-\lambda\beta_l}+1-\pi_h\beta_h-\pi_l\beta_l(1-\Theta(P(\delta)))\right) > 0.$$

Finally, implicitly differentiate $\Theta(P(\delta))P(\delta)$:

$$\frac{d^2\Theta(P(\delta))P(\delta)}{d\delta^2} = \frac{\lambda P''(\delta)(1-\pi_h\beta_h-\pi_l\beta_l)}{\pi_l(\beta_h-\lambda\beta_l)} > 0$$

Since all three of these functions are convex and G_1 second order stochastically dominates G_2, the result immediately follows from the definition of liquidity, volume, and average asking price.

PROOF OF LEMMA 2:

Differentiate equations (12) and (13) to prove that for all a and δ, $P_a(\delta)$ and

$\Theta_a(P_a(\delta))$ are decreasing functions of λ. We find that

$$\frac{\partial P_a(\delta)}{\partial\lambda} = \frac{-\pi_l P_a(\delta)}{\beta_h(\lambda\pi_h+\pi_l)^2\left(\frac{\delta}{P_a(\delta)}+\frac{\Theta_a(P_a(\delta))\pi_l}{\lambda\pi_h+\pi_l}\right)} \times\Bigg(\frac{\Theta_a(P_a(\delta))\big(\pi_l\beta_h(1-\pi_h\beta_h-\pi_l\beta_l)+\lambda(1-\pi_h\beta_h)(\pi_h\beta_h+\pi_l\beta_l)\big)}{\lambda(1-\beta_h(\pi_h+\pi_l/\lambda))} + (1-\pi_h\beta_h-\pi_l\beta_l)-\beta_l(\lambda\pi_h+\pi_l)\Theta_a(P_a(\delta))\log\Theta_a(P_a(\delta))\Bigg)<0$$

and

$$\frac{\partial\Theta_a(P_a(\delta))}{\partial\lambda} = \frac{-\Theta_a(P_a(\delta))}{\lambda(\beta_h-\lambda\beta_l)\big(1-\pi_h\beta_h-\pi_l\beta_l(1-\Theta_a(P_a(\delta)))\big)} \times\left(\frac{\pi_l\beta_h^2(1-\Theta_a(P_a(\delta)))(1-\pi_h\beta_h-\pi_l\beta_l)}{\lambda(1-\beta_h(\pi_h+\pi_l/\lambda))}-\frac{\delta\beta_l\beta_h(\pi_l+\lambda\pi_h)\log\Theta_a(P_a(\delta))}{P_a(\delta)}\right)<0.$$

It follows immediately that liquidity, volume, and average asking price are decreasing in λ.

To prove average transaction price is decreasing in λ, we use the fact that when λ is higher, Θ_a falls more for higher values of δ, and so the generalized density $\Theta_a(P_a(\delta))dG_a(\delta)$ is higher in the sense of first order stochastic dominance when λ is lower. Again directly using equations (12), and (13), we obtain

$$\frac{\partial(\Theta_a(P_a(\delta_1))/\Theta_a(P_a(\delta_2)))}{\partial\lambda} = \frac{\beta_h}{z(\beta_h-\lambda\beta_l)}\Bigg(-\frac{\beta_l\beta_h}{(\beta_h-\lambda\beta_l)^2}\log z + \frac{\pi_l}{\lambda^2(1-\beta_h(\pi_h+\pi_l/\lambda))\big(1-\pi_h\beta_h-\pi_l\beta_l(1-\theta_1)\big)\big(1-\pi_h\beta_h-\pi_l\beta_l(1-z\theta_1)\big)} \times\Big(\beta_h\theta_1(1-\pi_h\beta_h)(1-\pi_h\beta_h-\pi_l\beta_l)(1-z+z\log z) - \theta_1\big(\pi_l\beta_l^2\lambda(1-\beta_h(\pi_h+\pi_l/\lambda))(1-\theta_1)+(\beta_h-\lambda\beta_l)(1-\pi_h\beta_h)^2\big)z\log z - \lambda\beta_l(1-\beta_h(\pi_h+\pi_l/\lambda))(1-\pi_h\beta_h-\pi_l\beta_l)(1-z)\theta_1\log\theta_1\Big)\Bigg),$$

where $\theta_1 \equiv \Theta_a(P_a(\delta_1))$ and $z \equiv \Theta_a(P_a(\delta_2))/\Theta_a(P_a(\delta_1))$. If $\delta_1 < \delta_2$, $z < 1$ and $\theta_1 \leq 1$, and so one can verify that each line in this expression is nonnegative and all but the last is positive. Thus higher values of λ both reduce the price of each quality and increase the relative likelihood that low quality goods are sold, reducing the average transaction price.

PROOF OF PROPOSITION 7:

By assumption the left hand side of condition (14) is unchanged while the right hand side, $\sum_{a=1}^{A} K_a V_a$, declines at the initial value of λ. If $1 < \lambda < \beta_h/\beta_l$, this is no longer an equilibrium, while if $\lambda = 1$ it is. If it is still an equilibrium, then λ does not change. If not, λ rises to restore equilibrium, as Lemma 2 implies that a reduction in λ raises V_a for all a. Moreover, Lemma 2 implies that this reduction in λ is associated with an increase in the liquidity, volume, average asking price, and average transaction price for all other assets.

PROOF OF PROPOSITION 8:

First suppose $1 < \lambda < \beta_h/\beta_l$. A reduction in π_h and increase in π_l must cause a reduction in $\sum_{a=1}^{A} K_a \int_{\underline{\delta}_a}^{\bar{\delta}_a} \Theta_a(P_a(\delta))P_a(\delta)dG_a(\delta)$, which in turn requires an increase in λ by Lemma 2. The same Lemma then implies that both average asking and transactions prices decrease. If $\lambda = 1$, a reduction in π_h and increase in π_l may either leave λ unchanged, in which case prices are unchanged, or may cause an increase in λ and hence a reduction in prices.

PROOF OF PROPOSITION 9:

It is straightforward to prove that the expressions in the statement of the proposition describe an equilibrium. We again can use the arguments in Veronica Guerrieri and Robert Shimer (2013) to prove that this is the unique equilibrium.

The next step in the proof is based on comparative statics in $\hat{p}$:

$$\frac{\partial \log \hat{P}(\delta)}{\partial \log \hat{p}} = \frac{\pi_l \beta_h \hat{\Theta}(\hat{P}(\delta))}{\lambda\big(1 - \pi_h\beta_h - \pi_l\beta_l(1 - \hat{\Theta}(\hat{P}(\delta)))\big)} > 0.$$

In addition, it is straightforward to verify that $\hat{\Theta}(\hat{P}(\gamma\hat{\delta})) = \Theta(P(\gamma\underline{\delta}))$ for all $\gamma > 1$. Since $\hat{\Theta}(\hat{P}(\delta))$ is a decreasing function, $\hat{\delta} > \underline{\delta}$ implies $\Theta(P(\gamma\underline{\delta})) < \hat{\Theta}(\hat{P}(\gamma\underline{\delta}))$ for all γ, so liquidity of all assets is higher under the asset purchase program.

Finally, we prove that $\hat{\Theta}(\hat{P}(\delta_1))/\hat{\Theta}(\hat{P}(\delta_2))$ and $\hat{P}(\delta_1)/\hat{P}(\delta_2)$ are increasing in $\hat{p}$. Using the expressions in the statement of the proposition, we have that for any $\delta_2 > \delta_1 \geq \hat{\delta}$,

$$\begin{aligned}\frac{\hat{P}(\delta_2)}{\hat{P}(\delta_1)} &= \frac{\delta_2}{\delta_1} \cdot \frac{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\hat{\Theta}(\hat{P}(\delta_1))}{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\hat{\Theta}(\hat{P}(\delta_2))} \\ &= \frac{\delta_2}{\delta_1} \cdot \frac{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\hat{\Theta}(\hat{P}(\delta_2))(\hat{P}(\delta_2)/\hat{P}(\delta_1))^{\frac{\beta_h}{\beta_h - \beta_l\lambda}}}{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\hat{\Theta}(\hat{P}(\delta_2))},\end{aligned}$$

where the second equation uses

$$\frac{\hat{\Theta}(\hat{P}(\delta_1))}{\hat{\Theta}(\hat{P}(\delta_2))} = \left(\frac{\hat{P}(\delta_2)}{\hat{P}(\delta_1)}\right)^{\frac{\beta_h}{\beta_h-\beta_l\lambda}}.$$

Implicitly differentiating these expressions proves that

$$\frac{\partial\big(\hat{P}(\delta_2)/\hat{P}(\delta_1)\big)}{\partial\hat{\Theta}(P(\delta_2))} < 0 < \frac{\partial\big(\hat{\Theta}(\hat{P}(\delta_2))/\hat{\Theta}(\hat{P}(\delta_1))\big)}{\partial\hat{\Theta}(P(\delta_2))}.$$

Since $\hat{\Theta}(P(\delta_2))$ is increasing in $\hat{p}$, this implies $\hat{P}(\delta_2)/\hat{P}(\delta_1)$ is decreasing and $\hat{\Theta}(\hat{P}(\delta_2))/\hat{\Theta}(\hat{P}(\delta_1))$ is increasing in $\hat{p}$. In particular, $\hat{P}(\delta_2)/\hat{P}(\delta_1) < P(\delta_2)/P(\delta_1)$ and $\hat{\Theta}(\hat{P}(\delta_2))/\hat{\Theta}(\hat{P}(\delta_1)) > \Theta(P(\delta_2))/\Theta(P(\delta_1))$ for all $\delta_2 > \delta_1 \geq \hat{\delta}$.

PROOF OF PROPOSITION 10:

Throughout this proof, let $\hat{G}(\delta) \equiv \frac{G(\delta)-G(\hat{\delta})}{1-G(\hat{\delta})}$ denote the quality distribution of the asset in the private market, with associated density $\hat{g}$.

Average Asking Price

By definition,

$$\hat{A} = \int_{\hat{\delta}}^{\bar{\delta}} \hat{P}(\delta)\hat{g}(\delta)d\delta > \int_{\hat{\delta}}^{\bar{\delta}} P(\delta)\hat{g}(\delta)d\delta > \int_{\underline{\delta}}^{\bar{\delta}} P(\delta)g(\delta)d\delta = A,$$

where the first inequality uses the fact that for every quality δ, $\hat{P}(\delta) > P(\delta)$ and the second uses the fact that $\hat{G}(\delta)$ first order stochastically dominates $G(\delta)$ and $P(\delta)$ is increasing.

Average Transaction Price

Let $D(p)$ and $\hat{D}(p)$ denote the inverse of $P(\delta)$ and $\hat{P}(\delta)$ respectively. Then

$$\hat{T} = \frac{\int_{\hat{p}}^{\hat{P}(\bar{\delta})} \hat{\Theta}(p)p\, dG(\hat{D}(p))}{\int_{\hat{p}}^{\hat{P}(\bar{\delta})} \hat{\Theta}(p)\, dG(\hat{D}(p))} > \frac{\int_{\hat{p}}^{P(\bar{\delta})} \Theta(p)p\, dG(D(p))}{\int_{\hat{p}}^{P(\bar{\delta})} \Theta(p)\, dG(D(p))} > T.$$

The first inequality uses the fact that $\hat{P}(\delta) > P(\delta)$ for all $\delta > \hat{\delta}$ implies $\hat{D}(p) < D(p)$ for all $p > \hat{p}$, and so $G(\hat{D}(p)) < G(D(p))$, i.e. the first distribution first order stochastically dominates the second. Since $\hat{\Theta}(p) \propto \Theta(p)$, the generalized density $\hat{\Theta}(p)dG(\hat{D}(p))$ therefore first order stochastically dominates $\Theta(p)dG(D(p))$. The second inequality holds because T is a weighted average of the expected price conditional on $p > \hat{p}$ and a smaller number, the expected price conditional on $p \in [\underline{p}, \hat{p}]$.

Liquidity

By definition,

$$\hat{L} = \pi_l \int_{\hat{\delta}}^{\infty} \hat{\Theta}(\hat{P}(\delta))\hat{g}(\delta)d\delta = \pi_l \int_{1}^{\infty} \Theta(P(\gamma\underline{\delta}))\hat{g}(\gamma\hat{\delta})d\gamma,$$

since $\hat{\Theta}(\hat{P}(\gamma\hat{\delta})) = \Theta(P(\gamma\underline{\delta}))$. Next, the log concavity condition is equivalent to $\frac{\delta G'(\delta)}{1-G(\delta)}$ nondecreasing. This in turn holds if and only if $(G(\gamma\delta) - G(\delta))/(1 - G(\delta))$ is nondecreasing in δ for all γ. Therefore $\hat{G}(\gamma\hat{\delta}) \geq G(\gamma\underline{\delta})$ for all $\gamma > 1$, so the distribution relevant for $\hat{L}$ is first order stochastically dominated by the distribution relevant for L. Since Θ is decreasing, this implies

$$\int_{1}^{\infty} \Theta(P(\gamma\underline{\delta}))\hat{g}(\gamma\hat{\delta})d\gamma \geq \int_{1}^{\infty} \Theta(P(\gamma\underline{\delta}))g(\gamma\underline{\delta})d\gamma) = L.$$

This proves $\hat{L} > L$.

Volume

This follows immediately from the increase in liquidity and average transaction price.

PROOF OF PROPOSITION 11:

Characterization

The pricing equation (16) is a natural extension of equation (12), obtained by combining the analogs of equations (7)–(9). The optimal sale price satisfies the seller's first order condition for solving

$$\max_p \big(\min\{\hat{\Theta}(p), 1\}(p + \sigma(p)) + (1 - \min\{\hat{\Theta}(p), 1\})\beta_l \bar{v}(\delta) \big).$$

This gives equation (17). With an arbitrary subsidy program, we cannot solve explicitly for $\hat{\Theta}(p)$, however.

Prices

Since $\Theta(P(\underline{\delta})) = \hat{\Theta}(\hat{P}(\underline{\delta})) = 1$, equations (12) and (16) imply $\hat{P}(\underline{\delta}) > P(\underline{\delta})$. Equation (13) implies Θ is decreasing and so $\Theta(\hat{P}(\underline{\delta})) < 1$. Next equations (13) and (17) imply that $\hat{\Theta}'(p)/\hat{\Theta}(p) > \Theta'(p)/\Theta(p)$ for all p and so $\hat{\Theta}(p) > \Theta(p)$ for all $p \geq \hat{P}(\underline{\delta})$.

Now suppose to find a contradiction that $\hat{P}(\delta) \leq P(\delta)$ for some $\delta \geq \underline{\delta}$. Then since $\hat{\Theta}(p) > \Theta(p)$ for all $p \geq \hat{P}(\underline{\delta})$ and both are decreasing functions, $\hat{\Theta}(\hat{P}(\delta)) > \Theta(P(\delta))$. Since $\sigma(\hat{P}(\delta)) \geq 0$,

$$\frac{\delta(\pi_l + \lambda\pi_h) + \pi_l\hat{\Theta}(\hat{P}(\delta))\sigma(\hat{P}(\delta))}{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\hat{\Theta}(\hat{P}(\delta))} > \frac{\delta(\pi_l + \lambda\pi_h)}{\lambda(1 - \pi_l\beta_l - \pi_h\beta_h) - \pi_l(\beta_h - \beta_l\lambda)\Theta(P(\delta))}.$$

Using equations (12) and (16), this implies $\hat{P}(\delta) > P(\delta)$, a contradiction.

Sale Probabilities

Totally differentiate equation (16) with respect to δ. Then use equation (17) to prove that

$$\hat{P}'(\delta) = \frac{\beta_h(\lambda\pi_h + \pi_l)}{\lambda\big(1 - \beta_h\pi_h - \beta_l\pi_l(1 - \hat{\Theta}(\hat{P}(\delta)))\big)},$$

a function of the subsidy only indirectly through $\hat{\Theta}(\hat{P}(\delta))$. Then using equation (17), we obtain

$$\frac{\partial \log \hat{\Theta}(\hat{P}(\delta))}{\partial \delta} = \frac{-\beta_h^2(\lambda\pi_h + \pi_l)(1 + \sigma'(\hat{P}(\delta)))}{\lambda\big(1 - \beta_h\pi_h - \beta_l\pi_l(1 - \hat{\Theta}(\hat{P}(\delta)))\big)\big(\hat{P}(\delta)(\beta_h - \lambda\beta_l) + \beta_h\sigma(\hat{P}(\delta))\big)},$$

and similarly for $\frac{\partial \log \Theta(P(\delta))}{\partial \delta}$. Since $\hat{\Theta}(\hat{P}(\underline{\delta})) = \Theta(P(\underline{\delta})) = 1$, $\hat{P}(\underline{\delta}) > P(\underline{\delta})$, and $\sigma(\hat{P}(\underline{\delta})) > 0 > \sigma'(\hat{P}(\underline{\delta}))$, this proves $\frac{\partial \log \hat{\Theta}(\hat{P}(\underline{\delta}))}{\partial \delta} > \frac{\partial \log \Theta(P(\underline{\delta}))}{\partial \delta}$. The same logic implies that at any $\delta > \underline{\delta}$, if $\hat{\Theta}(\hat{P}(\delta)) \geq \Theta(P(\delta))$, then $\frac{\partial \log \hat{\Theta}(\hat{P}(\delta))}{\partial \delta} > \frac{\partial \log \Theta(P(\delta))}{\partial \delta}$. This implies $\hat{\Theta}(\hat{P}(\delta)) > \Theta(P(\delta))$ for all $\delta > \underline{\delta}$, as we prove in the next paragraph.

First, note that $\hat{\Theta}(\hat{P}(\underline{\delta})) = \Theta(P(\underline{\delta}))$ and $\frac{\partial \log \hat{\Theta}(\hat{P}(\underline{\delta}))}{\partial \delta} > \frac{\partial \log \Theta(P(\underline{\delta}))}{\partial \delta}$ implies that there exists an $\varepsilon > 0$ such that for all $\delta \in (\underline{\delta}, \underline{\delta} + \varepsilon)$, $\hat{\Theta}(\hat{P}(\delta)) > \Theta(P(\delta))$. Fix any $\delta_1 \in (\underline{\delta}, \underline{\delta} + \varepsilon)$. Now suppose there is a $\delta > \underline{\delta}$ with $\hat{\Theta}(\hat{P}(\delta)) \leq \Theta(P(\delta))$. Let δ_2 denote the smallest such δ. Then

$$\log \hat{\Theta}(\hat{P}(\delta_2)) - \log \hat{\Theta}(\hat{P}(\delta_1)) = \int_{\delta_1}^{\delta_2} \frac{\partial \log \hat{\Theta}(\hat{P}(\delta))}{\partial \delta} d\delta > \int_{\delta_1}^{\delta_2} \frac{\partial \log \Theta(P(\delta))}{\partial \delta} d\delta = \log \Theta(P(\delta_2)) - \log \Theta(P(\delta_1)),$$

where the inequality uses the result from the previous paragraph that $\frac{\partial \log \hat{\Theta}(\hat{P}(\delta))}{\partial \delta} >$

$\frac{\partial \log \Theta(P(\delta))}{\partial \delta}$ for all $\delta \in [\delta_1, \delta_2]$ since $\hat{\Theta}(\hat{P}(\delta)) \geq \Theta(P(\delta))$ by construction. But since $\hat{\Theta}(\hat{P}(\delta_1)) > \Theta(P(\delta_1))$, this implies $\hat{\Theta}(\hat{P}(\delta_2)) > \Theta(P(\delta_2))$, a contradiction. This proves $\hat{\Theta}(\hat{P}(\delta)) > \Theta(P(\delta))$ for all $\delta > \underline{\delta}$ and so, using the result from the previous paragraph, $\frac{\partial \log \hat{\Theta}(\hat{P}(\delta))}{\partial \delta} > \frac{\partial \log \Theta(P(\delta))}{\partial \delta}$, which is equivalent to $\hat{\Theta}(\hat{P}(\delta))/\Theta(P(\delta))$ increasing.

PROOF OF PROPOSITION 12:

The statement about liquidity, volume, and average asking price follow trivially from the results in Proposition 11. The same proposition shows also that the likelihood ratio $\hat{\Theta}(\hat{P}(\delta))/\Theta(P(\delta))$ is increasing and so the generalized density $\hat{\Theta}(\hat{P}(\delta))g(\delta)$ first order stochastically dominates $\Theta(P(\delta))g(\delta)$. This implies that the average transaction price also increases with the subsidy program.