Misallocation and Growth

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Online Appendix

A. Proof of Proposition 1

We begin by showing that $\alpha_t'(x)$ exists; if $H_t'(x)$ and $\Phi_t'(y)$ exist for all (t,x,y) in their domains, then

$$\alpha_t'(x) = \frac{H_t'(x)}{\Phi_t'(\alpha_t[x])}. \tag{40}$$

exist for all (t,x). Now, F has density for each y, so does Φ_t as defined in (6). Since Φ_t has a density, α_t' exists if H_t' does. Moreover, the RHS of (11) preserves differentiability, so that if H_0 has a density, so does H_t. Therefore α' exists for all (t,x). Then $\alpha_t(x)$ must solve for s the first-order condition to the problem in (12):

$$\int [f(x,y) + \beta\pi_{t+1}(\phi[x,y])] \frac{\partial}{\partial s} d\tau_t(y|s) = V_t'(s). \tag{41}$$

The unknowns (π, w, V) must satisfy (8), (12) and (41). They must also satisfy the SOC

$$\int [f(x,y) + \beta\pi_{t+1}(\phi[x,y])] \frac{\partial^2}{\partial s^2} d\tau_t(y|s) \le V_t''(s). \tag{42}$$

The relation in (13) is a map from a set of function w' into itself. If w is differentiable so is V, which is defined in (8). The FOC reads

$$V_t'(\alpha_t[x]) = \int [f(x,y) + \beta\pi_{t+1}(\phi[x,y])] \frac{\partial}{\partial s} d\tau_t(y|\alpha_t[x]). \tag{43}$$

Now (8) implies that

$$\begin{aligned} V_t'(\alpha_t[x]) &= \left(w_t'(x) + \beta \int \pi_{t+1}'(\phi[x,y]) \phi_x(x,y)\, d\tau_t(y|\alpha_t[x]) \right) \left.\frac{d\alpha_t^{-1}}{ds}\right|_{s=\alpha_t(x)} \\ &\quad + \beta \int \pi_{t+1}(\phi[x,y]) \frac{\partial}{\partial s} d\tau_t(y|\alpha_t[x]). \end{aligned} \tag{44}$$

Combining (43) and (44), the term $\beta \int \pi_{t+1}(\phi[x,y]) \frac{\partial}{\partial s} d\tau_t(y|\alpha_t[x])$ cancels, leaving

$$\left(w_t'(x) + \beta \int \pi_{t+1}'(\phi[x,y]) \phi_x(x,y)\, d\tau_t(y|\alpha_t[x]) \right) \left.\frac{d\alpha_t^{-1}}{ds}\right|_{s=\alpha_t(x)} = \int f(x,y) \frac{\partial}{\partial s} d\tau_t(y|\alpha_t[x]).$$

Now $\left.\frac{d\alpha_t^{-1}}{ds}\right|_{s=\alpha_t(x)} = \frac{1}{\alpha_t'(x)}$, implying that

$$w_t'(x) = \alpha_t'(x) \int f(x,y) \frac{\partial}{\partial s} d\tau_t(y|\alpha_t[x]) - \beta \int \pi_{t+1}'(\phi[x,y]) \phi_x(x,y) d\tau_t(y|\alpha_t[x]). \tag{45}$$

Finally using the envelope theorem in (9) and updating to $t+1$ yields

$$\pi_{t+1}'(x) = \int f_x(x,y') d\tau_{t+1}(y'|\alpha_{t+1}(x)) + \alpha_{t+1}'(x) \int f(x,y') \frac{\partial}{\partial s'} d\tau_{t+1}(y'|s'=\alpha_{t+1}(x)) - w_{t+1}'(x)$$

(using y', s' for the next period y, s), so that replacing x by $\phi(x,y)$

$$\begin{aligned} \pi_{t+1}'(\phi[x,y]) &= \int f_x(\phi[x,y], y') d\tau_{t+1}(y'|s'=\alpha_{t+1}(\phi[x,y])) \\ &\quad + \alpha_{t+1}'(\phi[x,y]) \int f(\phi[x,y], y') \frac{\partial}{\partial s'} d\tau_{t+1}(y'|\alpha_{t+1}[\phi(x,y)]) \\ &\quad - w_{t+1}'(\phi[x,y]). \end{aligned}$$

Substituting for π_{t+1} into (45), it becomes

$$\begin{aligned} w_t'(x) &= \alpha_t'(x) \int f(x,y) \frac{\partial}{\partial s} d\tau_t(y|\alpha_t[x]) \\ &\quad - \beta \int \phi_x(x,y) \int f_x(\phi[x,y], y') d\tau_{t+1}(y'|\alpha_{t+1}(\phi[x,y])) d\tau_t(y|\alpha_t[x]) \\ &\quad - \beta \int \alpha_{t+1}'(\phi[x,y]) \phi_x(x,y) \int f(\phi[x,y], y') \frac{\partial}{\partial s'} d\tau_{t+1}(y'|\alpha_{t+1}[\phi(x,y)]) d\tau_t(y|\alpha_t[x]) \\ &\quad + \beta \int w_{t+1}'(\phi[x,y]) \phi_x(x,y) d\tau_t(y|\alpha_t[x]). \end{aligned}$$

This is the Bellman equation to be solved for $w_t'(x)$. Finally, the standard contraction mapping argument for continuous functions applies when U is bounded and (15) holds. That completes the proof.

B. Proof of Proposition 2

Suppose $y = \alpha^* x$ for some $\alpha^* > 0$. But then (3)) implies that $\bar{y} = \alpha^* \bar{x} = b\bar{x}\bar{\varepsilon} \Rightarrow \alpha^* = b\bar{\varepsilon}$, i.e., (17). Since ϕ is linear homogenous, $\Gamma = \frac{1}{x}\phi(x, \alpha^* x) = \phi(1, b\bar{\varepsilon})$, i.e., (16) and (18). Next, if (19) and (20) are correct, then lifetime utility in (8) simplifies to

$$V(y) = \left(\omega + \beta(1-\omega)\frac{\Gamma}{b\bar{\varepsilon}}\right) f\left(\frac{1}{b\bar{\varepsilon}}, 1\right) y. \tag{46}$$

It remains to be shown that in the Bellman equation (8), the solution for y of the FOC is then given by (17), and that the firm's objective is strictly concave in y. With the substitution from (20) and (46), (8) reads

$$(1-\omega)\,\bar{f}x=\max_{y}\left\{f\left(x,y\right)-V\left(y\right)+\beta\left(1-\omega\right)\bar{f}\phi\left(x,y\right)\right\},\tag{47}$$

where $\bar{f}=f\left(1,b\bar{\varepsilon}\right)$. The FOC is

$$\begin{aligned}0 &= f_y\left(1,\frac{y}{x}\right)-V_y+\beta\left(1-\omega\right)\bar{f}\phi_y\left(1,\frac{y}{x}\right)\\ &= f_y\left(1,b\bar{\varepsilon}\right)-\omega f\left(\frac{1}{b\bar{\varepsilon}},1\right)-\beta\left(1-\omega\right)\bar{f}\phi\left(\frac{1}{b\bar{\varepsilon}},1\right)+\beta\left(1-\omega\right)\bar{f}\phi_y\left(1,b\bar{\varepsilon}\right),\quad(48)\end{aligned}$$

where (48) after substituting for the assignment in (17). For any $\omega\in[0,1]$, the SOC holds in that the RHS of (47) is concave in y because V is linear in y, while and f and ϕ are strictly concave in y. Hence, (42) holds for any as long as $\omega\le 1$. Evaluating the RHS of (47) at $y=b\bar{\varepsilon}x$, and noting that $f\left(\frac{1}{b\bar{\varepsilon}},1\right)y=f\left(\frac{1}{b\bar{\varepsilon}},1\right)b\bar{\varepsilon}x=f\left(1,b\bar{\varepsilon}\right)x$ and that $\phi\left(\frac{1}{b\bar{\varepsilon}},1\right)y=\phi\left(1,b\bar{\varepsilon}\right)x=\Gamma x$, it reads

$$(1-\omega)\,\bar{f}x=\bar{f}\bar{x}-V\left(b\bar{\varepsilon}x\right)+\beta\left(1-\omega\right)\bar{f}\Gamma x=\bar{f}x-\omega\bar{f}x-\beta\left(1-\omega\right)\bar{f}\Gamma x+\beta\left(1-\omega\right)\bar{f}\Gamma x=\left(1-\omega\right)\bar{f}x,$$

i.e., (8) evidently holds for all (x,ω). Therefore we can just solve the FOC (48) for the one unknown, ω. Noting that $\phi\left(\frac{1}{b\bar{\varepsilon}},1\right)=\frac{1}{b\bar{\varepsilon}}\Gamma$ and that $f\left(\frac{1}{b\bar{\varepsilon}},1\right)=\frac{1}{b\bar{\varepsilon}}\bar{f}$, we get

$$\omega=\frac{f_y\left(1,b\bar{\varepsilon}\right)+\beta\bar{f}\phi_y\left(1,b\bar{\varepsilon}\right)-\frac{1}{b\bar{\varepsilon}}\beta\bar{f}\Gamma}{\bar{f}\frac{1}{b\bar{\varepsilon}}+\beta\bar{f}\phi_y\left(1,b\bar{\varepsilon}\right)-\frac{1}{b\bar{\varepsilon}}\beta\bar{f}\Gamma}=\frac{\frac{f_y(1,b\bar{\varepsilon})}{f(1,b\bar{\varepsilon})}+\beta\phi_y\left(1,b\bar{\varepsilon}\right)-\frac{1}{b\bar{\varepsilon}}\beta\Gamma}{\frac{1}{b\bar{\varepsilon}}+\beta\phi_y\left(1,b\bar{\varepsilon}\right)-\frac{1}{b\bar{\varepsilon}}\beta\Gamma}.$$

But since ϕ is linear homogeneous, $\Gamma=\phi\left(1,b\bar{\varepsilon}\right)=\phi_x+\phi_y b\bar{\varepsilon}$ so that $\beta\phi_y\left(1,b\bar{\varepsilon}\right)-\frac{1}{b\bar{\varepsilon}}\beta\Gamma=-\beta\frac{1}{b\bar{\varepsilon}}\phi_x\left(1,b\bar{\varepsilon}\right)$, and substituting into the above equation we have (19). Next, we show that $\omega<1$: By linear homogeneity of f and Euler's theorem, in (19), $\frac{f_y b\bar{\varepsilon}}{f}=\frac{f_y b\bar{\varepsilon}}{f_x+f_y b\bar{\varepsilon}}<1\Rightarrow\omega<1$.

Finally, since we derived ω directly, it remains to be shown that (13) and (14) are consistent with the claim of Proposition 2.

C. Proof of Proposition 3

Eq. (23) implies that $\int f\left(x,y\right)\frac{\partial}{\partial s}d\tau_t\left(y|s=\alpha_t\left[x\right]\right)$ and $\int f\left(\phi\left[x,y\right],y'\right)\frac{\partial}{\partial s'}d\tau_{t+1}\left(y'|s=\alpha_{t+1}\left[\phi\left(x,y\right)\right]\right)$ both converge to zero. Then the first and third terms on the RHS of (14) converge to zero, and only the second term stays positive. Therefore (13) reads

$$w_t'\left(x\right)=\beta\int\left(w_{t+1}'\left(\phi\left[x,y\right]\right)-\int f_x\left(\phi\left[x,y\right],y'\right)dG_{t+1}\left(y'\right)\right)\phi_x\left(x,y\right)dG_t\left(y\right).$$

In the Cobb-Douglas case (21), it reads

$$\begin{aligned}w_t'\left(x\right) &= \beta\left(1-\theta\right)A\int\left(w_{t+1}'\left(Ax^{1-\theta}y^{\theta}\right)-\left(1-\rho\right)\int\left(\frac{y'}{Ax^{1-\theta}y^{\theta}}\right)^{\rho}dG_{t+1}\left(y'\right)\right)\left(\frac{y}{x}\right)^{\theta}dG_t\left(y\right),\\ &= \beta\left(1-\theta\right)A\int\left(w_{t+1}'\left(Ax^{1-\theta}y^{\theta}\right)x^{-\theta}-\left(1-\rho\right)x^{-\rho(1-\theta)-\theta}\int\left(\frac{y'}{Ay^{\theta}}\right)^{\rho}dG_{t+1}\left(y'\right)\right)y^{\theta}dG_t\left(y\right).\end{aligned}$$

If $w_t'(x)$ is of the form $B_t x^{-C}$, it must be that $C = (\theta + \rho(1-\theta))$, and that $C = (1-\theta)(\theta + \rho(1-\theta))$. But the equality

$$(1-\theta)(\theta + \rho(1-\theta)) + \theta = \rho(1-\theta) + \theta$$

implies that $\rho = 1$. Then let

$$w_t'(x;\rho) = \frac{1}{1-\rho} w_t'(x),$$

so that

$$w_t'(x;\rho) = \beta(1-\theta) A \int \left(w_{t+1}'\left(Ax^{1-\theta}y^{\theta};\rho\right) x^{-\theta} - x^{-\rho(1-\theta)-\theta} \int \left(\frac{y'}{Ay^{\theta}}\right)^{\rho} dG_{t+1}(y') \right) y^{\theta} dG_t(y).$$

As $\rho \to 1$, we have

$$w_t'(x;\rho) = \beta(1-\theta) A \int \left(w_{t+1}'\left(Ax^{1-\theta}y^{\theta};\rho\right) x^{-\theta} - x^{-1} \int \left(\frac{y'}{Ay^{\theta}}\right)^{\rho} dG_{t+1}(y') \right) y^{\theta} dG_t(y).$$

Then the solution is

$$\lim_{\rho\to 1} w_t'(x;\rho) = -\frac{B_t}{x},$$

where

$$\begin{aligned} B_t &= \lim_{\rho\to 1} \beta(1-\theta) A \int \left(B_{t+1} A^{-\rho} y^{-\rho\theta} + \int \left(\frac{y'}{Ay^{\theta}}\right)^{\rho} dG_{t+1}(y') \right) y^{\theta} dG_t(y) \\ &= \beta(1-\theta) \left(B_{t+1} + \int y' dG_{t+1}(y') \right) \quad \text{(because } A \text{ and } y \text{ cancel)} \\ &= \sum_{j=1}^{\infty} [\beta(1-\theta)]^j E(y_{t+j}). \end{aligned}$$

But since $\lim_{\rho\to 1} x^{1-\rho} y^{\rho} = y$, $\lim_{\rho\to 1} q = y$, the second equality in (27) follows.

D. Proof of Proposition 4

With ϕ as in (21),

$$\Pr(x_{t+1} \le x' \mid x_t = x) = \Pr\left(y \le \left(\frac{x'}{Ax^{1-\theta}}\right)^{1/\theta} \mid s_t = \alpha_t(x) \right) = \tau\left(\left(\frac{x'}{Ax^{1-\theta}}\right)^{1/\theta} \mid \Phi_t^{-1}(H_t(x)) \right),$$

and therefore the law of motion for H in (11) becomes

$$H_{t+1}(x') = \int \tau\left(\left(\frac{x'}{Ax^{1-\theta}}\right)^{1/\theta} \mid \Phi_t^{-1}(H_t(x)) \right) dH_t(x). \tag{49}$$

Now in (6) and (7), Φ is log normal and therefore if H_t is log normal, $\alpha_t(x)$ is log linear with $\hat{s}$ given in (29). Thus, since τ is log normal, if H_t is log normal, so is H_{t+1}. Thus, if H_0 is log-normal, H_t remains log-normal for all t, and is fully described by its first two moments.

Deriving (30).—To keep notation as simple as possible, we shall omit the t subscript from $\hat{x}, \hat{y}, \hat{s}, \mu_{\hat{y}}$, and $\mu_{\hat{s}}$. Letting $\zeta \sim N(0,1)$,[11]

$$\begin{aligned}\hat{y} &= \left(1-r^2\right)\mu_{\hat{y}} + r^2\hat{s} + \sqrt{1-r^2}\sigma_{\hat{\varepsilon}}\zeta \\ &= \left(1-r^2\right)\mu_{\hat{y}} + r^2\left(\mu_{\hat{s}} + \frac{\sigma_{\hat{s}}}{\sigma_t}\left(\hat{x}-\mu_t\right)\right) + \sqrt{1-r^2}\sigma_{\hat{\varepsilon}}\zeta \\ &= \mu_{\hat{y}} + r\frac{\sigma_{\hat{\varepsilon}}}{\sigma_t}\left(\hat{x}-\mu_t\right) + \sqrt{1-r^2}\sigma_{\hat{\varepsilon}}\zeta, \end{aligned} \tag{50}$$

because $\mu_{\hat{y}} = \mu_{\hat{s}}$, $\sigma_{\hat{y}} = \sigma_{\hat{\varepsilon}}$, and (since $r^2\frac{\sigma_{\hat{s}}}{\sigma_t} = r^2\frac{\sqrt{\sigma_{\hat{y}}^2+\sigma_\eta^2}}{\sigma_t} = r^2\frac{\sigma_{\hat{y}}}{\sigma_t}\sqrt{\frac{\sigma_{\hat{y}}^2+\sigma_\eta^2}{\sigma_{\hat{y}}^2}} = r\frac{\sigma_{\hat{y}}}{\sigma_t} = r\frac{\sigma_{\hat{\varepsilon}}}{\sigma_t}$), because

$$r^2\frac{\sigma_{\hat{s}}}{\sigma_t} = r\frac{\sigma_{\hat{\varepsilon}}}{\sigma_t}. \tag{51}$$

This proves (38) Since $\phi(x,y) = Ax^{1-\theta}y^\theta$, $\hat{x}$ evolves as follows:

$$\hat{x}' = \hat{A} + (1-\theta)\hat{x} + \theta\hat{y} = \hat{A} + (1-\theta)\hat{x} + \theta\left(\mu_{\hat{y}} + r\frac{\sigma_{\hat{\varepsilon}}}{\sigma_t}\left(\hat{x}-\mu_t\right) + \sqrt{1-r^2}\sigma_{\hat{\varepsilon}}\zeta\right). \tag{52}$$

We integrate both sides over H, to get $\mu_{t+1} = \hat{A} + \theta\mu_{\hat{y}} + (1-\theta)\mu_t$. In (3) we note that $y = b\bar{x}_t\varepsilon = b\varepsilon\exp\left(\mu_t + \frac{1}{2}\sigma_t^2\right)$, so that

$$\begin{aligned}\mu_{t+1} &= \hat{A} + \theta\mu_{\hat{y}} + (1-\theta)\mu_t = \hat{A} + \theta\left(\hat{b} + \mu_{\hat{\varepsilon}} + \mu_t + \frac{1}{2}\sigma_t^2\right) + (1-\theta)\mu_t \\ &= \mu_t + \hat{A} + \theta\left(\hat{b} + \mu_{\hat{\varepsilon}} + \frac{1}{2}\sigma_t^2\right), \text{ i.e., (30).}\end{aligned} \tag{53}$$

Deriving (31).—The evolution of σ_t does not depend on μ_t. Taking the variance of both sides of (52) we get the difference equation for σ_t^2:

$$\begin{aligned}\sigma_{t+1}^2 &= \left(1-\theta+\theta r\frac{\sigma_{\hat{\varepsilon}}}{\sigma_t}\right)^2\sigma_t^2 + \left(1-r^2\right)\theta^2\sigma_{\hat{\varepsilon}}^2 \\ &= (1-\theta)^2\sigma_t^2 + 2\theta(1-\theta)r\sigma_{\hat{\varepsilon}}\sigma_t + \theta^2\sigma_{\hat{\varepsilon}}^2,\end{aligned}$$

[11] The distribution

$$p(\hat{y}\mid\hat{s}) = \frac{p(\hat{s}\mid\hat{y})\,p(\hat{y})}{p(\hat{s})} \qquad \text{has variance} \qquad \frac{\sigma_{\hat{y}}^2\sigma_\eta^2}{\sigma_{\hat{y}}^2+\sigma_\eta^2} = \left(1-r^2\right)\sigma_{\hat{y}}^2 = r^2\sigma_\eta^2.$$

Therefore this is the variance of $\hat{y}$ conditional on s.

and (31) follows.

*Deriving (*32).—That BGP value of σ_t solves the equation $\sigma = \chi_2(\sigma)$ which, since $1-(1-\theta)^2 = \theta(2-\theta)$ is the quadratic.

$$0 = (2-\theta)\sigma^2 - 2(1-\theta) r\sigma_{\hat{\varepsilon}}\sigma - \theta\sigma_{\hat{\varepsilon}}^2.$$

The unique positive solution for σ (after rearrangement and then division by θ) is

$$\sigma(r) = \frac{\sigma_{\hat{\varepsilon}}}{2-\theta}\left((1-\theta) r + \sqrt{(1-\theta)^2 r^2 + (2-\theta)\theta}\right).$$

Therefore, for $\theta \in (0,1]$, we have (32). This completes the proof of Proposition 4

E. Deriving (34) and (35)

Conditional on (x, s), output is

$$\int f(x,y) d\tau_t(y|s) = \exp\left\{\rho(1-r^2)\mu_{\hat{y}} + \frac{\rho^2}{2} r^2 \sigma_\eta^2\right\} x^{1-\rho} s^{\rho r^2}.$$

Now (29) implies

$$\alpha_t(x) = \exp\left(\mu_{\hat{s}} - \frac{\sigma_{\hat{s}}}{\sigma_t}\mu_t\right) x^{\sigma_{\hat{s}}/\sigma_t}.$$

Conditional on x alone, output is

$$\exp\left\{\rho(1-r^2)\mu_{\hat{y}} + \frac{\rho^2}{2} r^2 \sigma_\eta^2\right\} x^{1-\rho+\rho r^2 \sigma_{\hat{s}}/\sigma_t} \equiv C x^D,$$

where $1-\rho+\rho r^2 \frac{\sigma_{\hat{s}}}{\sigma_t}$

$$C \equiv \exp\left\{\rho(1-r^2)\mu_{\hat{y}} + \frac{\rho^2}{2} r^2 \sigma_\eta^2\right\}, \quad \text{and} \quad D \equiv 1-\rho+\rho r^2 \frac{\sigma_{\hat{s}}}{\sigma_t}.$$

Therefore aggregate output in (34) is gotten by integrating out $\ln x \,\tilde{}\, N(\mu_t, \sigma_t^2)$.

F. Computation of the solution for $w(x)$ to (13)

We want to solve the following integral

$$\begin{aligned} w_t'(x) &= \frac{\partial}{\partial s}\int f(x,y) d\tau_t(y|s)\bigg|_{s=\alpha_t(x)} \alpha_t'(x) & (54)\\ &- \beta\frac{\partial}{\partial x}\int_{y_t}\int_{y_{t+1}} f(\phi[x,y_t], y_{t+1}) d\tau_{t+1}(y_{t+1}|\alpha_{t+1}(\phi[x,y_t])) d\tau_t(y_t|\alpha_t(x)) & (55)\\ &+ \beta\int w_{t+1}'(\phi[x,y_t])\frac{\partial\phi[x,y_t]}{\partial x} d\tau_t(y_t|s)\bigg|_{s=\alpha_t(x)}. & (56)\end{aligned}$$

We will show next how to compute each line separately. First some equations we shall use the following equations

$$\begin{aligned}\mu_{\hat{y}} &= \hat{b}+\mu_t+\frac{\sigma_t^2}{2}\\ \sigma_{\hat{y}}^2 &= \sigma_{\hat{\varepsilon}}^2\\ \mu_{\hat{s}} &= \mu_{\hat{y}}\\ \sigma_{\hat{s}}^2 &= \sigma_{\hat{y}}^2+\sigma_\eta^2\\ \hat{\alpha}_t(x) &= \mu_{\hat{s}}+\frac{\sigma_{\hat{s}}}{\sigma_t}(\hat{x}-\mu_t)\end{aligned}$$

This last equation then implies

$$\begin{aligned}\alpha_t(x) &= \exp\{\mu_{\hat{s}}+\frac{\sigma_{\hat{s}}}{\sigma_t}(\hat{x}-\mu_t)\}, \quad \text{so that}\\ \alpha_t'(x) &= \frac{1}{x}\frac{\sigma_{\hat{s}}}{\sigma_t}\alpha_t(x).\end{aligned}$$

Finally, Bayes rule implies

$$\hat{y}|\hat{s}\sim N\left((1-r^2)\mu_{\hat{y}}+r^2\hat{s}, r^2\sigma_\eta^2\right).$$

Since the posterior of y is log-normal, we have that for any constant κ

$$\begin{aligned}\int y^\kappa d\tau_t(y|s) &= \exp\left\{\kappa[(1-r^2)\mu_{\hat{y}}+r^2\hat{s}]+\frac{\kappa^2}{2}r^2\sigma_\eta^2\right\}\\ &= \exp\left\{\kappa(1-r^2)\mu_{\hat{y}}+\frac{\kappa^2}{2}r^2\sigma_\eta^2\right\}s^{\kappa r^2}. \qquad (57)\end{aligned}$$

For the first term, using that $f(x,y)=x^{1-\rho}y^\rho$ note that

$$\begin{aligned}\int f(x,y)d\tau_t(y|s) &= x^{1-\rho}\int y^\rho d\tau_t(y|s)\\ &= x^{1-\rho}\exp\left\{\rho(1-r^2)\mu_{\hat{y}}+\frac{\rho^2}{2}r^2\sigma_\eta^2\right\}s^{\rho r^2},\end{aligned}$$

where the second line uses equation (57) with $\kappa=\rho$. Then

$$\frac{\partial}{\partial s}\int f(x,y)d\tau_t(y|s)=\rho r^2 D_t s^{\rho r^2-1},$$

where $D_t=\exp\left\{\rho(1-r^2)\mu_{\hat{y},t}+\frac{\rho^2}{2}r^2\sigma_\eta^2\right\}$. So the first line of the expression for $w_t'(x)$ is given by

$$\text{line (54)}=\rho r^2\frac{\sigma_{\hat{s}}}{\sigma_t}D_t\exp\left\{\rho r^2\left(\mu_{\hat{s}}-\frac{\sigma_{\hat{s}}}{\sigma_t}\mu_t\right)\right\}x^{\rho\left(r^2\frac{\sigma_{\hat{s}}}{\sigma_t}-1\right)}.$$

We move now to the second line, i.e., line (55). We start with the inside integral.

$$\begin{aligned}\int_{y_{t+1}} f(\phi[x,y_t],y_{t+1})d\tau_{t+1}(y_{t+1}|\alpha_{t+1}(\phi[x,y_t])) &= \int_{y_{t+1}} \phi[x,y_t]^{1-\rho} y_{t+1}^{\rho} d\tau_{t+1}(y_{t+1}|\alpha_{t+1}(\phi[x,y_t])) \\ &= \left(Ax^{1-\theta}y_t^{\theta}\right)^{1-\rho} \int_{y_{t+1}} y_{t+1}^{\rho} d\tau_{t+1}(y_{t+1}|\alpha_{t+1}(\phi[x,y_t])) \\ &= \left(Ax^{1-\theta}y_t^{\theta}\right)^{1-\rho} D_{t+1}\alpha_{t+1}(\phi[x,y_t])^{\rho r^2} \\ &= C_{t+1}\left(Ax^{1-\theta}y_t^{\theta}\right)^{a_{t+1}},\end{aligned}$$

where $C_t = D_t \exp\left\{\rho r^2[\mu_{\hat{s},t} - \frac{\sigma_{\hat{s}}}{\sigma_t}\mu_t]\right\}$ and $a_t = 1-\rho+\rho r^2\frac{\sigma_{\hat{s}}}{\sigma_t}$, where σ_t is the standard deviation of x at date t.
Now to compute the integral outside we have

$$\begin{aligned}&\int_{y_t} C_{t+1}\left(Ax^{1-\theta}y_t^{\theta}\right)^{a_{t+1}} d\tau_t(y_t|\alpha_t(x)) \\ =\ & C_{t+1}\left(Ax^{1-\theta}\right)^{a_{t+1}} \int_{y_t} y_t^{\theta a_{t+1}} d\tau_t(y_t|\alpha_t(x)) \\ =\ & C_{t+1}A^{a_{t+1}} \exp\left\{a_{t+1}\theta(1-r^2)\mu_{\hat{y},t} + \frac{\theta^2 a_{t+1}^2}{2} r^2\sigma_\eta^2\right\} x^{a_{t+1}(1-\theta)}\alpha_t(x)^{a_{t+1}\theta r^2} \\ =\ & \tilde{B}_{t+1}A^{a_{t+1}}x^{a_{t+1}(1-\theta)}\alpha_t(x)^{a_{t+1}\theta r^2},\end{aligned}$$

where

$$\tilde{B}_{t+1} = C_{t+1}\exp\left\{a_{t+1}\theta(1-r^2)\mu_{\hat{y},t} + \frac{a_{t+1}^2\theta^2}{2}r^2\sigma_\eta^2\right\}.$$

Next, we now take the derivative with respect to x but *keep the assignment function* $\alpha_t(x)$ *constant*. We obtain

$$\begin{aligned}&\frac{\partial}{\partial x}\int_{y_t}\int_{y_{t+1}} f(\phi[x,y_t],y_{t+1})d\tau_{t+1}(y_{t+1}|\alpha_{t+1}(\phi[x,y_t]))d\tau_t(y_t|\alpha_t(x)) \\ =\ & a_{t+1}(1-\theta)\tilde{B}_{t+1}A^{a_{t+1}}x^{a_{t+1}(1-\theta)-1}\alpha_t(x)^{a_{t+1}\theta r^2}.\end{aligned}$$

Finally, using the function for $\alpha_t(x)$, line (55) reads

$$\text{line (55)} = -\beta(1-\theta)a_{t+1}B_{t+1}A^{a_{t+1}}x^{a_{t+1}\left(1-\theta+\theta r^2\frac{\sigma_{\hat{s}}}{\sigma_t}\right)-1}, \tag{58}$$

where

$$B_{t+1} = \tilde{B}_{t+1}\exp\left\{a_{t+1}\theta r^2\left(\mu_{\hat{s},t} - \frac{\sigma_{\hat{s}}}{\sigma_t}\mu_t\right)\right\}.$$

Notice that $\tilde{B}_{t+1}$ has in it a $\mu_{\hat{y},t}$ component, i.e., the expectation of $\hat{y}$ at date t.

Last, we need to solve for (56). This part cannot be done analytically since we have the term $w'_{t+1}(x)$ which is the function we have to find. We write it here as

$$\text{line (56)} = \beta(1-\theta)Ax^{-\theta}\int w'_{t+1}(Ax^{\theta}y^{1-\theta})y^{\theta}d\tau_t(y|\alpha_t(x)),$$

and simply use $\hat{y}|\hat{\alpha}_t(x) \sim N\left((1-r^2)\mu_{\hat{y}} + r^2\hat{\alpha}_t(x), r^2\sigma_\eta^2\right)$.

The limit $\sigma_\eta \to 0$ One way to check the validity of the simulation is to compare it to the analytic solution in (22), which obtains in the economy with no noise.

To simplify the calculations, we choose parameters so that the economy does not grow, which means that time indexes t drop out. Lets start with term 1 in (54). Recall, the solution found is

$$(54) = \rho r^2 \frac{\sigma_{\hat{s}}}{\sigma_t} D \exp\left\{\rho r^2[\mu_{\hat{s}} - \frac{\sigma_{\hat{s}}}{\sigma_t}\mu_t]\right\} x^{\rho\left(r^2\frac{\sigma_{\hat{s}}}{\sigma_t}-1\right)}, \tag{59}$$

where $D_t = \exp\left\{\rho(1-r^2)\mu_{\hat{y}} + \frac{\rho^2}{2}r^2\sigma_\eta^2\right\}$. Since $\sigma_\eta \approx 0$, then $r \approx 1$ and $D \approx 1$. Also, since $\sigma_\eta \approx 0$, we have $\sigma_{\hat{s}} \approx \sigma_t$ and thus the exponent of x yields 0: $\left(r^2\frac{\sigma_{\hat{s}}}{\sigma_t} - 1\right) \approx 0$. Similarly, note that with $\sigma_\eta \approx 0$ we have $\mu_{\hat{s}} \approx \mu_t$ and thus $\left(\mu_{\hat{s}} - \frac{\sigma_{\hat{s}}}{\sigma_t}\mu_t\right) \approx 0$. Then, is easy to see that (59) $\approx \rho$.

Next, we consider term 2 in (55). Recall, the solution found is

$$(55) = -\beta(1-\theta)aBA^a x^{a\left(1-\theta+\theta r^2\frac{\sigma_{\hat{s}}}{\sigma_t}\right)-1}, \tag{60}$$

where

$$\begin{aligned}
a &= 1-\rho+\rho r^2\frac{\sigma_{\hat{s}}}{\sigma_t} \\
C &= D\exp\left\{\rho r^2\left(\mu_{\hat{s}} - \frac{\sigma_{\hat{s}}}{\sigma_t}\mu_t\right)\right\} \\
\tilde{B} &= C\exp\left\{a\theta(1-r^2)\mu_{\hat{y}} + \frac{a^2\theta^2}{2}r^2\sigma_\eta^2\right\} \\
B &= \tilde{B}\exp\left\{a\theta r^2\left(\mu_{\hat{s}} - \frac{\sigma_{\hat{s}}}{\sigma_t}\mu_t\right)\right\}.
\end{aligned}$$

Since $\sigma_\eta \approx 0$, we have both $r \approx 1$, $\sigma_{\hat{s}} \approx \sigma_t$ and thus $a \approx 1$. Also, since $\mu_{\hat{s}} \approx \mu_t$ and $D \approx 1$, we have $C \approx 1$. Then, since $\sigma_\eta \approx 0$ and $r \approx 0$, we have that $\tilde{B} \approx 1$ and $B \approx 1$ as well. Finally, note that $\left(1-\theta+\theta r^2\frac{\sigma_{\hat{s}}}{\sigma_t}\right) \approx 1$, and in the economy with no growth $A=1$, so that the second term in (55)$\approx -\beta(1-\theta)$, which is what the code delivers.

So far, (54) + (55) $\approx \rho - \beta(1-\theta)$. We consider the last term now. We look for a fixed point in which the wage is linear in its argument $w(x) = \omega x$, the result reported in (22) for the economy with no noise. Then, (56) reads

$$\begin{aligned}(56) &= \beta(1-\theta)Ax^{-\theta}\int \omega y^{\theta} d\tau_t(y|\alpha(x)) \\ &= \beta(1-\theta)Ax^{-\theta}\omega \exp\theta(1-r^2)\mu_{\hat{y}} + \frac{\theta^2}{2}r^2\sigma_\eta^2\alpha_t(x)^{\theta r^2}.\end{aligned} \tag{61}$$

When $\sigma_\eta \approx 0$ we have $\exp\theta(1-r^2)\mu_{\hat{y}} + \frac{\theta^2}{2}r^2\sigma_\eta^2 \approx 1$ and $\alpha_t(x)^{\theta r^2} \approx x^\theta$, then

$$(56) \approx \beta(1-\theta)A\omega.$$

Finally, our wage equation is a constant ω given by

$$\omega = \rho - \beta(1-\theta) + \beta(1-\theta)A\omega, \tag{62}$$

the solution to which reads

$$\omega = \frac{\rho - \beta(1-\theta)A}{1-\beta(1-\theta)A}. \tag{63}$$

The code delivers $\lim_{\sigma_\eta \to 0} w'(x) \approx 0.5853\ \forall x$. This is very close to the analytical result in (22) for the economy with no noise.