Spatial Development: Online Appendix

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PROOFS OF PROPOSITIONS

1. Proof of Proposition 1

Proof. Given the optimally chosen $\hat{L}_i(\ell,t)$, the objective function of a firm in a given location ℓ at time t_0 is

$$\max_{\{\phi_i(\ell,t)\}_{t_0}^{\infty}} E_{t_0}\left[\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left(\begin{array}{c} p_i(\ell,t)\left(\left(\frac{\phi_i(\ell,t)}{a-1}+1\right)Z_i^-(\ell,t)\right)^{\gamma}\hat{L}_i(\ell,t)^{\mu_i} \\ -w(\ell,t)\hat{L}_i(\ell,t)-R(\ell,t)-\psi(\phi_i(\ell,t))\end{array}\right)\right]$$

with $Z_i^-(\ell,t_0)$ given. Note that $\hat{L}_i(\ell,t)$ is a function of $\phi_i(\ell,t)$, but since $\hat{L}_i(\ell,t)$ was optimally chosen, the envelope theorem applies. Similarly, although firms choose their bid for land, the rent that makes firms win the competition for land is determined in the market, so the equilibrium $R(\ell,t)$ is taken as given. Hence, the problem above can be simplified to

$$\begin{aligned}(13)\qquad &\max_{\{\phi_i(\ell,t)\}_{t_0}^{\infty}} E_{t_0}\left[\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left(p_i(\ell,t)\left(\frac{\phi_i(\ell,t)}{a-1}Z_i^-(\ell,t)\right)^{\gamma}\hat{L}_i(\ell,t)^{\mu_i}-\psi(\phi_i(\ell,t))\right)\right]\\ &=\max_{\{\phi_i(\ell,t)\}_{t_0}^{\infty}} p_i(\ell,t_0)\left(\frac{\phi_i(\ell,t_0)}{a-1}Z_i^-(\ell,t_0)\right)\gamma\hat{L}_i(\ell,t_0)^{\mu_i}-\psi(\phi_i(\ell,t_0))\\ &+E_{t_0}\left[\sum_{t=t_0+1}^{\infty}\beta^{t-t_0+1}\left(p_i(\ell,t)\left(\frac{\phi_i(\ell,t)}{a-1}Z_i^-(\ell,t)\right)^{\gamma}\hat{L}_i(\ell,t)^{\mu_i}-\psi(\phi_i(\ell,t))\right)\right]\end{aligned}$$

Note that the last term in (13) is independent of the decision in t_0 except for its dependence on the technology early in period t_0+1, $Z_i^-(\ell,t_0+1)$. However, this technology is independent of the choice $\phi_i(\ell,t_0)$. To prove this, first note that by (3) technology at the beginning of the period is continuous in ℓ, and so are $\hat{L}_i(\ell,t)$ and $R_i(\ell,t)$. Furthermore, a location is small so it does not change $p_i(\ell,t)$. Since $\lim_{\ell\to\ell'} s(\ell,\ell') = 1$ from above or below, (3) implies that if $\ell = \arg\max_{r\in[0,1]} e^{-\delta|\ell-r|} Z_i^+(r,t_0)$ there exists an ℓ' arbitrarily close to ℓ such that $\ell' = \arg\max_{r\in[0,1]} e^{-\delta|\ell-r|} Z_i^+(r,t_0)$. Hence, $Z_i^-(\ell,t_0+1) = Z_i^+(\ell',t_0)$ and so innovation decisions at ℓ do not determine $Z_i^-(\ell,t_0+1)$, which implies that the last term in (13) does not depend on $\phi_i(\ell,t_0)$. Hence, to determine $\phi_i(\ell,t_0)$, maximizing (13) is equivalent to solving

$$
\begin{aligned}
&\max_{\phi_i(\ell,t_0)} p_i(\ell,t_0)\left(\frac{\phi_i(\ell,t_0)}{a-1} Z_i^-(\ell,t_0)\right)^{\gamma} \hat{L}_i(\ell,t_0)^{\mu_i} - \psi(\phi_i(\ell,t_0)) \\
= \;& \max_{\phi_i(\ell,t_0)} \Bigg(p_i(\ell,t_0)\left(\left(\frac{\phi_i(\ell,t_0)}{a-1}+1\right) Z_i^-(\ell,t_0)\right)^{\gamma} \hat{L}_i(\ell,t_0)^{\mu_i} \\
& -w(\ell,t_0)\hat{L}_i(\ell,t_0) - R(\ell,t_0) - \psi(\phi_i(\ell,t_0))\Bigg)
\end{aligned}
$$

which denotes current expected profits. ■

2. Proof of Proposition 2

Proof. The main part of the proof consists of showing that the solution to the problem of choosing labor and a technology levels, conditional on the result of Proposition 1 (which holds irrespective of uniqueness), exists and is unique. Proposition 1 in Rossi-Hansberg (2005) then implies that conditional on productivity $Z_i^-(\cdot,t)$, all i, the allocation in period t is uniquely determined. Given the unique allocation in period t, which results in a pair of functions $Z_i^+(\cdot,t)$, all i, equation (3) can be used to obtain a unique pair of functions $Z_i^-(\cdot,t+1)$, all i. Hence, given initial productivity functions $Z_i^-(\cdot,1)$, for $i\in\{M,S\}$, the whole path of productivities and allocations exists and is unique, which proves the result.

So consider the problem[1]

$$
\begin{aligned}
&\max_{L_i(\ell,t),\phi_i(\ell,t)\geq 0} \Phi(\phi_i(\ell,t), L_i(\ell,t)) \equiv \\
&\max_{L_i(\ell,t),\phi_i(\ell,t)\geq 0} p_i(\ell,t)\left(\left(\frac{\phi_i(\ell,t)}{a-1}+1\right) Z_i(\ell,t)(\ell,t)\right)^{\gamma} L_i(\ell,t)^{\mu_i} \\
&-w(\ell,t)L - \psi(\phi_i(\ell,t))
\end{aligned}
$$

[1]We do not make explicit the dependence of $\psi(\cdot)$ on $w(\ell,t)$ since it is never used in the proof.

for $\gamma \in (0,1)$. As in the text we keep the assumption that $\psi'(\phi), \psi''(\phi) > 0$ and $\mu_i \in (0,1)$. The first order conditions of the problem are given by

$$\begin{aligned}
\Phi\left(\phi_i(\ell,t), L_i(\ell,t)\right)_\phi &= \frac{\gamma p_i(\ell,t)}{a-1} Z_i(\ell,t)^\gamma \left(\frac{\phi_i(\ell,t)}{a-1}+1\right)^{\gamma-1} L_i(\ell,t)^{\mu_i} \\
&\quad -\psi'\left(\phi_i(\ell,t)\right) = 0, \\
\Phi\left(\phi_i(\ell,t), L_i(\ell,t)\right)_L &= \mu_i p_i(\ell,t) Z_i(\ell,t)^\gamma \left(\frac{\phi_i(\ell,t)}{a-1}+1\right)^{\gamma} L^{\mu_i-1} - w(\ell,t) = 0.
\end{aligned}$$

The second order derivatives of the problem are given by

$$\begin{aligned}
\Phi\left(\phi_i(\ell,t), L_i(\ell,t)\right)_{\phi\phi} &= \frac{\gamma(\gamma-1) p_i(\ell,t)}{(a-1)^2} Z_i(\ell,t)^\gamma \left(\frac{\phi_i(\ell,t)}{a-1}+1\right)^{\gamma-2} L_i(\ell,t)^{\mu_i} \\
&\quad -\psi''\left(\phi_i(\ell,t)\right) < 0, \\
\Phi\left(\phi_i(\ell,t), L_i(\ell,t)\right)_{LL} &= \mu_i(\mu_i-1) p_i(\ell,t) Z_i(\ell,t)^\gamma \left(\frac{\phi_i(\ell,t)}{a-1}+1\right)^{\gamma} L_i(\ell,t)^{\mu_i-2} < 0, \\
\Phi\left(\phi_i(\ell,t), L_i(\ell,t)\right)_{\phi L} &= \Phi\left(\phi_i(\ell,t), L_i(\ell,t)\right)_{L\phi} \\
&= \frac{\gamma \mu_i p_i(\ell,t)}{a-1} Z_i(\ell,t)^\gamma \left(\frac{\phi_i(\ell,t)}{a-1}+1\right)^{\gamma-1} L_i(\ell,t)^{\mu_i-1} > 0.
\end{aligned}$$

The problem above has a unique optimum if the Hessian is a negative-definite matrix. This is equivalent to the matrix having a negative determinant of the kth minor for k odd and positive determinant for k even. In the case above $k = 1$ or 2. So the determinant for $k = 1$ is just the second derivative with respect to any of the terms, both of which are negative. The determinant for $k = 2$ is given by

$$\begin{aligned}
& \Phi(\phi, L)_{\phi\phi} \Phi(\phi, L)_{LL} - \Phi(\phi, L)_{L\phi} \\
= & -\psi''(\phi) \mu_i(\mu_i-1) p(\ell,t) Z(\ell,t)^\gamma \left(\frac{\phi}{a-1}+1\right)^{\gamma} L^{\mu_i-2} \\
& + \frac{\gamma(\gamma-1)\mu_i(\mu_i-1)}{(a-1)^2} p(\ell,t)^2 Z(\ell,t)^{2\gamma} \left(\frac{\phi}{a-1}+1\right)^{2\gamma-2} L^{2\mu_i-2} \\
& - \frac{\gamma^2 \mu_i^2 p(\ell,t)^2}{(a-1)^2} Z(\ell,t)^{2\gamma} \left(\frac{\phi}{a-1}+1\right)^{2\gamma-2} L^{2\mu_i-2}
\end{aligned}$$

and so

$$\begin{aligned}
&\Phi\left(\phi_i(\ell,t), L_i(\ell,t)\right)_{\phi\phi} \Phi\left(\phi_i(\ell,t), L_i(\ell,t)\right)_{LL} - \Phi\left(\phi_i(\ell,t), L_i(\ell,t)\right)_{L\phi} \\
= \; & -\psi''\left(\phi_i(\ell,t)\right)\mu_i\left(\mu_i-1\right)p_i\left(\ell,t\right)Z_i\left(\ell,t\right)^{\gamma}\left(\frac{\phi_i(\ell,t)}{a-1}+1\right)^{\gamma}L_i(\ell,t)^{\mu_i-2} \\
& +\frac{\gamma\mu_i\left(-\gamma-\mu_i+1\right)}{\left(a-1\right)^2}p_i\left(\ell,t\right)^2 Z_i\left(\ell,t\right)^{2\gamma}\left(\frac{\phi_i(\ell,t)}{a-1}+1\right)^{2\gamma-2}L_i(\ell,t)^{2\mu_i-2}
\end{aligned}$$

which is necessarily positive if

$$1-\gamma-\mu_i \geq 0.$$

Hence if $1-\gamma-\mu_i \geq 0$, the problem has a unique solution. ■

3. Proof of Proposition 3

Proof. Let $m_\ell = \{\ell : \lim_{\ell' \nearrow \ell} \theta_i\left(\ell',t\right) \neq \lim_{\ell' \searrow \ell} \theta_i\left(\ell',t\right)\}$ denote the locations in which specialization changes from one industry to the other. Take $\ell \in m_\ell$ and ℓ' such that $\theta_i\left(\ell',t\right)=1$ and $\ell = \arg\min\{|l-\ell'| \text{ for } l \in m_\ell\}$. Then, there exists a $B>0$ such that $\partial p_i\left(\ell',t\right)/\partial\kappa > 0$ for $|\ell-\ell'|<B$. The reason is that by (11), $H_M\left(\ell',t\right)$ will be lower everywhere for unchanged local surpluses, $\theta_i\left(\ell,t\right)x_i\left(\ell,t\right)-c_i\left(\ell,t\right)\left(\sum_i \theta_i\left(\ell,t\right)\hat{L}_i\left(\ell,t\right)\right)$, so the average price of the good, $\int p_i\left(\ell,t\right)d\ell$, needs to increase in order to increase production and decrease consumption of the good, so that local surpluses increase and the market clears with higher transport costs (higher transport costs consume goods, so the average price of goods increases). By (10), $\partial\frac{p_i(\ell',t)}{p_i(\ell'+\varepsilon,t)}/\partial\varepsilon = \kappa e^{\kappa\varepsilon}$ for ε small enough, such that $\ell = \arg\min\{|l-\ell'+\varepsilon| \text{ for } l \in m_\ell\}$, and so the slope of $p_i\left(\ell',t\right)$ increases with κ for B small enough. Hence, $\partial p_i\left(\ell',t\right)/\partial\kappa>0$ for $|\ell-\ell'|<B$. Given $Z_i\left(\ell',t-1\right)$, this result and (3) imply by (5) that $\partial L_i\left(\ell',t\right)/\partial\kappa>0$ for $|\ell-\ell'|<B$ and so by (9), $\partial\phi_i\left(\ell',t\right)/\partial\kappa \geq 0$ for $|\ell-\ell'|<B$. Since at κ by assumption $\phi_i\left(\ell,t\right)=0$ all ℓ, $\partial\max_\ell \phi_i\left(\ell,t\right)/\partial\kappa \geq 0$. Now consider κ^* such that (9) imply that $\max_\ell \phi_i\left(\ell,t\right)=0$, but a marginal increase in $Z_i\left(\ell',t\right)p_i\left(\ell',t\right)L_i\left(\ell',t\right)^{\mu_i}$ would yield $\phi_i\left(\ell,t\right)>0$ for some ℓ, then the argument above guarantees that an increase in transport costs above κ^* increases aggregate productivity. ■