

A MACROECONOMIC MODEL WITH A FINANCIAL SECTOR
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 ONLINE APPENDIX

A1. Microfoundation of Balance Sheets and Intermediation

This section describes the connection between balance sheets in our model and agency problems. Extensive corporate finance literature (see Townsend (1979), Bolton and Scharfstein (1990), DeMarzo and Sannikov (2006), Biais et al. (2007), or Sannikov (2012) for a survey of these models) suggests that agency frictions increase when the agent's net worth falls. In a macroeconomic setting, this logic points to the aggregate net worth of end borrowers, as well as that of intermediaries.

Incentive provision requires the agent to have some "skin in the game" in the projects he manages. When projects are risky, it follows that the agent must absorb some of project risk through net worth. Some of the risks may be identified and hedged, reducing the agent's risk exposure. However, whenever some aggregate risk exposures of constrained agents cannot be hedged, macroeconomic fluctuations due to financial frictions arise, as these residual risks have aggregate impact on the net-worth-constrained agents.

Our baseline model assumes the simplest form of balance sheets, in which constrained agents (experts) absorb all risk and issue just risk-free debt. Qualitatively, however, our results still hold if experts can issue some outside equity and even hedge some of their risks, as long as they *cannot* hedge *all* the risks. Quantitatively, the assumption regarding equity issuance matters: If experts can issue more equity or hedge more risks, then they can operate efficiently with much lower net worths. This does not necessarily lead to a more stable system because, as we saw in Section 5, the steady state in our model is endogenous. Agents who can function with lower wealth accumulate lower net worth buffers. Thus, we expect that our baseline model with simple balance sheets captures many characteristics of equilibria of more general models.

To illustrate the connection between balance sheets and agency models, first, we discuss the agency problem with direct lending from investors to a single agent. Second, we illustrate agency problems that arise with intermediaries. In this case, the net worth of intermediaries matters as well. At the end of this section, we discuss contracting with idiosyncratic jump risk, which is relevant for Section 5.

AGENCY FRICTIONS BETWEEN AN EXPERT AND HOUSEHOLDS

Assume that experts are able to divert capital returns at rate $b_t \in [0, \infty)$. Diversion is inefficient: Of the funds b_t diverted, an expert is able to recover only a portion $h(b_t) \in [0, b_t]$, where $h(0) = 0$, $h' \leq 1$, $h'' \leq 0$. Net of diverted funds, capital generates the return of

$$dr_t^k - b_t dt$$

and the expert receives an income flow of $h(b_t) dt$.

If capital is partially financed by outside equity held by households, then households receive the return of

$$dr_t^k - b_t dt - f_t dt,$$

where f_t is the fee paid to the expert. When the expert holds a fraction Δ_t of equity, then per dollar invested in capital he gets

$$\Delta_t(dr_t^k - b_t dt) + (1 - \Delta_t)f_t dt + h(b_t) dt.$$

The incentives with respect to diversion are summarized by the first-order condition $\Delta_t = h'(b_t)$, which leads to a weakly decreasing function $b(\Delta)$ with $b(1) = 0$. In equilibrium, the fee f_t is chosen so that household investors get the expected required return of r on their investment, i.e., $f_t = E[dr_t^k]/dt - b_t - r$. As a result, the return on the expert's equity stake in capital (including the benefits of diversion) is

$$\frac{\Delta_t(dr_t^k - b_t dt) + (1 - \Delta_t)f_t dt + h(b_t) dt}{\Delta_t} = \frac{E[dr_t^k] - r dt}{\Delta_t} + r dt + (\sigma + \sigma_t^q) dZ_t - \frac{b(\Delta_t) - h(b(\Delta_t))}{\Delta_t} dt,$$

where $\frac{b(\Delta_t) - h(b(\Delta_t))}{\Delta_t}$ is the deadweight loss rate due to the agency problem. The law of motion of the expert's net worth in this setting is of the form

$$(A1) \quad \frac{dn_t}{n_t} = x_t \left(\frac{E[dr_t^k] - r dt}{\Delta_t} + r dt + (\sigma + \sigma_t^q) dZ_t - \frac{b(\Delta_t) - h(b(\Delta_t))}{\Delta_t} dt \right) + (1 - x_t)r dt - d\zeta_t,$$

where x_t is the portfolio allocation to inside equity and $d\zeta_t$ is the expert's consumption rate. It is convenient to view equation (A1) as capturing the issuance of equity and risk-free debt. However, it is possible to reinterpret this capital structure in many other ways, since securities such as risky debt can be replicated by continuous trading in the firm's stock and risk-free debt.

Equation (EK) generalizes to

$$\max_{\Delta} \frac{E[dr_t^k]/dt - r - (b(\Delta) - h(b(\Delta)))}{\Delta} = -\sigma_t^\theta (\sigma + \sigma_t^q)$$

and determines optimal equity issuance. Our results suggest that, as risk premia $E[dr_t^k]/dt - r$ rise in downturns, the experts' equity retention Δ decreases and deadweight losses increase.³²

Our baseline setting is a special case of this formulation, in which there are no costs to the diversion of funds, i.e., $h(b) = b$ for all $b \geq 0$. In this case, the agency problem can be solved only by setting $\Delta = 1$, i.e., experts can finance

³²One natural way to interpret this is through a capital structure that involves risky debt, as it becomes riskier (more equity-like) after experts suffer losses. Many agency problems become worse when experts are "under water."

themselves only through risk-free debt. Our analysis can be generalized easily, but for expositional purposes we keep our baseline model as simple as possible.

We would like to be clear about our assumptions regarding the space of acceptable contracts, which specify how observable cash flows are divided between the expert and household investors. We make the following two restrictions on the contracting space:

- A The allocation of profit is determined by the total value of capital, and shocks to k_t or q_t separately are not contractible, and
- B lockups are not allowed; at any moment of time, any party can break the contractual relationship. The value of assets is divided among the parties the same way, independently of who breaks the relationship.

Condition B simplifies analysis, as it allows us to focus only on expert net worth rather than a summary of the expert's individual past performance history. It assumes a degree of anonymity, so that once the relationship breaks, parties never meet again and the outcome of the relationship that just ended affects future relationships only through net worth. This condition prevents commitment to long-term contracts, such as in the setting of Myerson (2010). However, in many settings this restriction alone does not rule out optimal contracts: Fudenberg, Holmström and Milgrom (1990) show that it is possible to implement the optimal long-term contract through short-term contracts with continuous marking-to-market.

Condition A requires that contracts have to be written on the total return of capital and that innovations in k_t , q_t , or the aggregate risk dZ_t cannot be hedged separately. This assumption is clearly restrictive, but it creates a convenient and simple way to capture important phenomena that we observe in practice. Specifically, condition A creates an amplification channel, in which market prices affect the agents' net worth, and is consistent with the models of Kiyotaki and Moore (1997) and Bernanke, Gertler and Gilchrist (1999). Informally, contracting directly on k_t is difficult because we view k_t not as something objective and static like the number of machines, but rather something much more forward looking, like the expected NPV of assets under a particular management strategy. Moreover, even though in our model there is a one-to-one correspondence between k_t and output, in a more general model this relationship could differ across projects, depend on the expert's private information, and be manipulable, e.g., through underinvestment.

More generally, we could assume that aggregate shocks dZ_t to the experts' balance sheets can be hedged partially. As long as it is impossible to design a perfect hedge and to perfectly share all aggregate risks with households, the model will generate economic fluctuations driven by the shocks to the net worth of the constrained agents. Thus, to generate economic fluctuations, we make assumptions that would otherwise allow agents to write optimal contracts, but

place restrictions on hedging. Experts can still choose their risk exposure Δ_t , but cannot hedge aggregate shocks dZ_t .

INTERMEDIARY SECTOR

It is possible to reinterpret our model to discuss the capitalization of intermediaries as well as end borrowers.

One natural model of intermediation involves a double moral-hazard problem motivated by Holmström and Tirole (1997). Let us, like Meh and Moran (2010) separate experts into two classes of agents: entrepreneurs who manage capital under the productive technology, and intermediaries who can channel funds from households to entrepreneurs. Through costly monitoring actions that are unobservable by outside investors, intermediaries are able to reduce the benefits that entrepreneurs get from the diversion of funds. Specifically, the entrepreneurs' marginal benefit of fund diversion $\frac{\partial}{\partial b}h(b_t|m_t)$ is continuously decreasing with the proportional cost of monitoring $m_t \geq 0$, i.e. $\frac{\partial^2}{\partial b \partial m}h(b_t|m_t) < 0$. Thus, for a fixed equity stake Δ_t of the entrepreneur, higher monitoring intensity m_t leads to a lower diversion rate $b_t = b(\Delta_t|m_t)$. Assuming that $\frac{\partial^2}{\partial b^2}h(\cdot|m_t) < 0$, the entrepreneur's optimal diversion rate b_t is uniquely determined by the first-order condition $\frac{\partial}{\partial b}h(b_t|m_t) = \Delta_t$ and is continuously decreasing in Δ_t .

Intermediaries have no incentives to exert costly monitoring effort unless they themselves have a stake in the entrepreneur's project. An intermediary who holds a fraction Δ_t^I of the entrepreneur's equity optimally chooses the monitoring intensity m_t that solves

$$\min_m \Delta_t^I b(\Delta_t|m) + m.$$

The solution to this problem determines how the rates of monitoring $m(\Delta_t, \Delta_t^I)$ and cash flow diversion $b(\Delta_t, \Delta_t^I)$ depend on the allocations of equity to the entrepreneur and the intermediary.

By reducing the entrepreneurs' agency problem through monitoring, intermediaries are able to increase the amount of financing available to entrepreneurs. However, intermediation itself requires risk-taking, as the intermediaries need to absorb the risk in their equity stake Δ_t^I through their net worth. Thus, the aggregate net worth of intermediaries becomes related to the amount of financing available to entrepreneurs. Figure A1 depicts the interlinked balance sheets of entrepreneurs, intermediaries, and households. Fraction $\Delta_t + \Delta_t^I$ of entrepreneur risk gets absorbed by the entrepreneur and intermediary net worths, while fraction $1 - \Delta_t - \Delta_t^I$ is held by households.

The marginal values of entrepreneur and intermediary net worths, θ_t and θ_t^I , can easily differ in this economy. If so, then the capital-pricing equation (EK) generalizes to

$$\max_{\Delta, \Delta^I} E[dr_t^k]/dt - r - (b(\Delta, \Delta^I) - h(b(\Delta, \Delta^I))) - m(\Delta, \Delta^I) + (\Delta\sigma_t^\theta + \Delta^I\sigma_t^{\theta, I})(\sigma + \sigma_t^q) = 0.$$

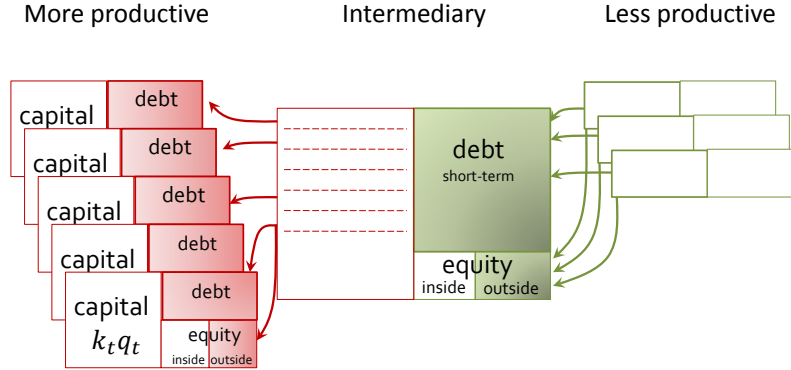


FIGURE A1. BALANCE SHEET STRUCTURES OF ENTREPRENEURS AND FINANCIAL INTERMEDIARIES.

Equilibrium dynamics in this economy depend on two state variables, the shares of net worth that belong to the entrepreneurs η_t and intermediaries η_t^I . Generally, these are imperfect substitutes, as intermediaries can reduce the entrepreneurs' required risk exposure by taking on risk and monitoring. However, several special cases can be reduced to a single state variable. For example, if entrepreneurs and intermediaries can write contracts on aggregate shocks among themselves (but not with households), then the two groups of agents have identical risk premia (i.e., $\sigma_t^\theta = \sigma_t^{\theta,I}$) and the sum $\eta_t + \eta_t^I$ determines the equilibrium dynamics.

CONTRACTING WITH IDIOSYNCRATIC LOSSES AND COSTLY STATE VERIFICATION

Next, we discuss contracting in an environment of Section 5, where experts may suffer idiosyncratic loss shocks. For simplicity, we focus on the simplest form of the agency problem without intermediaries, in which $h(b) = b$ for all $b \geq 0$. As discussed earlier in the Appendix, in our *baseline* model this assumption leads to a simple capital structure, in which experts can borrow only through risk-free debt.

Assume, as in Section 5, that, in the absence of benefit extraction, capital managed by expert $i \in \mathbb{I}$ evolves according to

$$dk_t = (\Phi(\iota_t) - \delta) k_t dt + \sigma k_t dZ_t + k_t dJ_t^i,$$

where dJ_t^i is a *compensated* loss process with intensity λ and jump distribution $F(y)$, $y \in [-1, 0]$. Then, in the absence of jumps, J_t^i has a positive drift of

$$dJ_t^i = \left(\lambda \int_0^1 (-y) dF(y) \right) dt,$$

so that $E[dJ_t^i] = 0$.

The entrepreneur can extract benefits continuously or via discrete jumps. Benefit extraction is described by a non-decreasing process $\{B_t, t \geq 0\}$, which changes the law of motion of capital to

$$dk_t = (\Phi(\iota) - \delta) k_t dt + \sigma k_t dZ_t + k_t dJ_t^i - dB_t$$

and gives entrepreneur benefits at the rate of dB_t units of capital. The jumps in B_t are bounded by k_{t-} , the total amount of capital under the entrepreneur's management just before time t .

Unlike in our earlier specification of the agency problem, in which the entrepreneur's *rate* of benefit extraction $b_t = \frac{dB_t}{(q_t k_t) dt}$ must be finite, now the entrepreneur can also extract benefits discontinuously, including in quantities that reduce the value of capital under management below the value of debt.

We assume a verification technology that can be employed in the event of discrete drops in capital. In particular, if a verification action is triggered by outside investors when capital drops from k_{t-} to k_t at time t , then investors

- (i) learn whether a drop in capital was caused partially by the entrepreneur's benefit extraction at time t and in what amount,
- (ii) recover all capital that was diverted by the entrepreneur at time t , and
- (iii) pay a cost of $(q_t k_{t-})c(dJ_t^i)$, that is proportional to the value of the investment prior to verification³³.

If verification reveals that the drop in capital at time t was partially caused by benefit extraction, i.e., $k_{t-}(1 + dJ_t^i) > k_t$, then the entrepreneur cannot extract any benefits, as diverted capital $k_{t-}(1 + dJ_t^i) - k_t$ is returned to the investors.

We maintain the same assumptions as before about the form of the contract in the absence of verification, i.e., (A) the contract determines how the total market value of assets is divided between the entrepreneur and outside investors, and (B) at any moment either party can break the relationship and walk away with its share of assets. In particular, contracting on k_t or q_t separately is not possible. In addition, the contract specifies conditions, under which a sudden drop in the market value of the expert's assets $q_t k_t$ triggers a verification action. In this event, the contract specifies how the remaining assets, net of verification costs, are divided among the contracting parties conditional on the amount of capital that was diverted at time t . We assume that the monitoring action is not randomized, i.e., it is completely determined by the asset value history.

PROPOSITION A.1: *With idiosyncratic jump risk, it is optimal to trigger verification only in the event that the market value of the expert's assets $q_t k_t$ falls*

³³The assumption that the verification cost depends only on the amount of capital recovered, regardless of the diverted amount, is without loss of generality since on the equilibrium path the entrepreneur does not divert funds.

below the value of debt. In the event of verification, it is optimal for debt holders to receive the value of the remaining assets net of verification costs.

PROOF:

Because jumps are idiosyncratic, they carry no risk premium. Therefore, it is better to deter fund diversion that does not bankrupt the expert by requiring him to absorb jump risk through equity rather than triggering costly state verification (which leads to a deadweight loss). However, verification is required to deter the expert from diverting more funds than his net worth at a single moment of time.

The division of value between debt holders and the expert in the event of verification matters for the expert's incentives only if it is in fact revealed that the expert diverted cash. If no cash was diverted (i.e. it is clear that the loss was caused by an exogenous jump), the division of value between debt holders and the expert can be arbitrary (as long as the expected return of debt holders, net of verification costs, is r) since idiosyncratic jump risk carries no risk premium. Without loss of generality we can assume that debt holders receive the entire remaining value in case of verification.

Proposition A.1 implies that with idiosyncratic jump risk, debt is no longer risk-free.

STATIONARY DISTRIBUTION AND TIME TO REACH

Suppose that X_t is a stochastic process that evolves on the state space $[x_L, x_R]$ according to the equation

$$(B1) \quad dX_t = \mu^x(X_t) dt + \sigma^x(X_t) dZ_t.$$

If, at time $t = 0$, X_t is distributed according to the density $d(x, 0)$, then the density of X_t at all future dates $t \geq 0$ is described by the Kolmogorov forward equation (see, e.g., Ghosh (2010)):

$$(B2) \quad \frac{\partial}{\partial t} d(x, t) = -\frac{\partial}{\partial x} (\mu^x(x) d(x, t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma^x(x)^2 d(x, t)).$$

A *stationary density* is a function that solves (B2) and does not change with time, i.e., $\frac{\partial d(x, t)}{\partial t} = 0$ on the left-hand side of (B2). If so, then integration over x yields the first-order ordinary differential equation

$$0 = F - \mu^x(x) d(x) + \frac{1}{2} \frac{\partial}{\partial x} (\sigma^x(x)^2 d(x)),$$

where the constant of integration F is the “flow” of the density in the positive direction. If one of the endpoints of the interval $[x_L, x_R]$ is reflecting (as η^* in our model), then the flow is $F = 0$.

To compute the stationary density numerically, it is convenient to work with the function $D(x) = \sigma^x(x)^2 d(x)$, which satisfies the ODE

$$(B3) \quad D'(x) = 2 \frac{\mu^x(x)}{\sigma^x(x)^2} D(x).$$

Then $d(x)$ can be found from $D(x)$ using $d(x) = \frac{D(x)}{\sigma^x(x)^2}$.

With an absorbing boundary, the process X_t eventually ends up absorbed (and so the stationary distribution is degenerate) unless the law of motion (B1) prevents X_t from hitting the absorbing boundary with probability one. A non-degenerate stationary density, with an absorbing boundary at x_L , exists if the boundary condition $D(x_L) = 0$ can be satisfied together with $D(x_0) > 0$ for $x_0 > x_L$. For this to happen, we need

$$\log D(x) = \log D(x_0) - \int_x^{x_0} \frac{2\mu^x(x')}{\sigma^x(x')^2} dx' \rightarrow -\infty, \text{ as } x \rightarrow x_L,$$

i.e., $\int_{x_L}^{x_0} \frac{2\mu^x(x)}{\sigma^x(x)^2} dx = \infty$. This condition is satisfied whenever the drift $\mu_t^x(x)$ is positive near x_L (i.e., it pushes X_t away from the boundary x_L) and strong enough working against the volatility that may move X_t toward x_L . For example, if X_t behaves as a geometric Brownian motion near the boundary $x_L = 0$, i.e., $\mu^x(x) = \mu x$ and $\sigma^x(x) = \sigma x$, with $\mu > 0$, then $\int_0^{x_0} \frac{2\mu^x(x)}{\sigma^x(x)^2} dx = \int_0^{x_0} \frac{2\mu}{\sigma^2 x} dx = \infty$.

The following proposition characterizes the expected amount of time it takes to reach any point $x \leq x_R$ starting from x_R .

PROPOSITION B.1: *Suppose that X_t follows (B1) and x_R is a reflecting boundary. Then the expected amount of time $g(x)$ that it takes to reach $x \leq x_R$ from x_R solves equation*

$$(B4) \quad 1 - g'(x)\mu^x(x) - \frac{\sigma^x(x)^2}{2} g''(x) = 0$$

with boundary conditions $g(x_R) = 0$ and $g'(x_R) = 0$.

PROOF:

Denote by $f_{x_0}(y)$ the expected amount of time it takes to reach a point x_0 starting from $y \geq x_0$. Then, to reach x_0 from x_R (expected time $f_{x_0}(x_R) = g(x_0)$), one has to reach $x \in (x_0, x_R)$ first (expected time $g(x)$) and then reach x_0 from x (additional expected time $f_{x_0}(x)$). Therefore,³⁴

$$(B5) \quad g(x) = g(x_0) - f_{x_0}(x).$$

³⁴Equation (B5) implies that, in expectation to reach any point x_0 starting from $x > x_0$, it takes time $g(x_0) - g(x)$.

Since $t + f_{x_0}(X_t)$ is a martingale, it follows that f_{x_0} satisfies the ordinary differential equation

$$1 + f'_{x_0}(x)\mu^x(x) + \frac{\sigma^x(x)^2}{2}f''_{x_0}(x) = 0.$$

Since $g'(x) = -f'_{x_0}(x)$ and $g''(x) = -f''_{x_0}(x)$, it follows that g must satisfy (B4). It takes no time to reach x_R from x_R , so $g(x_R) = 0$. Furthermore, since x_R is a reflecting barrier, due to volatility, $g'(x_R) = 0$.

PROOFS

PROOF OF LEMMA II.1:

Let us show that if the process θ_t satisfies (11) and the transversality condition holds, then θ_t represents the expert's continuation payoff, i.e., satisfies (10). Consider the process

$$\Theta_t = \int_0^t e^{-\rho s} n_s d\zeta_s + e^{-\rho t} \theta_t n_t.$$

Differentiating Θ_t with respect to t using Ito's lemma, we find

$$d\Theta_t = e^{-\rho t} (n_t d\zeta_t - \rho \theta_t n_t dt + d(\theta_t n_t)).$$

If (11) holds, then $E[d\Theta_t] = 0$, so Θ_t is a martingale under the strategy $\{x_t, d\zeta_t\}$. Therefore,

$$\theta_0 n_0 = \Theta_0 = E[\Theta_t] = E \left[\int_0^t e^{-\rho s} n_s d\zeta_s \right] + E [e^{-\rho t} \theta_t n_t].$$

Taking the limit $t \rightarrow \infty$ and using the transversality condition, we find that

$$\theta_0 n_0 = E \left[\int_0^\infty e^{-\rho s} n_s d\zeta_s \right],$$

and the same calculation can be done for any other time t instead of 0.

Conversely, if θ_t satisfies (10), then Θ_t is a martingale since

$$\Theta_t = E_t \left[\int_0^\infty e^{-\rho s} n_s d\zeta_s \right].$$

Therefore, the drift of Θ_t must be zero, and so (11) holds.

Next, let us show that the strategy $\{x_t, d\zeta_t\}$ is optimal if and only if the Bellman equation (12) holds. Under any alternative strategy $\{\hat{x}_t, d\hat{\zeta}_t\}$, define

$$\hat{\Theta}_t = \int_0^t e^{-\rho s} n_s d\hat{\zeta}_s + e^{-\rho t} \theta_t n_t, \quad \text{so that} \quad d\hat{\Theta}_t = e^{-\rho t} (n_t d\hat{\zeta}_t - \rho \theta_t n_t dt + d(\theta_t n_t)).$$

If the Bellman equation (12) holds, then $\hat{\Theta}_t$ is a supermartingale under an arbitrary alternative strategy, so

$$\theta_0 n_0 = \hat{\Theta}_0 \geq E[\hat{\Theta}_t] \geq E \left[\int_0^t e^{-\rho s} n_s d\hat{\zeta}_s \right].$$

Taking the limit $t \rightarrow \infty$, we find that $\theta_0 n_0$ is an upper bound on the expert's payoff from an arbitrary strategy.

Conversely, if the Bellman equation (12) fails, then there exists a strategy $\{\hat{x}_t, d\hat{\zeta}_t\}$ such that

$$n_t d\hat{\zeta}_t - \rho \theta_t n_t dt + E[d(\theta_t n_t)] \geq 0,$$

with a strict inequality on the set of positive measures. Then, for large enough t ,

$$\theta_0 n_0 = \hat{\Theta}_0 < E[\hat{\Theta}_t]$$

and so the expert's expected payoff from following the strategy $\{\hat{x}_t, d\hat{\zeta}_t\}$ until time t , and $\{x_t, d\zeta_t\}$ thereafter, exceeds that from following $\{x_t, d\zeta_t\}$ throughout.

PROOF OF PROPOSITION II.2:

Using the laws of motion of θ_t and n_t as well as Ito's lemma, we can transform the Bellman equation (12) into

$$\rho \theta_t n_t dt = \max_{\hat{x}_t \geq 0, d\hat{\zeta}_t \geq 0} (1 - \theta_t) n_t d\hat{\zeta}_t + r \theta_t n_t dt + n_t E_t[d\theta_t] + \hat{x}_t \theta_t n_t \left(E_t[dr_t^k] - r dt + \sigma_t^\theta (\sigma + \sigma_t^q) dt \right).$$

Assume that $n_t \theta_t$ represents the expert's maximal expected future payoff, so that by Lemma II.1 the Bellman equation holds, and let us justify (i) through (iii). The Bellman equation cannot hold unless $1 \leq \theta_t$ and $E_t[dr_t^k]/dt - r + \sigma_t^\theta (\sigma + \sigma_t^q) \leq 0$, since otherwise the right-hand side of the Bellman equation can be made arbitrarily large. If so, then the choices $d\hat{\zeta}_t = 0$ and $\hat{x}_t = 0$ maximize the right-hand side, which becomes equal to $r \theta_t n_t dt + \theta_t n_t \mu_t^\theta dt$. Thus,

$$\rho \theta_t n_t dt = r \theta_t n_t dt + \theta_t n_t \mu_t^\theta dt \quad \Rightarrow \quad (\text{E}).$$

Furthermore, any $d\hat{\zeta}_t > 0$ maximizes the right-hand side only if $\theta_t = 1$, and $\hat{x}_t > 0$ does only if $E_t[dr_t^k]/dt - r + \sigma_t^\theta (\sigma + \sigma_t^q) \leq 0$. This proves (i) through (iii). Finally, Lemma II.1 implies that the transversality condition must hold for any strategy that attains value $n_t \theta_t$, proving (iv).

Conversely, it is easy to show that if (i) through (iii) hold, then the Bellman equation also holds and the strategy $\{x_t, d\zeta_t\}$ satisfies (11). Thus, by Lemma II.1, the strategy $\{x_t, d\zeta_t\}$ is optimal and attains value $\theta_t n_t$.

PROOF OF LEMMA II.3:

Aggregating over all experts, the law of motion of N_t is

$$(C1) \quad dN_t = rN_t dt + \psi_t q_t K_t (dr_t^k - r dt) - dC_t,$$

where C_t are aggregate payouts. Furthermore, note that $d(q_t K_t)/(q_t K_t)$ are the capital gains earned by a world portfolio of capital, with weight ψ_t on expert capital and $1 - \psi_t$ on household capital. Thus, from (5) and (6),

$$\frac{d(q_t K_t)}{q_t K_t} = \underbrace{dr_t^k - \frac{a - \iota(q_t)}{q_t} dt}_{\text{expert capital gains}} - \underbrace{(1 - \psi_t)(\underline{\delta} - \delta) dt}_{\text{adjustment for household-held capital}},$$

since household capital gains are less than those of experts by $\underline{\delta} - \delta$. Using Ito's lemma,

$$\frac{d(1/(q_t K_t))}{1/(q_t K_t)} = -dr_t^k + \frac{a - \iota(q_t)}{q_t} dt + (1 - \psi_t)(\underline{\delta} - \delta) dt + (\sigma + \sigma_t^q)^2 dt.$$

Combining this equation with (C1) and using Ito's lemma, we get

$$\begin{aligned} d\eta_t &= (dN_t) \frac{1}{q_t K_t} + N_t d\left(\frac{1}{q_t K_t}\right) + \psi_t q_t K_t (\sigma + \sigma_t^q) \frac{-1}{q_t K_t} (\sigma + \sigma_t^q) dt = \\ &(\psi_t - \eta_t)(dr_t^k - r dt - (\sigma + \sigma_t^q)^2 dt) + \eta_t \left(\frac{a - \iota(q_t)}{q_t} + (1 - \psi_t)(\underline{\delta} - \delta) \right) dt - \eta_t d\zeta_t, \end{aligned}$$

where $d\zeta_t = dC_t/N_t$. If $\psi_t > 0$, then Proposition II.2 implies that $E[dr_t^k] - r dt = -\sigma_t^\theta (\sigma + \sigma_t^q) dt$, and the law of motion of η_t can be written as in (15).

PROOF OF PROPOSITION II.4:

First, we derive expressions for the volatilities of η_t , q_t , and θ_t . Using (15), the law of motion of η_t , and Ito's lemma, the volatility of q_t is given by

$$\sigma_t^q q(\eta) = q'(\eta) \underbrace{(\psi - \eta)(\sigma + \sigma_t^q)}_{\sigma_t^\eta \eta} \Rightarrow \sigma_t^q = \frac{q'(\eta)}{q(\eta)} \underbrace{\frac{(\psi - \eta)\sigma}{1 - \frac{q'(\eta)}{q(\eta)}(\psi - \eta)}}_{\sigma_t^\eta \eta}.$$

The expressions for σ_t^η and σ_t^θ follow immediately from Ito's lemma.

Second, note that from (EK) and (H), it follows that

$$(C2) \quad \frac{a - \underline{a}}{q(\eta)} + \underline{\delta} - \delta + (\sigma + \sigma_t^q)\sigma_t^\theta \geq 0,$$

with equality if $\psi < 1$. Moreover, when $q(\eta), q'(\eta), \theta(\eta) > 0$ and $\theta'(\eta) < 0$, then $\sigma_t^q, \sigma_t^\eta > 0$ are increasing in ψ while $\sigma_t^\theta < 0$ is decreasing in ψ . Thus, the left-hand

side of (C2) is decreasing from $\frac{a-\underline{q}}{q(\eta)} + \underline{\delta} - \delta$ at $\psi = \eta$ to $-\infty$ at $\psi = \eta + \frac{q(\eta)}{q'(\eta)}$, justifying our procedure for determining ψ .

We get μ_t^η from (15), μ_t^q from (EK), and μ_t^θ from (E). The expressions for $q''(\eta)$ and $\theta''(\eta)$ then follow directly from Ito's lemma and (15), the law of motion of η_t .

Finally, let us justify the five boundary conditions. First, because in the event that η_t drops to 0 experts are pushed to the solvency constraint and must liquidate any capital holdings to households, we have $q(0) = \underline{q}$. The households are willing to pay this price for capital if they have to hold it forever. Second, because η^* is defined as the point where experts consume, expert optimization implies that $\theta(\eta^*) = 1$ (see Proposition 1). Third and fourth, $q'(\eta^*) = 0$ and $\theta'(\eta^*) = 0$ are the standard boundary conditions at a reflecting boundary. If one of these conditions were violated, e.g., if $q'(\eta^*) < 0$, then any expert holding capital when $\eta_t = \eta^*$ would suffer losses at an infinite expected rate.³⁵ Likewise, if $\theta'(\eta^*) < 0$, then the drift of $\theta(\eta_t)$ would be infinite at the moment when $\eta_t = \eta^*$, contradicting Proposition 1. Fifth, if η_t ever reaches 0, it becomes absorbed there. If any expert had an infinitesimal amount of capital at that point, he would face a permanent price of capital of \underline{q} . At this price, he is able to generate the return on capital of

$$\frac{a - \iota(\underline{q})}{\underline{q}} + \Phi(\iota(\underline{q})) - \delta > r$$

without leverage, and arbitrarily high return with leverage. In particular, with high enough leverage this expert can generate a return that exceeds his rate of time preference ρ , and since he is risk-neutral he can attain infinite utility. It follows that $\theta(0) = \infty$.

Note that we have five boundary conditions required to solve a system of two second-order ordinary differential equations with an unknown boundary η^* .

PROOF OF PROPOSITION III.2:

Since $q'(\eta^*) = \theta'(\eta^*) = 0$, the drift and volatility of η at η^* are given by

$$\mu_t^\eta(\eta^*)\eta^* = (1 - \eta^*)\sigma^2 + \frac{a - \iota(q(\eta^*))}{q(\eta^*)}\eta^* > 0 \quad \text{and} \quad \sigma_t^\eta(\eta^*)\eta^* = (1 - \eta^*)\sigma.$$

Hence, $D'(\eta^*) = 2\mu_t^\eta(\eta^*)\eta^*/(\sigma_t^\eta(\eta^*)\eta^*)^2 D(\eta^*) > 0$, where $D(\eta) = d(\eta)(\sigma_t^\eta(\eta)\eta)^2$. Furthermore, because in the neighborhood of η^* ,

$$\sigma_t^\eta(\eta)\eta = \frac{(1 - \eta)\sigma}{1 - (1 - \eta)q'(\eta)/q(\eta)}$$

is *decreasing* in η , it follows that the density $d(\eta)$ must be *increasing* in η .

³⁵To see intuition behind this result, if $\eta_t = \eta^*$ then $\eta_{t+\epsilon}$ is approximately distributed as $\eta^* - \bar{\omega}$, where $\bar{\omega}$ is the absolute value of a normal random variable with mean 0 and variance $(\sigma_t^\eta)^2 \epsilon$. As a result, $\eta_{t+\epsilon} \sim \eta^* - \sigma_t^\eta \sqrt{\epsilon}$, so $q(\eta^*) - q'(\eta^*)\sigma_t^\eta \sqrt{\epsilon}$. Thus, the loss per unit of time ϵ is $q'(\eta^*)\sigma_t^\eta \sqrt{\epsilon}$, and the average rate of loss is $q'(\eta^*)\sigma_t^\eta/\sqrt{\epsilon} \rightarrow \infty$ as $\epsilon \rightarrow 0$.

The dynamics near $\eta = 0$ are more difficult to characterize because of the singularity there. We will do that by conjecturing, and then verifying, that asymptotically as $\eta \rightarrow 0$,

$$\mu_t^\eta = \hat{\mu} + o(1) \quad \text{and} \quad \sigma_t^\eta = \hat{\sigma} + o(1),$$

i.e., η_t evolves as a geometric Brownian motion, and that

$$\psi(\eta) = C_\psi \eta + o(\eta), \quad q(\eta) = \underline{q} + C_q \eta^\alpha + o(\eta^\alpha), \quad \text{and} \quad \theta(\eta) = C_\theta \eta^{-\beta} + o(\eta^{-\beta})$$

for some constants $C_\psi > 1$, $C_q, C_\theta > 0$, $\alpha, \beta \in (0, 1)$. We need to verify that the equilibrium equations hold, up to terms of smaller order. Using the equations of Proposition II.4, we have

$$\sigma_t^\eta = \frac{(C_\psi - 1)\sigma + o(1)}{1 - O(\eta^\alpha)} \Rightarrow \hat{\sigma} = (C_\psi - 1)\sigma,$$

$$\sigma_t^q = \frac{\alpha C_q \eta^\alpha}{\underline{q}} \hat{\sigma} + o(\eta^\alpha) = o(1), \quad \sigma_t^\theta = -\beta \hat{\sigma} + o(1),$$

$$(C3) \quad (17) \Rightarrow \beta \hat{\sigma} \sigma = \Lambda \Rightarrow \hat{\sigma} = (C_\psi - 1)\sigma = \frac{\Lambda}{\beta \sigma} \quad \text{and}$$

$$\hat{\mu} = -\hat{\sigma}(\sigma - \beta \hat{\sigma}) + \frac{a - \iota(\underline{q})}{\underline{q}} + \underline{\delta} - \delta = -\frac{\Lambda}{\beta \sigma} \left(\sigma - \frac{\Lambda}{\sigma} \right) + \frac{a - \iota(\underline{q})}{\underline{q}} + \Lambda.$$

We can determine μ_t^q from the household valuation equation

$$\frac{a - \iota(q_t)}{q_t} + \Phi(\iota(q_t)) - \underline{\delta} + \mu_t^q + \sigma \sigma_t^q = r$$

instead of that of experts, because we already took into account (17). By the envelope theorem,

$$\frac{a - \iota(q(\eta))}{q(\eta)} + \Phi(q(\eta)) - \underline{\delta} = \underbrace{\frac{a - \iota(\underline{q})}{\underline{q}} + \Phi(\underline{q}) - \underline{\delta}}_r - \frac{a - \iota(\underline{q})}{\underline{q}^2} (C_q \eta^\alpha + o(\eta^\alpha)) + o(\eta^\alpha).$$

Therefore,

$$\mu_t^q = \frac{a - \iota(\underline{q})}{\underline{q}^2} C_q \eta^\alpha - \frac{\alpha C_q \eta^\alpha}{\underline{q}} \hat{\sigma} \sigma + o(\eta^\alpha).$$

Our conjecture is valid if equations

$$\mu^q q(\eta) = q'(\eta) \mu_t^\eta \eta + \frac{1}{2} q''(\eta) (\sigma_t^\eta \eta)^2 \quad \text{and} \quad \mu^\theta \theta(\eta) = \theta'(\eta) \mu_t^\eta \eta + \frac{1}{2} \theta''(\eta) (\sigma_t^\eta \eta)^2$$

hold up to higher-order terms of $o(\eta^\alpha)$ and $o(\eta^{-\beta})$, respectively. Ignoring those terms, we need

$$\frac{\underline{a} - \iota(\underline{q})}{\underline{q}} C_q \eta^\alpha - \alpha C_q \eta^\alpha \hat{\sigma} \sigma = \alpha C_q \eta^\alpha \hat{\mu} + \frac{1}{2} \alpha (\alpha - 1) C_q \eta^\alpha \hat{\sigma}^2 \quad \text{and}$$

$$(C4) \quad (\rho - r) C_\theta \eta^{-\beta} = -\beta C_\theta \eta^{-\beta} \hat{\mu} + \frac{1}{2} \beta (\beta + 1) C_\theta \eta^{-\beta} \hat{\sigma}^2.$$

These equations lead to

$$\frac{\underline{a} - \iota(\underline{q})}{\underline{q}} - \alpha \frac{\Lambda}{\beta} = \alpha \left(-\frac{\Lambda}{\beta \sigma} \left(\sigma - \frac{\Lambda}{\sigma} \right) + \frac{\underline{a} - \iota(\underline{q})}{\underline{q}} + \Lambda \right) + \frac{1}{2} \alpha (\alpha - 1) \frac{\Lambda^2}{\beta^2 \sigma^2} \quad \text{and}$$

$$\rho - r = -\beta \left(-\frac{\Lambda}{\beta \sigma} \left(\sigma - \frac{\Lambda}{\sigma} \right) + \frac{\underline{a} - \iota(\underline{q})}{\underline{q}} + \Lambda \right) + \frac{1}{2} \beta (\beta + 1) \frac{\Lambda^2}{\beta^2 \sigma^2} \quad \Rightarrow$$

$$(C5) \quad \alpha \left(\frac{\Lambda^2}{\beta \sigma^2} + \Lambda \right) + \frac{\underline{a} - \iota(\underline{q})}{\underline{q}} (\alpha - 1) + \frac{1}{2} \alpha (\alpha - 1) \frac{\Lambda^2}{\beta^2 \sigma^2} = 0 \quad \text{and}$$

$$(C6) \quad \rho - r = \Lambda - \beta \left(\frac{\underline{a} - \iota(\underline{q})}{\underline{q}} + \Lambda \right) + \frac{(1 - \beta) \Lambda^2}{2 \beta \sigma^2}$$

We can solve for β , C_ψ , and α in the following order. First, equation (C6) has a solution $\beta \in (0, 1)$. To see this, note that, as $\beta \rightarrow 0$ from above, the right-hand side of (C6) converges to infinity. For $\beta = 1$, the right-hand side becomes

$$-\frac{\underline{a} - \iota(\underline{q})}{\underline{q}} < 0.$$

We have $\underline{a} - \iota(\underline{q}) > 0$, since the net rate of output that households receive at $\eta = 0$ must be positive. Second, equation (C3) determines the value of $C_\psi > 1$ for any $\beta > 0$. Lastly, equation (C5) has a solution $\alpha \in (0, 1)$. To see this, note that the left-hand side is negative when $\alpha = 0$ and positive when $\alpha = 1$.

This confirms our conjecture about the asymptotic form of the equilibria near $\eta = 0$. Arbitrary values of constants C_q and C_θ are consistent with these asymptotic dynamics. The value of C_q has to be chosen to ensure that functions $q(\eta)$ and $\theta(\eta)$ reach slope 0 at the same point η^* , and the C_θ , to ensure that $\theta(\eta^*) = 1$.

We are now ready to characterize the asymptotic form of the stationary distri-

bution near $\eta = 0$. We have $D'(\eta) = 2\hat{\mu}/\hat{\sigma}^2 D(\eta)/\eta$, so

$$(C7) \quad D(\eta) = C_D \eta^{2\hat{\mu}/\hat{\sigma}^2} \quad \text{and} \quad d(\eta) = D(\eta)/(\hat{\sigma}\eta)^2 = C_d \eta^{2\hat{\mu}/\hat{\sigma}^2 - 2}.$$

Equation (C4) implies that

$$\frac{2(\rho - r)}{\hat{\sigma}^2} = -\beta \frac{2\hat{\mu}}{\hat{\sigma}^2} + \beta(\beta + 1) \quad \Rightarrow \quad \frac{2\hat{\mu}}{\hat{\sigma}^2} - 2 = \beta - 1 - \frac{2(\rho - r)}{\Lambda^2} \sigma^2 \beta.$$

We see that $2\hat{\mu}/(\hat{\sigma}^2) - 2 < 0$, and so $d(\eta) = C_d \eta^{2\hat{\mu}/\hat{\sigma}^2 - 2} \rightarrow \infty$ as $\eta \rightarrow 0$. Furthermore, if

$$\frac{2\hat{\mu}}{\hat{\sigma}^2} - 2 > -1 \quad \Leftrightarrow \quad 1 - \frac{2(\rho - r)}{\Lambda^2} \sigma^2 > 0 \quad \Leftrightarrow \quad 2(\rho - r) \sigma^2 < \Lambda^2,$$

then the stationary density exists and has a hump near $\eta = 0$. Otherwise, if $2(\rho - r) \sigma^2 \geq \Lambda^2$, then the integral of $d(\eta)$ is infinity, implying that the stationary density does not exist and in the long run η_t ends up in an arbitrarily small neighborhood of 0 with probability close to 1.

PROOF OF PROPOSITION III.3:

Boundary conditions (23) as well as equation (24) follow from the market-clearing condition for consumption goods,

$$(C8) \quad r(q_t K_t - N_t) + \rho N_t = (\psi a + (1 - \psi)\underline{a} - \iota(q_t)) K_t.$$

Furthermore, since the volatilities of expert and household net worths are $\frac{\psi_t}{\eta_t}(\sigma + \sigma_t^q)$ and $\frac{1 - \psi_t}{1 - \eta_t}(\sigma + \sigma_t^q)$, respectively, the portfolio optimization conditions imply that

$$E[dr_t^k - dr_t]/dt = \frac{\psi_t}{\eta_t}(\sigma + \sigma_t^q)^2 \quad \text{and} \quad E[dr_t^k - dr_t]/dt \leq \frac{1 - \psi_t}{1 - \eta_t}(\sigma + \sigma_t^q)^2, \quad \text{with equality if } \psi_t < 1.$$

As $E[dr_t^k - dr_t^k]/dt = (a - \underline{a})/q_t + \underline{\delta} - \delta$, these two conditions together imply that, when $\psi_t < 1$,

$$(C9) \quad \frac{a - \underline{a}}{q_t} + \underline{\delta} - \delta = \left(\frac{\psi_t}{\eta_t} - \frac{1 - \psi_t}{1 - \eta_t} \right) (\sigma + \sigma_t^q)^2.$$

This leads to the first equation in (25). The second equation in (25) holds because, as in the risk-neutral model,

$$\sigma + \sigma_t^q = \frac{\sigma}{1 - (\psi_t - \eta_t)q'(\eta)/q(\eta)}.$$

Equations (26) hold because, by Lemma II.3,

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t} (dr_t^k - r dt - (\sigma + \sigma_t^q)^2 dt) + \frac{a - \iota(q_t)}{q_t} dt + (1 - \psi_t)(\underline{\delta} - \delta) dt - \rho dt.$$

LEMMA C.1: *Under the logarithmic utility model, the stationary density exists if*

$$2\sigma^2(\Lambda + r - \rho) + \Lambda^2 > 0$$

and has a hump at 0 if also $\rho > r + \Lambda$, where $\Lambda = (a - \underline{a})/q(0) + \underline{\delta} - \delta$.

PROOF:

Note that asymptotically $\sigma_t^q \rightarrow 0$ as $\eta \rightarrow 0$. Thus, from equation (C9),

$$\psi_t = \eta_t \frac{\Lambda}{\sigma^2} + o(\eta_t).$$

Therefore, equation (26) implies that

$$\sigma_t^\eta = \frac{\Lambda}{\sigma} + o(1) \quad \text{and} \quad \mu_t^\eta = (\sigma_t^\eta)^2 + \Lambda + r - \rho + o(1).$$

The Kolmogorov forward equation (see (C7)) implies that asymptotically the stationary density of η_t takes the form

$$d(\eta) = C_d \eta^{\beta_d}, \quad \text{where} \quad \beta_d = 2 \left(\frac{\mu_t^\eta}{(\sigma_t^\eta)^2} - 1 \right) = 2\sigma^2 \frac{\Lambda + r - \rho}{\Lambda^2}.$$

Thus, unlike in the risk-neutral case, the stationary density is nonsingular if $2\sigma^2(\Lambda + r - \rho) + \Lambda^2 > 0$ and has a hump at 0 if $\rho > r + \Lambda$.

PROOF OF PROPOSITION IV.1:

From the proof of Lemma C.1,

$$\psi_t = \eta_t \frac{\Lambda}{\sigma^2} + o(\eta_t)$$

under logarithmic utility. Under risk neutrality,

$$\psi_t = C_\psi \eta_t + o(\eta_t), \quad \text{where} \quad C_\psi = \frac{\Lambda}{\beta \sigma^2} + 1.$$

The variable β is determined by equation (C6), which implies that $\beta = 1 + O(\sigma^2)$ when σ is small. Thus,

$$\psi_t = \eta_t \left(\frac{\Lambda}{\sigma^2} + O(1) \right) + o(\eta_t).$$

In both cases,

$$\sigma_t^\eta = \frac{\psi_t - \eta_t}{\eta_t} (\sigma + \sigma_t^q),$$

and $\sigma_t^q \rightarrow 0$ as $\eta \rightarrow 0$. Thus,

$$\sigma_t^\eta = \frac{\Lambda}{\sigma} + O(\sigma)$$

as $\eta \rightarrow 0$.