

The General Equilibrium Incidence of Environmental Mandates

By Don Fullerton and Garth Heutel

Web Appendix

Appendix OA1: Finding the Substitution Elasticities b_{ij}

The b_{ij} elasticities are evaluated from the production function in the dirty sector in a manner analogous to Allen (1938, p. 505-508). We are solving for the derivatives of input demands with respect to changes in either input prices or δ , the policy parameter. These input demand equations come from the firm's cost minimization problem, where the total quantity to be produced is exogenous. First consider a small change in the price of capital, dr . If we differentiate the production function with respect to r we get

$$Y_K \frac{dK_Y}{dr} + Y_L \frac{dL_Y}{dr} + Y_Z \frac{dZ}{dr} = \frac{dY}{dr} = 0,$$

where the last equation comes from the fact that total output demanded is exogenous and not a function of the rental rate.

The first order condition of the minimization problem with respect to the choice of K_Y is $\frac{P_Y}{1 - \delta Y_Z} Y_K = r$. Differentiate this equation with respect to r , multiply through by $\frac{1 - \delta Y_Z}{P_Y}$, and collect terms to get

$$\frac{Y_K}{P_Y} \frac{dp_Y}{dr} + [Y_{KK} + Y_K Y_{ZK} \xi] \frac{dK_Y}{dr} + [Y_{KL} + Y_K Y_{ZL} \xi] \frac{dL_Y}{dr} + [Y_{KZ} + Y_K Y_{ZK} \xi] \frac{dZ}{dr} = \frac{1 - \delta Y_Z}{P_Y},$$

where $\xi \equiv \frac{\delta}{1 - \delta Y_Z}$. Similarly, differentiate the next first order condition,

$\frac{P_Y}{1 - \delta Y_Z} Y_L = w$, with respect to r and rearrange to get

$$\frac{Y_L}{P_Y} \frac{dp_Y}{dr} + [Y_{LK} + Y_L Y_{ZK} \xi] \frac{dK_Y}{dr} + [Y_{LL} + Y_L Y_{ZL} \xi] \frac{dL_Y}{dr} + [Y_{LZ} + Y_L Y_{ZK} \xi] \frac{dZ}{dr} = 0.$$

Note that the right hand side of this equation is zero, since a change in r has no effect on w , which is exogenous to this input demand system. Finally, the policy constraint binds,

so $Z = \delta Y$. Since Y and δ are both exogenous variables in the input demand system, a change in r has no effect on their values. Hence, differentiating this equation with respect to r yields $\frac{dZ}{dr} = 0$.

Writing these four equations in matrix form allows use of Cramer's rule to evaluate the derivatives. This equation is

$$\begin{bmatrix} 0 & Y_K & Y_L & Y_Z \\ Y_K & Y_{KK} + Y_K Y_{ZK} \zeta & Y_{KL} + Y_K Y_{ZL} \zeta & Y_{KZ} + Y_K Y_{ZZ} \zeta \\ Y_L & Y_{LK} + Y_L Y_{ZK} \zeta & Y_{LL} + Y_L Y_{ZL} \zeta & Y_{LZ} + Y_L Y_{ZZ} \zeta \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{p_Y} \frac{dp_Y}{dr} \\ \frac{dK_Y}{dr} \\ \frac{dL_Y}{dr} \\ \frac{dZ}{dr} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1 - \delta Y_Z}{p_Y} \\ 0 \\ 0 \end{bmatrix}$$

Follow the notation of Allen (1938) and use F to denote the determinant of the bordered Hessian of the production function, and use F_{ij} to denote the cofactor of element i,j of that matrix. The determinant of the matrix of coefficients in the above equation simplifies to F_{ZZ} (the terms with ζ all cancel each other out). With an odd number of inputs, the assumption of constant returns to scale (linear homogeneity) implies that $F < 0$ and $F_{ZZ} > 0$. Using Cramer's rule, we solve for the derivatives of interest:

$$\frac{dK_Y}{dr} = \frac{-(Y_L)^2 (1 - \delta Y_Z)}{p_Y F_{ZZ}} < 0, \quad \frac{dL_Y}{dr} = \frac{Y_L Y_K (1 - \delta Y_Z)}{p_Y F_{ZZ}} > 0.$$

These signs indicate that $b_{KK} < 0$ and $b_{LK} > 0$, as we now show. The term $1 - \delta Y_Z$ is strictly positive for the following reason. The policy parameter $\delta = Z/Y$ is the inverse of *average* output per unit of Z . It is multiplied by Y_Z , the *marginal* output per unit of Z . Since production is constant returns to scale, the average output must exceed the marginal output, and hence $\delta Y_Z < 1$. Furthermore, both first derivatives of Y are positive, and $F_{ZZ} < 0$ as mentioned before. Thus $b_{KK} < 0$ and $b_{LK} > 0$.

We take the production function, the first order conditions for the cost minimization problem, and the binding constraint, and then we differentiate all, this time with respect to w . Writing these four equations in matrix form yields a similar system of equations:

$$\begin{bmatrix} 0 & Y_K & Y_L & Y_Z \\ Y_K & Y_{KK} + Y_K Y_{ZK} \zeta & Y_{KL} + Y_K Y_{ZL} \zeta & Y_{KZ} + Y_K Y_{ZZ} \zeta \\ Y_L & Y_{LK} + Y_L Y_{ZK} \zeta & Y_{LL} + Y_L Y_{ZL} \zeta & Y_{LZ} + Y_L Y_{ZZ} \zeta \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{p_Y} \frac{dp_Y}{dw} \\ \frac{dK_Y}{dw} \\ \frac{dL_Y}{dw} \\ \frac{dZ}{dw} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1 - \delta Y_Z}{p_Y} \\ 0 \end{bmatrix}.$$

The matrix of coefficients is the same as for dr above; the only difference is in which element of the vector of constants is nonzero. Here it is the element corresponding to the differentiation of the first order condition for labor input, since w is changing. Solving this system yields

$$\frac{dK_Y}{dw} = \frac{Y_K Y_L (1 - \delta Y_Z)}{p_Y F_{ZZ}} > 0, \quad \frac{dL_Y}{dw} = \frac{-(Y_K)^2 (1 - \delta Y_Z)}{p_Y F_{ZZ}} < 0.$$

These solutions can be used to evaluate the input demand elasticities.

$$b_r = b_{KK} - b_{LK} = \frac{r}{K_Y} \frac{dK_Y}{dr} - \frac{r}{L_Y} \frac{dL_Y}{dr} = \frac{r(1 - \delta Y_Z) Y_L}{p_Y F_{ZZ}} \left(-\frac{Y_L}{K_Y} - \frac{Y_K}{L_Y} \right) < 0$$

$$b_w = b_{KL} - b_{LL} = \frac{w}{K_Y} \frac{dK_Y}{dw} - \frac{w}{L_Y} \frac{dL_Y}{dw} = \frac{w(1 - \delta Y_Z) Y_K}{p_Y F_{ZZ}} \left(\frac{Y_L}{K_Y} + \frac{Y_K}{L_Y} \right) > 0.$$

We can substitute in the first order conditions $p_Y Y_K = r(1 - \delta Y_Z)$ and $p_Y Y_L = w(1 - \delta Y_Z)$ to simplify these expressions.

$$b_r = -\frac{Y_K Y_L}{F_{ZZ}} \left(\frac{Y_L}{K_Y} + \frac{Y_K}{L_Y} \right), \quad b_w = \frac{Y_K Y_L}{F_{ZZ}} \left(\frac{Y_L}{K_Y} + \frac{Y_K}{L_Y} \right).$$

This substitution demonstrates that $b_r = -b_w$.

Lastly, we want to find the derivatives of factor demands with respect to a change in the policy parameter δ . Again, differentiate the production function and the first order conditions, here with respect to δ . The policy constraint ($Z = \delta Y$) differentiated with respect to δ yields $dZ/d\delta = Y$. The matrix form of this system of equations is

$$\begin{bmatrix} 0 & Y_K & Y_L & Y_Z \\ Y_K & Y_{KK} + Y_K Y_{ZK} \zeta & Y_{KL} + Y_K Y_{ZL} \zeta & Y_{KZ} + Y_K Y_{ZZ} \zeta \\ Y_L & Y_{LK} + Y_L Y_{ZK} \zeta & Y_{LL} + Y_L Y_{ZL} \zeta & Y_{LZ} + Y_L Y_{ZZ} \zeta \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{p_Y} \frac{dp_Y}{d\delta} \\ \frac{dK_Y}{d\delta} \\ \frac{dL_Y}{d\delta} \\ \frac{dZ}{d\delta} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{Y_Z Y_K}{1 - \delta Y_Z} \\ -\frac{Y_Z Y_L}{1 - \delta Y_Z} \\ Y \end{bmatrix}.$$

Again the matrix of coefficients is the same, with determinant F_{ZZ} . Solving for the derivatives of interest yields:

$$\frac{dK_Y}{d\delta} = Y \frac{F_{KZ}}{F_{ZZ}}, \quad \frac{dL_Y}{d\delta} = Y \frac{F_{LZ}}{F_{ZZ}},$$

where again F_{ij} denotes the cofactor of element i,j in the bordered Hessian of the production function. These cofactors are not immediately interpretable, but they are an integral part of the definition of the Allen elasticities. They are defined as:

$$e_{ij} \equiv \frac{p_Y Y}{i_Y j_Y} \cdot \frac{F_{ij}}{F}, \quad \text{where } i_Y \text{ is the quantity of input } i \text{ used. With these definitions we can}$$

calculate the remaining input demand elasticities:

$$b_\delta = b_{KZ} - b_{LZ} = \frac{\delta}{K_Y} \left(Y \frac{F_{KZ}}{F_{ZZ}} \right) - \frac{\delta}{L_Y} \left(Y \frac{F_{LZ}}{F_{ZZ}} \right) = \frac{\delta Z}{p_Y} \frac{F}{F_{ZZ}} (e_{KZ} - e_{LZ}),$$

where e_{ij} is the Allen elasticity of substitution between inputs i and j . Since $F/F_{ZZ} < 0$, the sign of b_δ is opposite the sign of $e_{KZ} - e_{LZ}$; if capital is a better substitute for pollution than is labor, then b_δ is negative.

Appendix OA2: Finding the Substitution Elasticities c_{ij}

We calculate these elasticities using a method similar to the one in Appendix OA1. First, consider the effect of small changes in the capital rental rate. If we differentiate the production function with respect to r we get, as before:

$$Y_K \frac{dK_Y}{dr} + Y_L \frac{dL_Y}{dr} + Y_Z \frac{dZ}{dr} = \frac{dY}{dr} = 0$$

The first order condition from the maximization problem with respect to capital is $r = p_Y(Y_K + \zeta Y_Z)$. Differentiate this with respect to r , divide through by p_Y , and rearrange terms to get:

$$\frac{Y_K + \zeta Y_Z}{p_Y} \frac{dp_Y}{dr} + [Y_{KK} + \zeta Y_{ZK}] \frac{dK_Y}{dr} + [Y_{KL} + \zeta Y_{ZL}] \frac{dL_Y}{dr} + [Y_{KZ} + \zeta Y_{ZK}] \frac{dZ}{dr} = \frac{1}{p_Y}.$$

The first order condition for labor is $w = p_Y Y_L$. Differentiating this equation by r and similarly rearranging yields

$$\frac{Y_L}{p_Y} \frac{dp_Y}{dr} + Y_{LK} \frac{dK_Y}{dr} + Y_{LL} \frac{dL_Y}{dr} + Y_{LZ} \frac{dZ}{dr} = 0.$$

Finally, differentiate the policy constraint $Z = \zeta K_Y$ by r to obtain

$$\frac{dZ}{dr} = \zeta \frac{dK_Y}{dr}.$$

Combining these four equations into matrix form allows us to solve for any of the derivatives. This matrix equation is

$$\begin{bmatrix} 0 & Y_K & Y_L & Y_Z \\ Y_K + \zeta Y_Z & Y_{KK} + \zeta Y_{ZK} & Y_{KL} + \zeta Y_{ZL} & Y_{KZ} + \zeta Y_{ZZ} \\ Y_L & Y_{LK} & Y_{LL} & Y_{LZ} \\ 0 & \zeta & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{p_Y} \frac{dp_Y}{dr} \\ \frac{dK_Y}{dr} \\ \frac{dL_Y}{dr} \\ \frac{dZ}{dr} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/p_Y \\ 0 \\ 0 \end{bmatrix}.$$

We solve for these derivatives using Cramer's Rule, where the denominator is the determinant of the matrix of coefficients. Call this denominator D . Solving along the bottom row, and using known properties of determinants, we get:

$$D = \zeta(F_{KZ} + \zeta(-F_{KK})) - (F_{ZZ} + \zeta(-F_{KZ})) = -\zeta^2 F_{KK} - F_{ZZ} + 2\zeta F_{KZ},$$

where the F_{ij} notation is from Allen (1938), just as in the previous section. We can solve for this denominator in terms of the Allen elasticities using their definitions:

$$D = -\zeta^2 \frac{F e_{KK} K_Y^2}{p_Y Y} - \frac{F e_{ZZ} Z^2}{p_Y Y} + 2\zeta \frac{F e_{KZ} K_Y Z}{p_Y Y}.$$

And, since $\zeta = Z/K_Y$,

$$D = \frac{FZ^2}{p_Y Y} (-e_{KK} - e_{ZZ} + 2e_{KZ}).$$

We can sign the denominator with information about these three Allen elasticities. The ratio in the front of this expression is negative, since $F < 0$ and all of the other constants are positive. The own-price elasticities e_{KK} and e_{ZZ} must be negative. Hence, D is negative if and only if e_{KZ} is not too negative:

$$\text{Condition 1: } e_{KZ} > \frac{e_{KK} + e_{ZZ}}{2}.$$

Since the right hand side of this inequality is strictly negative, a sufficient condition for D to be negative is capital and pollution are substitutes in production ($e_{KZ} > 0$).

However, D is still negative if K and Z are not too complementary.

We now use Cramer's Rule to solve for the derivatives.

$$\frac{dK_Y}{dr} = \frac{1}{D} \frac{Y_L^2}{p_Y}, \quad \frac{dL_Y}{dr} = -\frac{1}{D} \frac{Y_L(Y_K + \zeta Y_Z)}{p_Y}.$$

When $D < 0$, then $dK_Y/dr < 0$ and $dL_Y/dr > 0$. We can also use Cramer's rule to solve for dZ/dr , but differentiation of the policy constraint provides it as a function of dK_Y/dr .

Now, we solve for the elasticities c_{KK} and c_{LK} , and the difference (which is defined as c_r in the text):

$$c_r \equiv c_{KK} - c_{LK} \equiv \frac{r}{K_Y} \frac{dK_Y}{dr} - \frac{r}{L_Y} \frac{dL_Y}{dr} = \frac{r}{K_Y} \frac{Y_L^2}{D p_Y} + \frac{r}{L_Y} \frac{Y_L(Y_K + \zeta Y_Z)}{D p_Y} = \frac{r Y_L}{D p_Y} \left(\frac{Y_L}{K_Y} + \frac{Y_K + \zeta Y_Z}{L_Y} \right)$$

The sign of c_r is thus equal to the sign of D .

The same method is used to solve for the derivatives with respect to w and ζ .

Differentiating the four equations with respect to w yields:

$$\begin{bmatrix} 0 & Y_K & Y_L & Y_Z \\ Y_K + \zeta Y_Z & Y_{KK} + \zeta Y_{ZK} & Y_{KL} + \zeta Y_{ZL} & Y_{KZ} + \zeta Y_{ZZ} \\ Y_L & Y_{LK} & Y_{LL} & Y_{LZ} \\ 0 & \zeta & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{p_Y} \frac{dp_Y}{dw} \\ \frac{dK_Y}{dw} \\ \frac{dL_Y}{dw} \\ \frac{dZ}{dw} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/p_Y \\ 0 \end{bmatrix}.$$

The denominator again is D . Solving for the derivatives gives:

$$\frac{dK_Y}{dw} = -\frac{1}{D} \frac{Y_L(Y_K + \zeta Y_Z)}{p_Y}, \quad \frac{dL_Y}{dw} = \frac{1}{D} \frac{(Y_K + \zeta Y_Z)^2}{p_Y}.$$

So if $D < 0$, then $dK_Y/dw > 0$ and $dL_Y/dw < 0$. This gives us an expression for c_w :

$$c_w \equiv c_{LK} - c_{LL} = \frac{w}{K_Y} \frac{-Y_L(Y_K + \zeta Y_Z)}{D p_Y} - \frac{w}{L_Y} \frac{(Y_K + \zeta Y_Z)^2}{D p_Y} = -\frac{w(Y_K + \zeta Y_Z)}{D p_Y} \left(\frac{Y_L}{K_Y} + \frac{(Y_K + \zeta Y_Z)}{L_Y} \right)$$

The sign of c_w is the opposite of the sign of D .

Finally, we differentiate the four equations with respect to ζ to generate:

$$\begin{bmatrix} 0 & Y_K & Y_L & Y_Z \\ Y_K + \zeta Y_Z & Y_{KK} + \zeta Y_{ZK} & Y_{KL} + \zeta Y_{ZL} & Y_{KZ} + \zeta Y_{ZZ} \\ Y_L & Y_{LK} & Y_{LL} & Y_{LZ} \\ 0 & \zeta & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{p_Y} \frac{dp_Y}{d\zeta} \\ \frac{dK_Y}{d\zeta} \\ \frac{dL_Y}{d\zeta} \\ \frac{dZ}{d\zeta} \end{bmatrix} = \begin{bmatrix} 0 \\ -Y_Z \\ 0 \\ -K_Y \end{bmatrix}.$$

The difference on the right hand side comes from the fact that, when differentiating with respect to ζ , the term ζ can no longer be treated as a constant. For example, the policy constraint $Z = \zeta K_Y$ when differentiated yields $\frac{dZ}{d\zeta} = K_Y + \zeta \frac{dK_Y}{d\zeta}$, the bottom row of the matrix equation.

The denominator is the same as in earlier cases. Solving for the derivatives gives:

$$\frac{dK_Y}{d\zeta} = \frac{1}{D} (-Y_L^2 Y_Z - K_Y (F_{KZ} - \zeta F_{KK})),$$

$$\frac{dL_Y}{d\zeta} = \frac{1}{D} (Y_L Y_Z (Y_K + \zeta Y_Z) - K_Y (F_{LZ} - \zeta F_{KL})).$$

The first derivative above consists of two offsetting terms whenever capital and pollution are substitutes, since $D < 0$, $F_{KZ} < 0$, and $F_{KK} > 0$. Therefore, when policy is tightened and ζ falls, then demand for capital may fall or rise. The sign of the derivative of labor demand with respect to ζ is also ambiguous. It depends on both D and the relative magnitude of F_{KZ} and F_{LZ} , or e_{KZ} and e_{LZ} .

Solving for the elasticity $c_\zeta \equiv c_{ZK} - c_{ZL} \equiv \frac{\zeta}{K_Y} \frac{dK_Y}{d\zeta} - \frac{\zeta}{L_Y} \frac{dL_Y}{d\zeta}$, we get:

$$\begin{aligned} c_\zeta &= \frac{\zeta}{K_Y} \frac{-Y_L^2 Y_Z - K_Y (F_{KZ} - \zeta F_{KK})}{D} - \frac{\zeta}{L_Y} \frac{Y_L Y_Z (Y_K + \zeta Y_Z) - K_Y (F_{LZ} - \zeta F_{KL})}{D} \\ &= \frac{\zeta}{D} (-Y_L Y_Z (\frac{Y_L}{K_Y} + \frac{Y_K + \zeta Y_Z}{L_Y}) + \frac{F K_Y Z}{p_Y Y} (-e_{KZ} + e_{KK} + e_{LZ} - e_{KL})) \\ &= M + H(e_{LZ} - e_{KZ}) + H(e_{KK} - e_{KL}) \end{aligned}$$

The constants $M \equiv -\frac{\zeta}{D} (Y_L Y_Z (\frac{Y_L}{K_Y} + \frac{Y_K + \zeta Y_Z}{L_Y}))$ and $H \equiv \frac{\zeta F K_Y Z}{D p_Y Y}$ are both positive

when Condition 1 holds ($D < 0$), because $F < 0$ and all first derivatives of the production function Y_K , Y_L , and Y_Z are positive.

Finally, the text uses three relationships: $c_{KK} - c_{ZK} = 0$, $c_{KL} - c_{ZL} = 0$, and $c_{KZ} - c_{ZZ} = -1$. These can be verified using the derivations of the appropriate elasticities.