Calculation of a Population Externality: Online Appendix

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This appendix provides additional derivations and details. Claims made in the text and footnotes are stated here as **Propositions**. Additional results are stated as **Remarks**. **Lemmas** are intermediate results.

Throughout, parameters (θ, b) are admissible if they satisfy either $0 < \theta < 1$ and $0 < b < 1$ (Barro-Becker) or $\theta > 1$ and $b > 1$ (Jones-Schoonbroodt). Both imply $\omega = \frac{1-\theta}{1-b} > 0$. In some proofs, the transformation $\xi \equiv \frac{1}{\omega} - 1 = \frac{\theta-b}{1-\theta}$ will be convenient. Note that $\xi \geq 0$ if and only if $\omega = \frac{1}{1+\xi} \leq 1$.

A. *Power β (Section I)*

We claimed that power β is equivalent to assuming the utility an adult derives from grandchildren, $\beta(n_t)\beta(n_{t+1})U_{t+2}$, is independent of the number of children.

Proof: Independence implies $\beta(n_t)\beta(n_{t+1}) = \beta(1)\beta(n_t n_{t+1})$. Differentiating this with respect to n_t and n_{t+1} yields $n_t\beta'(n_t)/\beta(n_t) = n_{t+1}\beta'(n_{t+1})/\beta(n_{t+1})$ for any $n_t > 0$ and $n_{t+1} > 0$, so $n\beta'(n)/\beta(n)$ is a constant, denoted $1 - b$. The solution to the differential equation $\beta'(n)/\beta(n) = (1-b)/n$ is $\beta(n) = b_0 n^{1-b}$. Conversely, $\beta(n_t) = b_0 n_t^{1-b}$ implies $\beta(n_t)\beta(n_{t+1}) = b_0 n_t^{1-b} b_0 n_{t+1}^{1-b} = \beta(1)\beta(n_t n_{t+1})$. QED.

B. *The General Policy Problem (Sections I and VI)*

(1) We claimed in Section I-E that assumptions about (δ, Θ) are needed to obtain insightful results from the general problem (4). As one justification, note that dynamic programming yields very limited results unless the problem is concave, and that the return function $\beta(N_t)u(c_t)$ in (4) is not concave in $\mathbf{X}_t$ unless $\delta(\mathbf{X}_t)$ is convex. The latter needs to be assumed.

Failure of concavity/convexity is a serious concern, because Nordhaus-type damage specifications (e.g., Nordhaus and Boyer 2000), which are widely in the environmental literature, are inherently non-convex: if $\delta = \frac{\kappa_0 \cdot x^2}{1+\kappa_0 \cdot x^2}$ depends on temperature (x) with some parameter $\kappa_0 > 0$, $\delta(x)$ is convex for low x and concave for high x. To ensure concavity, one would need to restrict the domain of x, but this would require assumptions about Θ. Either way, assumptions about (δ, Θ) are needed.

(2) Note that the damage function $\delta(X_t)$ defined by (22) in Section VI-E is not convex. The derivation of $\delta(X_t)$ illustrates how non-convexity in the damage-to-temperature relation carries over into non-convexity of damages as function of greenhouse gas stocks. In Section VI-E, we sidestep the non-convexity problem by restricting the domain of (22) to $X_t \leq \hat{X}$ and by choosing parameters so that (22) is convex for all $X_t \leq \hat{X}$. Hence all optimization problems studied in Section VI-E are concave. We study population under assumptions S1-S3 in part to avoid such distracting technical complications.

(3) We claimed in Section I-E that the general problem (4) reduces to (5) when assumptions (S1-S3) are imposed and catastrophe is undesirable. To be precise, the claim that (4) reduces to (5) holds under the following conditions:

Proposition 1: Assume S1-S3. Assume (5) has solution V^* with an optimal policy correspondence H. For $\theta, b < 1$ assume $V^*(N_t) \geq \beta(N_t)u(f(e^+))$ for all N_t. Then (4) is solved by: (a) $V(N_t, \mathbf{X}_t) = V_{\mathcal{X}}$ for $\mathbf{X}_t \in \mathcal{X}$, where $V_{\mathcal{X}} = 0$ for $\theta, b < 1$ and $V_{\mathcal{X}} = -\infty$ for $\theta, b > 1$; and (b) $V(N_t, \mathbf{X}_t) = V^*(N_t)$ for $\mathbf{X}_t \notin \mathcal{X}$ with optimal policy $H \times E_t^*$, where $E_t^* = \min\{e^+ N_t, \hat{E}\}$.

Proof: (a) For $\mathbf{X}_t \in \mathcal{X}$, S1 implies $\delta(\mathbf{X}_t) = 1$, so $y_t = f(e_t)(1 - \delta(\mathbf{X}_t)) = 0$ for all (N_{t+1}, E_t), which implies $c_t = 0$ and $N_{t+i} = 0, c_{t+i} = 0 \forall i \geq 1$. For $\theta, b < 1$, $\beta(N)u(0) = \beta(0)u(c) = 0$, so $V = 0$. For $\theta, b > 1$, $\beta(N)u(0) = \beta(0)u(c) = -\infty$. Combining cases, (4) for $\mathbf{X}_t \in \mathcal{X}$ is solved by $V(N_t, \mathbf{X}_t) = V_{\mathcal{X}}$.

(b) For $\mathbf{X}_t \notin \mathcal{X}$, S3 implies $\delta(\mathbf{X}_t) = 0$, so the return function $v = v(N_t, N_{t+1}, E_t, \mathbf{X}_t) = \beta(N_t)u(f(\frac{E_t}{N_t})(1 - \delta(\mathbf{X}_t)) - \chi \frac{N_{t+1}}{N_t}) = \beta(N_t)u(f(\frac{E_t}{N_t}) - \chi \frac{N_{t+1}}{N_t})$ in (4) does not depend on $\mathbf{X}_t$. Since u, f are increasing, $\arg\max_{E_t \leq e^+ N_t}\{v\} = e^+ N_t$ and $\arg\max_{E_t \leq \hat{E}}\{v\} = \min\{\hat{E}, e^+ N_t\} = E_t^*$. To prove the recursion, suppose $V(N_{t+1}, \mathbf{X}_{t+1})$ has the claimed properties. Two cases arise:

(i) If $e^+ N_t \leq \hat{E}$, then $\mathbf{X}_{t+1} \notin \mathcal{X}$ for all E_t so $V(N_{t+1}, \mathbf{X}_{t+1}) = V^*(N_{t+1})$. By construction of V^*, $V(N_t, \mathbf{X}_t) = \max_{E_t, N_{t+1}}\{\upsilon + b_0 V(N_{t+1}, \mathbf{X}_{t+1})\} = \max_{N_{t+1}}\{\max_{E_t}\{\upsilon\} + bV^*(N_{t+1})\} = V^*(N_t)$ with policy $H \times E_t^*$, which proves the recursion for case (i).

(ii) If $e^+ N_t > \hat{E}$, consider the consequences of $E_t \in [0, \hat{E}]$ and $E_t \in (\hat{E}, e^+ N_t]$ separately. For $E_t \leq \hat{E}$, reasoning analogous to (i) implies $V_{E_t \leq \hat{E}} \equiv \max_{E_t \leq \hat{E}, N_{t+1}}\{\upsilon + b_0 V(N_{t+1}, \mathbf{X}_{t+1})\} = \max_{N_{t+1}}\{\max_{E_t \leq \hat{E}}\{\upsilon\} + bV^*(N_{t+1})\} = V^*(N_t)$. For $E_t > \hat{E}$, S2 implies $\mathbf{X}_{t+1} \in \mathcal{X}$, so $\upsilon + b_0 V(N_{t+1}, \mathbf{X}_{t+1}) = \upsilon + b_0 V_{\mathcal{X}}$. Note that $\max_{E_t > \hat{E}, N_{t+1}}\{\upsilon\} = \beta(N_t) u(f(e^+))$, so $V_{E_t > \hat{E}} \equiv \max_{E_t > \hat{E}, N_{t+1}}\{\upsilon + b_0 V(N_{t+1}, \mathbf{X}_{t+1})\} = \beta(N_t) u(f(e^+)) + b_0 V_{\mathcal{X}}$.

For $\theta, b < 1$, $V_{\mathcal{X}} = 0$ implies $V_{E_t > \hat{E}} = \beta(N_t) u(f(e^+))$, and assumption $V^*(N_t) \geq \beta(N_t) u(f(e^+))$ implies $V(N_t, \mathbf{X}_t) = \max\{V_{E_t \leq \hat{E}}, V_{E_t > \hat{E}}\} = V_{E_t \leq \hat{E}}$. For $\theta, b < 1$, $V_{\mathcal{X}} = -\infty$ implies $V_{E_t > \hat{E}} = -\infty$, so $V(N_t, \mathbf{X}_t) = \max\{V_{E_t \leq \hat{E}}, V_{E_t > \hat{E}}\} = V_{E_t \leq \hat{E}}$ requires no extra assumption. Thus $V(N_t, \mathbf{X}_t) = V_{E_t \leq \hat{E}} = V^*(N_t)$ apply for all (θ, b), and again $E_t^* = \min\{\hat{E}, e^+ N_t\}$, which proves the recursion for case (ii). In summary V^* solves (4) in all cases, and $H \times E_t^*$ attains the maximum. QED.

Remarks: Properties of V^* and H are examined in Section IV (proofs below). The point of Proposition 1 is to provide a motivation for problem (5) by showing that the population policy H that solves V^* is also an element of the solution to the more general problem (4). The caveat that catastrophe must be undesirable is non-trivial for $\theta, b < 1$, because $u(0) = 0$ places a lower bound on the utility of accepting catastrophic damages. Hence one must *assume* $V^*(N_t) \geq \beta(N_t) u(f(e^+))$ to rule out unbounded emissions and no children, which would yield utility $\beta(N_t) u(f(e^+))$. For $\theta, b > 1$, no such caveat is needed because $u(0) = -\infty$. The condition $V^*(N_t) \geq \beta(N_t) u(f(e^+))$ is easily verifiable numerically and holds in all cases studied in Section VI.

C. Existence and Uniqueness of n° (Section I)

We claimed that the first-order condition (7) defines a unique individually-rational fertility strictly between zero and y/χ.

Proof: The second-order condition, $U^\circ_{nn} = u''\chi^2 + \beta'' U < 0$, holds by assumptions on primitives (strict concavity of u, concave β and $U > 0$ for $\theta, b < 1$, convex β and $U < 0$

for $\theta, b > 1$). The marginal value of children ($\beta' U$) in (7) becomes infinite as $n \to 0$ and marginal cost of children ($u'\chi$) becomes infinite as $n \to y/\chi$ (so $c \to 0$). Therefore $U_n^\circ \to \infty$ as $n \to 0$ and $U_n^\circ \to -\infty$ as $n \to y/\chi$. Continuity of U_n° then implies that for any finite $y > 0$ and finite U, there is a unique individually-optimal fertility strictly between zero and y/χ. QED.

D. Existence and Uniqueness of n^+ (Section II)

We claimed that $S(n, e^+) = 0$ has a unique solution for n, where

$$S(n,e) = -u'(f(e) - \chi n)\chi + \frac{\beta'(n)}{1-\beta(n)} u(f(e) - \chi n). \tag{A.1}$$

Proof: As $n \to f(e^+)/\chi$, $u' \to \infty$, so $S(n, e^+) \to -\infty$. When $\theta < 1$, $\beta' \to \infty$ as $n \to 0$ so $S(n, e^+) \to \infty$. Because S is continuous, $S(n^+, e^+) = 0$ for some $n^+ \in (0, f(e^+)/\chi)$. When $\theta > 1$, $1/(1-\beta(n)) \to \infty$ as $n \to n^\circ$ from above (where $\beta(n^\circ) = 1$) so $S(n, e^+) \to \infty$. Because S is continuous, $S(n^+, e^+) = 0$ for some $n^+ \in (n^\circ, f(e^+)/\chi)$. (For $n < n^\circ$, $\beta(n) > 1$, so $S < 0$.) From (A.1), $\frac{\partial S}{\partial n} = u''\chi^2 - \frac{\beta' u' \chi}{1-\beta} + \frac{\beta''(1-\beta)-(\beta')^2}{1-\beta} u$, which reduces to $u''\chi^2 + \beta'' u$ at n such that $S = 0$. Because $\beta'' < 0$ and $u > 0$ when $\theta < 1$, and $\beta'' > 0$ and $u < 0$ when $\theta > 1$, $\frac{\partial S}{\partial n} < 0$. Hence S crosses zero only once. QED.

E. Population in the Cap Era (Section III)

Proposition 2A (Steady state exists and is unique): $S^\circ(e)$ crosses zero exactly once on $[f^{-1}(\chi), e^+]$.

Proof: Recall $S^\circ(e) = S(1, e)$. From (A.1), $S(1, e) \to -\infty$ as $e \to f^{-1}(\chi)$. Because S is continuous and $S(1, e^+) > 0$, there is at least one value on $(f^{-1}(\chi), e^+)$ at which $S(1, e) = 0$. The partial derivative of S is

$$\frac{\partial S(n,e)}{\partial e} = \left[-u''\chi + \frac{\beta'}{1-\beta} u'\right] f' = \left[-\frac{u''\chi}{u'} + \frac{\beta'}{1-\beta}\right] u' f'.$$

When $\theta < 1$ and $\beta' > 0$, we have $\partial S(1,e)/\partial e > 0$ for $e \in (f^{-1}(\chi), e^{+})$, so $S(1,e)$ crosses zero only once. When $\theta > 1$, power utility and $n = 1$ imply that

$$-\frac{u''\chi}{u'} + \frac{\beta'}{1-\beta} = \frac{\theta\chi}{f(e)-\chi} + \frac{\beta'(1)}{1-\beta(1)}$$

approaches $+\infty$ as $e \to f^{-1}(\chi)$ and decreases strictly in e. Thus either $\partial S(1,e)/\partial e$ is strictly positive for all $e \in (f^{-1}(\chi), e^{+})$, in which case $S(1,e)$ crosses zero only once, or $\partial S(1,e)/\partial e$ changes sign once at a value $e^{\circ} < e^{+}$, from positive for $e < e^{\circ}$ to negative for $e > e^{\circ}$. In the latter case, $S(1,e) \geq S(1,e^{+}) > 0$ for $e \geq e^{\circ}$ so $S(1,e)$ cannot cross zero on $[e^{\circ}, e^{+}]$, which means it crosses on $(f^{-1}(\chi), e^{\circ})$. Because $\partial S(1,e)/\partial e > 0$ on $(f^{-1}(\chi), e^{\circ})$, the crossing is unique. QED.

Proposition 2B (Convergence to steady state): Assume

$$\text{(A.2)} \qquad \varepsilon_{n_t,y_t}\left(\frac{f'(e_t)e_t}{f(e_t)}\right) < 1 \text{ where } \varepsilon_{n_t,y_t} = \left(\frac{b}{\theta}\frac{c_t}{f(e_t)} + \frac{\chi n_t}{f(e_t)}\right)^{-1}.$$

(a) (A.2) at the steady state is necessary and sufficient for the system of difference equations in (U, N) to have two strictly positive real roots that straddle one; and (b) (A.2) for all t along the saddle path implies that population converges to N_{ss} and that convergence is monotone.

Proof: (a) The characteristic roots of the system

$$\text{(A.3)} \qquad U_t = u(f(e(N_t)) - \chi\frac{N_{t+1}}{N_t}) + \beta(\frac{N_{t+1}}{N_t})U_{t+1},$$

$$\text{(A.4)} \qquad U_n^{\circ}(t) = \beta'\left(\frac{N_{t+1}}{N_t}\right)U_{t+1} - u'\left(f(e(N_t)) - \frac{N_{t+1}}{N_t}\chi\right)\chi = 0$$

are obtained by differentiating the system at (N_{ss}, U_{ss}). Differentiate $U_n^{\circ}(t)$ to obtain

$$-\beta'(n_t)dU_{t+1} - U_{nn}^{\circ}n_t\frac{dN_{t+1}}{N_{t+1}} = (1 + \frac{U_{ny}^{\circ}}{U_{nn}^{\circ}n_t}f'(e_t)e_t)(-U_{nn}^{\circ})n_t\frac{dN_t}{N_t}. \tag{A.5}$$

Define $z_{0t} \equiv -\frac{U_{ny}^{\circ}}{U_{nn}^{\circ}n_t}f'(e_t)e_t = \epsilon_{n,y}\frac{f'(e_t)e_t}{f(e_t)}$ and $z_{1t} \equiv 1 - z_{0t}$, where $z_{0t} = 0$ for $N_t \leq \hat{E}/e^+$ and $z_{0t} > 0$ for $N_t > \hat{E}/e^+$. Define $z_{2t} \equiv \frac{\beta'(n_t)}{(-U_{nn}^{\circ})n_t}$, which is positive when $\theta < 1$ and negative when $\theta > 1$. Then (A.5) can be written

$$-z_{2t}dU_{t+1} + \frac{dN_{t+1}}{N_{t+1}} = z_{1t}\frac{dN_t}{N_t}. \tag{A.6}$$

Similarly differentiate $U_t = u(f(e(N_t)) - \chi\frac{N_{t+1}}{N_t}) + \beta(\frac{N_{t+1}}{N_t}))U_{t+1}$ to obtain

$$\begin{aligned} dU_t &= u'(c_t)\left[-f'(e_t)e_t\frac{dN_t}{N_t} - \chi n_t^2(\frac{dN_{t+1}}{N_{t+1}} - \frac{dN_t}{N_t})\right] \\ &\quad +\beta(n_t)dU_{t+1} + U_{t+1}\beta'(n_t)n_t(\frac{dN_{t+1}}{N_{t+1}} - \frac{dN_t}{N_t}). \end{aligned}$$

Using (A.4), this can be written

$$\beta(n_t)dU_{t+1} = dU_t + z_{3t}\frac{dN_t}{N_t}, \tag{A.7}$$

where $z_{3t} \equiv u'(c_t)f'(e_t)e_t \geq 0$.

The system consisting of (A.6) and (A.7):

$$\begin{pmatrix} -z_{2t} & 1 \\ \beta(n_t) & 0 \end{pmatrix}\begin{pmatrix} dU_{t+1} \\ \frac{dN_{t+1}}{N_{t+1}} \end{pmatrix} = \begin{pmatrix} 0 & z_{1t} \\ 1 & z_{3t} \end{pmatrix}\begin{pmatrix} dU_t \\ \frac{dN_t}{N_t} \end{pmatrix},$$

has roots μ that satisfy the characteristic equation

$$\left|\begin{pmatrix} -z_{2t}\mu & \mu - z_{1t}) \\ \beta(n_t)\mu - 1 & -z_{3t} \end{pmatrix}\right| = 0,$$

which implies

$$\mu_{1t,2t} = \frac{1}{2}(Z_t + 1) \pm \sqrt{D_t},$$

where $z_{4t} = \frac{z_{2t}z_{3t}}{\beta(n_t)}$, $Z_t = \frac{1}{\beta(n_t)} + z_{4t} - z_{0t}$, and $D_t = \frac{1}{4}(Z_t + 1)^2 - \frac{z_{1t}}{\beta(n_t)}$. Because $(Z_t + 1)^2 = (Z_t - 1)^2 + 4Z_t$, it follows that

$$D_t = \frac{1}{4}(Z_t - 1)^2 + z_{4t} + \frac{(1 - \beta(n_t))}{\beta(n_t)} z_{0t}. \tag{A.8}$$

Moreover,

$$z_{4t} = \frac{z_{2t}z_{3t}}{\beta(n_t)} = \frac{\beta'(n_t)u' f' e_t}{(-V_{nn}n_t)\beta(n_t)} = \frac{\beta'(n_t)u'}{V_{ny}\beta(n_t)} z_{0t} = \frac{1-b}{\theta}\frac{f(e_t)/n_t - \chi}{\chi} z_{0t},$$

using $U^{\circ}_{ny} = -u''\chi = \frac{u'\chi}{\theta c_t}$. In steady state, $S(1, e_{ss}) = 0$ with power utility implies $(1-\theta)\chi = (1-b)\frac{\beta(1)}{1-\beta(1)}(f(e_{ss}) - \chi)$. Hence

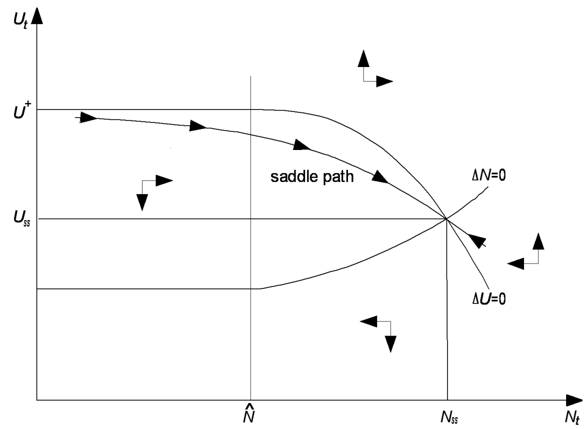
$$z_{4ss}\frac{\beta(1)}{1-\beta(1)} + z_{0ss} = \frac{z_{0ss}}{\theta\chi}[\chi\theta - (b-1)\frac{\beta(1)}{1-\beta(1)}(f(e_{ss}) - \chi)] = \frac{z_{0ss}}{\theta} > 0,$$

where $z_{0ss} > 0$ because $N_{ss} > \hat{E}/e^{+}$. Thus $D_{ss} = \frac{1}{4}(Z_{ss} - 1)^2 + \frac{(1-\beta(1))}{\beta(1)}\frac{z_{0ss}}{\theta} > 0$. This implies that both roots are real and that $\sqrt{D_{ss}} > \frac{1}{2}|Z_{ss} - 1|$. Hence $\mu_{1ss} = \frac{1}{2}(Z_{ss}+1) - \sqrt{D_{ss}} < \min(Z_{ss}, 1) \leq 1$ and $\mu_{2ss} = \frac{1}{2}(Z_{ss} + 1) + \sqrt{D_{ss}} > \max(Z_{ss}, 1) \geq 1$. Also, $z_{4ss} = \frac{1-\beta(1)}{\beta(1)}(\frac{z_{0ss}}{\theta} - z_{0ss})$, which implies

$$Z_{ss} = \frac{1}{\beta(1)} + z_{4ss} - z_{0ss} = \frac{z_{1ss}}{\beta(1)} + \frac{z_{0ss}}{\theta}\frac{1-\beta(1)}{\beta(1)} \geq \frac{z_{1ss}}{\beta(1)}.$$

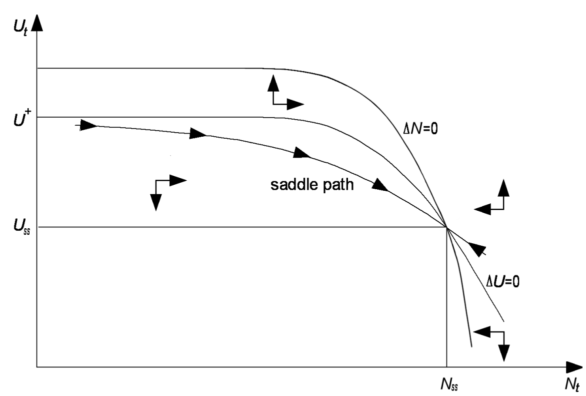
If (A.2) holds at the steady state, then $z_{1ss} > 0$ so $Z_{ss} > 0$, and $\sqrt{D_{ss}} < \frac{1}{2}(Z_{ss} + 1)$ so $\mu_{1ss} > 0$. Thus the roots μ_{1ss} and μ_{2ss} are real, strictly positive, and straddle one, proving sufficiency of (A.2). If (A.2) does not hold at the steady state, which means $\varepsilon_{n_t,y_t}\left(\frac{f'(e_t)e_t}{f(e_t)}\right) \geq 1$, then $z_{1ss} \leq 0$ and hence $\sqrt{D_{ss}} \geq \frac{1}{2}|Z_{ss}+1|$, which implies $\mu_{1ss} \leq 0$. Thus (A.2) is also necessary for strictly positive roots.

(b) Global convergence is conveniently established using phase diagrams. The phase diagram when $\theta < 1$ is:



Saddle-path stability and a unique steady state follow from Proposition 2A and Proposition 2B(a). Setting $U_{t+1} = U_t$ in (A.3) yields the phase arm labelled $\Delta U = 0$ that is horizontal at utility $U = U^+$ for $N \le \hat{N} \equiv \hat{E}/e^+$ and slopes downward for $N > \hat{E}/e^+$. Setting $N_{t+1} = N_t$ in (A.4) and using (A.3) to substitute out U_{t+1} yields the phase arm labelled $\Delta N = 0$ that is horizontal at utility $U < U^+$ for $N \le \hat{E}/e^+$ and cuts $\Delta U = 0$ from below at N_{ss}. (Note: for $N > \hat{E}/e^+$, the arm $\Delta N = 0$ can slope upward, as shown, or downward, but has less negative slope than $\Delta U = 0$, so the intersection is unique.) It is straightforward to show that the phase arrows are as shown, so there is a unique saddle path with strictly negative slope.

The phase diagram when $\theta > 1$ is:



The phase arm $\Delta U = 0$ has the same properties as when $\theta < 1$. However, because fertility declines in U, the arm $\Delta N = 0$ lies above $\Delta U = 0$ for $N \leq \hat{E}/e^{+}$, then declines and cuts $\Delta U = 0$ from above at N_{ss}. Inspecting the phase arrows, there is a unique saddle path with strictly negative slope that lies below $\Delta U = 0$ for $N < N_{ss}$ and above $\Delta U = 0$ for $N > N_{ss}$.

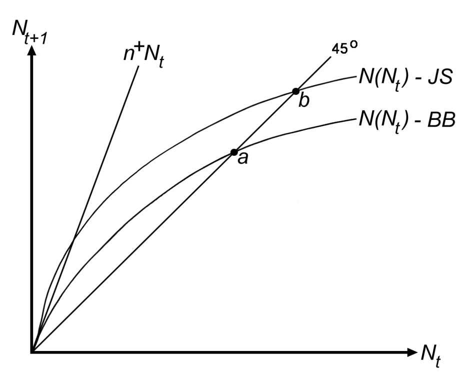
Let $U_t = U_{sp}(N_t)$ denote utilities on the saddle path and let $\nu(y, U)$ denote the fertility that solves the individual first-order condition $U_n^{\circ}(n, y, U) = 0$. Note that equilibrium fertility satisfies $\eta^{\circ}(N_t) = \nu(f(e(N_t)), U_{sp}(N(N_t))$ for all N_t, where $N_{t+1} = N(N_t) \equiv N_t\eta^{\circ}(N_t)$. By the implicit function theorem,

$$\frac{dN_{t+1}}{dN_t} = \eta^{\circ}(N(1 - \epsilon_{n,y}\frac{f'(e_t)e_t}{f(e_t)}) + \nu_U NU'(N)\frac{dN_{t+1}}{dN_t}, \text{ so}$$

$$\frac{dN_{t+1}}{dN_t} = \frac{\eta^{\circ}(N)}{1 - \nu_U N_t U'(N_t)}(1 - \epsilon_{n,y}\frac{f'(e_t)e_t}{f(e_t)}), \tag{A.9}$$

where saddle-path stability implies $1 - \nu_U N_t U'(N_t) > 0$. Hence (A.2) implies $N'(N_t) = \frac{dN_{t+1}}{dN_t} > 0$. Because N_{ss}° is stable and unique, $N(N_t)$ is upward sloping and crosses the 45-degree line at N_{ss} with slope $N'(N_{ss}) < 1$, which implies monotone convergence. QED.

Remark: The following diagram of N_t against N_{t+1} provides additional intuition about population dynamics and motivation for assumption (A.2):



Population dynamics in the no-cap era are on the ray with slope $n^+ > 1$. Steady states in the cap era are on the 45-degree line. The slope of $N(N_t)$ is given by (A.9).

When $\theta < 1$, $\nu_U > 0$ and $U'(N) < 0$ imply $0 < \frac{1}{1-N\nu_U U'(n)} < 1$. Hence $N(N_t)$ has an elasticity $\epsilon_{N_t N_{t+1}} \equiv N'(N_t)\frac{N_t}{N_{t+1}} < 1 - \epsilon_{n,y}\frac{f'(e_t)e_t}{f(e_t)} < 1$ for all N_t. The function starts out nearly parallel to the ray with slope n^+, then bends down until its meets the 45-degree line at point a. Assumption (A.2) implies $1 - \epsilon_{n,y}\frac{f'(e_t)e_t}{f(e_t)} > 0$, which ensures that $N'(N) > 0$ and hence the population dynamics obtained by iterating on $N_{t+1} = N(N_t)$ are monotone.

When $\theta > 1$, $\nu_U < 0$ and $U'(N) < 0$ imply $\frac{1}{1-N\nu_U U'(n)} > 1$. For $N_t \leq \hat{E}/e^+$, $\frac{f'e}{f} = 0$ implies $\epsilon_{N_t N_{t+1}} > 1$. Hence the $N(N_t)$ has a segment bending up above the uncapped path before bending down to meet the 45-degree line at b. Assumption (A.2) again implies $N'(N) > 0$ and hence monotone dynamics.

Remark on general (δ, Θ): We claimed at the end of Section III that a steady-state condition for e°_{ss} similar to $S^\circ = 0$ applies for general (δ, Θ), provided optimal policy leads to a steady state $(E^\circ_{ss}, \mathbf{X}^\circ_{ss})$.

Proof: From (1) and (7), a steady state requires $U^\circ_n(n, y, U) = 0, U = u(c)/(1-\beta(n))$ and $n = 1$. Hence $U^\circ_n(1, y, \frac{u(c)}{1-\beta(1)}) = -u'(y-\chi)\chi + \frac{b_0}{1-b_0}u(y-\chi) = 0$ applies for general (δ, Θ). If $\mathbf{X}^\circ_{ss}$ is constant, then $\delta^\circ_{ss} = \delta(\mathbf{X}^\circ_{ss})$ is constant, so $y = (1-\delta^\circ_{ss})f(e)$ depends only on e. Define

$$S^\circ(e|\delta) \equiv -u'((1-\delta)f(e)-\chi)\chi + \frac{b_0}{1-b_0}u((1-\delta)f(e)-\chi), \tag{A.10}$$

which is the same condition as in Section III except that f is scaled by $1-\delta$. By the same arguments that prove Proposition 2A, a root e°_{ss} exists and is unique. For given E°_{ss}, unique $N^\circ_{ss} = E^\circ_{ss}/e^\circ_{ss}$ follows. QED.

F. The Optimal Population (Section IV)

The claims in the text are proved as follows. Existence of a solution to the transformed problem is shown in Proposition 4(a). Necessity of $S^*(e^*_{ss}) = 0$ as the steady-state condition is in Proposition 6(a). For $\omega \leq 1$: properties of V^* and H are in Proposition 4(b); uniqueness of N^*_{ss} and e^*_{ss} are in Proposition 6(b); and sufficiency of (A.2) for convergence of the optimal population to N^*_{ss} is in Proposition 7(b). For $\omega > 1$, sufficient conditions for uniqueness of N^*_{ss} and e^*_{ss} are in Proposition 6(c); convergence of the optimal population to N^*_{ss} is in Proposition 7(b); and properties of V^* and H are in Proposition 7(c).

The proofs below use the Bellman equation (5). To be precise, define

$$\upsilon(N_t, N_{t+1}) = \beta(N_t) u\left(f(e(N_t)) - \chi \frac{N_{t+1}}{N_t} \right), \tag{A.11}$$

$N^{\max}(N_t) \equiv \frac{1}{\chi} N_t f((e(N_t))$ and $\mathcal{F}(N_t) \equiv [0, N^{\max}(N_t)] = [0, \frac{1}{\chi} N_t f((e(N_t))]$. Because $c_t \geq 0$ and $N_t \geq 0$ imply $N_{t+1} \leq N^{\max}(N_t)$, $\mathcal{F}(N_t)$ is the set of feasible choices for N_{t+1}. Then (5) can be written as

$$V^*(N_t) = \max_{N_{t+1} \in \mathcal{F}(N_t)} \{\upsilon(N_t, N_{t+1}) + b_0 V^*(N_{t+1})\}. \tag{A.12}$$

The optimal policy is denoted $H(N_t)$. As preliminary step to deriving properties of V^*, we show:

Lemma 3 (Properties of υ): (a) υ is twice continuously differentiable for all admissible (θ, b), except that $\frac{\partial^2 \upsilon}{\partial N_t^2}$ may have a discontinuity at $N_t = \hat{E}/e$.

(b) $\upsilon_{12}(N_t, N_{t+1}) = \frac{\partial^2 \upsilon(N_t, N_{t+1})}{\partial N_{t+1} \partial N_t} > 0$ if and only if (A.2) holds at (N_t, N_{t+1}), where $\varepsilon_{n_t, y_t} = \left(\frac{b}{\theta} \frac{f(e(N_t)) - \chi N_{t+1}/N_t}{f(e(N_t))} + \frac{\chi N_{t+1}/N_t}{f(e(N_t))} \right)^{-1}$.

(c) υ is strictly concave for $\omega < 1$ (or $\xi > 0$), and υ is concave for $\omega = 1$ (or $\xi = 0$).

(d) υ is strictly concave on the set $\mathbf{\Omega}_{con} \equiv \{(N_t, N_{t+1}) : \xi > -\varepsilon_{n_t, y_t} \varkappa(e(N_t))\}$, where $\varkappa(e) \equiv -\frac{f''(e) e^2}{f(e)} > 0$ for $e \in (0, e^+)$ and $\varkappa(e^+) \equiv 0$.

Remark: It would be technically convenient to restrict attention to (θ, b) such that $\omega = \frac{1-\theta}{1-b} \leq 1$ because, as shown below, this implies that V^* is strictly concave everywhere. The assumption $\omega \leq 1$ is restrictive, however, so we do not impose it. Instead, we rely on conditions involving the curvature of f and on the elasticity condition (A.2);

the latter is needed for convergence even if one assumes $\omega \leq 1$. The relevant measure of curvature is $\varkappa(e)$ in (d) above. Note that $\xi > -\varepsilon_{n_t,y_t}\varkappa(e)$ in (d) is equivalent to $\omega = \frac{1}{1+\xi} < \frac{1}{1-\varepsilon_{n_t,y_t}\varkappa(e)}$ where $\frac{1}{1-\varepsilon_{n_t,y_t}\varkappa(e)} > 1$ for $e < e^+$. Hence the condition in (d) covers values $\omega > 1$.

Proof: (a) Taking derivatives of υ:

$$\frac{\partial \upsilon}{\partial N_t} = [\xi c_t + w(e(N_t))]\, B(N_t, N_{t+1}),\ \frac{\partial \upsilon}{\partial N_{t+1}} = -\chi B(N_t, N_{t+1}),$$

where $B = B(N_t, N_{t+1}) \equiv \frac{\beta(N_t)}{N_t}u'(c_t) > 0$, $c_t = f(e(N_t)) - \chi N_{t+1}/N_t$, and $w(e_t) = f(e_t) - f'(e_t)e_t$. Note that $\frac{\partial \upsilon}{\partial N_t}$ is continuous at $N_t = \hat{E}/e^+$ because $\lim_{N_t \downarrow \hat{E}/e^+} w(e(N_t)) = f(e^+)$. Taking derivatives of $\frac{\partial \upsilon}{\partial N_{t+1}}$:

$$\begin{aligned} \frac{\partial^2 \upsilon}{\partial N_{t+1}\partial N_t} &= \chi \frac{\theta B}{N_t}\left(\frac{w_t}{c_t} - 1 + \frac{b}{\theta}\right), \\ \frac{\partial^2 \upsilon}{\partial N_{t+1}^2} &= -\chi^2 \frac{\theta B}{N_t c_t} < 0. \end{aligned} \tag{A.13}$$

To find $\frac{\partial^2 \upsilon}{\partial N_t^2}$, note that $\frac{dw(e(N_t))}{dN_t} = 0$ for $N_t < \hat{E}/e^+$ and $\frac{dw(e(N_t))}{dN_t} = -w'(e(N_t))\frac{e(N_t)}{N_t}$ for $N_t > \hat{E}/e^+$, where $w'(e) = \frac{\partial[f(e) - f'(e)e)]}{\partial e} = f'(e) - f''(e)e - f'(e) = -f''(e)e > 0$ for $e < e^+$. Because $\lim_{N_t \downarrow \hat{E}/e^+} \frac{dw(e(N_t))}{dN_t} \neq 0$ unless $\lim_{e \to e^+} f''(e) = 0$ (which we do not impose), $w(e(N_t))$ is not generally differentiable at $N_t = \hat{E}/e^+$. For $N_t \neq \hat{E}/e^+$, differentiating $\frac{\partial \upsilon}{\partial N_t}$ yields

$$\begin{aligned} \frac{\partial^2 \upsilon}{\partial N_t^2} &= -w'(e(N_t))\frac{e(N_t)}{N_t}B - \theta\frac{Bc_t}{N_t}\left[\left(\frac{w_t}{c_t} - 1 + \frac{b}{\theta}\right)^2 + \frac{b}{\theta^2}\xi\right] \\ &= \frac{B}{N_t}\left[\frac{bc_t}{\theta N_t}\xi - \frac{d}{dN_t}w(e(N_t))\right] - \theta\frac{Bc_t}{N_t}\left(\frac{w_t}{c_t} - 1 + \frac{b}{\theta}\right)^2, \end{aligned}$$

showing (a) by construction.

(b) This follows from (A.13) because $\chi\frac{\theta B}{N_t} > 0$ and because $\frac{w_t}{c_t} - 1 + \frac{b}{\theta} > 0$ is algebraically equivalent to (A.2).

(c) For $N_t \neq \hat{E}/e^+$:

$$Det(\upsilon) \equiv \frac{\partial^2 \upsilon}{\partial N_t^2}\frac{\partial^2 \upsilon}{\partial N_{t+1}^2} - \left(\frac{\partial^2 \upsilon}{\partial N_{t+1}\partial N_t}\right)^2 = \left[\frac{bc_t}{\theta N_t}\xi - \frac{dw(e(N_t))}{dN_t}\right]\left[\chi^2 \frac{\theta B^2}{N_t c_t}\right].$$

Note that $\frac{bc_t}{\theta N_t}\xi > -\frac{d}{dN_t}w(e(N_t))$ implies $Det(u^A) > 0$ and $\frac{\partial^2 u^A}{\partial N_t^2} < 0$. Hence u^A is strictly concave locally if $\frac{bc_t}{\theta N_t}\xi > -\frac{d}{dN_t}w(e(N_t))$.

For global concavity of u^A, a complication is that $\frac{d}{dN_t}w(e(N_t))$ may not exist at $N_t = \hat{E}/e^+$. To show concavity on a set, one must show that

$$\begin{aligned} & \lambda\upsilon(N_t^1, N_{t+1}^1) + (1-\lambda)\upsilon(N_t^2, N_{t+1}^2) \\ \leq \quad & \upsilon(\lambda N_t^1 + (1-\lambda)N_t^2, N_{t+1}^1 + (1-\lambda)N_{t+1}^2) \end{aligned}$$

for any $(N_t^1, N_{t+1}^1) \neq (N_t^2, N_{t+1}^2)$ in this set and for any $\lambda \in (0, 1)$, with strict inequality required for strict concavity.

Concavity of F and constant returns to scale imply that $N_t f(e(N_t)) = F(N_t, \min(e^+ N_t, \hat{E})$ is concave in N_t. For given (N_t^1, N_{t+1}^1) and (N_t^2, N_{t+1}^2) with $N_t^1 < N_t^2$, define

$$\bar{F}(N) \equiv N_t^1 f(e(N_t^1)) + \frac{N_t^2 f(e(N_t^2)) - N_t^1 f(e(N_t^1))}{N_t^2 - N_t^1}(N - N_t^1).$$

Note that $\bar{F}(N_t^1) = F(N_t^1)$, $\bar{F}(N_t^2) = F(N_t^2)$, and, by concavity of F, $\bar{F}(N) \leq Nf(e(N))$ for all $N \in [N_t^1, N_t^2]$. Also define

$$\bar{\upsilon}(N_t, N_{t+1}) \equiv \beta(N_t)u\left(\frac{1}{N_t}(\bar{F}(N_t) - \chi N_{t+1})\right),$$

and note that $\bar{\upsilon}(N_t^1, N_{t+1}^1) = \upsilon(N_t^1, N_{t+1}^1)$, $\bar{u}(N_t^2, N_{t+1}^2) = \upsilon(N_t^2, N_{t+1}^2)$, and

$$\bar{\upsilon}(\lambda N_t^1 + (1-\lambda)N_t^2, N_{t+1}^1 + (1-\lambda)N_{t+1}^2) \leq \upsilon(\lambda N_t^1 + (1-\lambda)N_t^2, N_{t+1}^1 + (1-\lambda)N_{t+1}^2).$$

Because $\bar{F}(N)$ is differentiable for all $N \in [N_t^1, N_t^2]$, $\bar{\upsilon}$ is twice continuously differentiable. Taking derivatives, $\frac{\partial^2 \bar{\upsilon}}{\partial N_t^2} = -\theta \frac{Bc_t}{N_t}\left[\left(\frac{w_t}{c_t} - 1 + \frac{b}{\theta}\right)^2 + \frac{b}{\theta^2}\xi\right]$ and $Det(\bar{\upsilon}) = \left[\frac{bc_t}{\theta N_t}\xi\right]\left[\chi^2 \frac{\theta B^2}{N_t c_t}\right]$, so $\bar{\upsilon}$ is concave everywhere for $\xi \geq 0$ and is strictly concave for $\xi > 0$. Hence for all $N \in [N_t^1, N_t^2]$ and all $\xi \geq 0$,

$$\begin{aligned} & \lambda \upsilon(N_t^1, N_{t+1}^1) + (1-\lambda)\upsilon(N_t^2, N_{t+1}^2) = \lambda \bar{\upsilon}(N_t^1, N_{t+1}^1) + (1-\lambda)\bar{\upsilon}(N_t^2, N_{t+1}^2) \\ \leq \quad & \bar{\upsilon}(\lambda N_t^1 + (1-\lambda)N_t^2, N_{t+1}^1 + (1-\lambda)N_{t+1}^2) \\ \leq \quad & \upsilon(\lambda N_t^1 + (1-\lambda)N_t^2, N_{t+1}^1 + (1-\lambda)N_{t+1}^2), \end{aligned}$$

where the first inequality follows from concavity of $\bar{\upsilon}$ and the second inequality follows from $\bar{F}(N) \leq Nf(e(N))$. Thus for all $\xi \geq 0$, υ is globally concave. For $\xi > 0$, the first inequality above is strict by strict concavity of $\bar{\upsilon}$, so υ is strictly concave. This proves (c).

(d) This is trivial for $\xi > 0$ because $-\varepsilon_{n_t,y_t}\varkappa(e(N_t)) \leq 0$. For $\xi \leq 0$, $\varkappa(e(N_t)) = 0$ for $N_t \leq \hat{E}/e^+$ implies $\Omega_{con} \subset \{(N_t, N_{t+1}) : N_t > \hat{E}/e^+\}$, so $\frac{d}{dN_t}w(e(N_t))$ exists. Note that $0 < \frac{b}{\theta}\frac{c_t}{f(e(N_t))} < \varepsilon_{n_t,y_t}^{-1}$, so $\xi > -\varepsilon_{n_t,y_t}\varkappa(e(N_t))$ implies $\xi > -\varkappa(e(N_t))/\left(\frac{b}{\theta}\frac{c_t}{f(e(N_t))}\right) = -\frac{d}{dN_t}w(e(N_t))/\left(\frac{bc_t}{\theta N_t}\right)$ and $\frac{bc_t}{\theta N_t}\xi > -\frac{d}{dN_t}w(e(N_t))$, which implies strict concavity of υ, proving (d). QED.

Proposition 4 (Value function V^*): In the optimal population problem of section IV:

(a) For any admissible (θ, b), there is a unique continuous function V^* that solves (A.12) for all $N_t \in (0, \infty)$. The associated optimal policy correspondence H is compact-valued and upper hemi-continuous (u.h.c.) on any compact subset of $(0, \infty)$. Moreover, H also solves the Bellman equation (15) for U^*.

(b) For $\omega \leq 1$, V^* is strictly concave and differentiable, and H is single-valued and continuous.

Remark: Because $N_0 = 0$ results trivially in a zero population sequence, we consider only cases with $N_t > 0$. Note that H is generally a correspondence; in the text we only consider cases where H reduces to a function.

Proof: (a) By construction, $\mathcal{F}(N_t)$ is non-empty, compact, and increasing in N_t for all N_t. Note that $N^{\max}(N_t) \leq \hat{E}/f^{-1}(\chi)$ for $N_t \leq \hat{E}/f^{-1}(\chi)$, and $N^{\max}(N_t) < N_t$ for $N_t > \hat{E}/f^{-1}(\chi)$, so population is bounded in $\mathbf{\Omega} = [0, \max(N_0, \hat{E}/f^{-1}(\chi))]$ for all t, and $\mathcal{F}(\mathbf{\Omega}) \subset \mathbf{\Omega}$ is compact. We consider the cases with $\theta < 1$ and $\theta > 1$ separately because when $\theta > 1$, υ is unbounded as $c_t \to 0$.

When $\theta < 1$, consider domain $[0, N^{\max}]$ for any $N^{\max} \geq \max(N_0, \hat{E}/f^{-1}(\chi))]$. Because $b_0 < 1$ and $0 \leq u^A(N_t, N_{t+1}) \leq u^A(N^{\max}, 0)$ is bounded for all $(N_t, N_{t+1}) \in [0, N^{\max}] \times [0, N^{\max}]$, the Contraction Mapping Theorem implies that a continuous V^* exists and that H is compact-valued and u.h.c.

When $\theta > 1$, consider domain $[\epsilon, N^{\max}]$ with arbitrary $0 \leq \epsilon < \min(N_0, \frac{\hat{E}/e^+}{f(e^+)/\chi}) < \hat{E}/e^+$. Define $\mathcal{F}_{\epsilon\alpha}(N_t) \equiv \{N_{t+1} \in [\epsilon, N^{\max}(N_t) - \alpha N_t]\}$ with $0 < \alpha < N^{\max}(\epsilon) - \epsilon$. Then: N_{t+1} is bounded away from zero and from $N^{\max}(N_t)$, so υ is bounded; $\mathcal{F}_{\epsilon\alpha}(N_t)$ is non-empty and compact; and $\mathcal{F}_{\epsilon\alpha}(\mathbf{\Omega}) \subset [\epsilon, N^{\max}]$, which implies the existence of V^* and H. For sufficiently small α and ϵ, the choice of $N_{t+1} \in \mathcal{F}(N_t)$ yields optimal values in the interior of $\mathcal{F}_{\epsilon\alpha}(N_t)$ so V^* and H do not depend on α and ϵ. The Theorem of the Maximum implies that H is compact-valued and u.h.c.

For any N_t, the Bellman equation for $U^*(N_t)$ is equivalent to the Bellman equation for $V^*(N_t)$ multiplied by the exogenous factor $1/\beta(N_t) > 0$. Hence N_{t+1} solves the former if and only if $N_{t+1} \in H(N_t)$, which means H is the optimal policy associated with U^*.

(b) For $\omega < 1$, Lemma 3(c) proves strict concavity of υ, so the claims in (b) follow by standard arguments (e.g., see Lucas, Robert, and Nancy Stokey, with Edward Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989). The case $\omega = 1$ requires a more detailed argument. First, Lemma 3(d) with $\xi = 0 > -\varepsilon_{n_t,y_t}\varkappa(e(N_t))$ for $N_t > \hat{E}/e^+$ implies strict concavity of υ on $\bar{\mathbf{\Omega}}_{con} = \{(N_t, N_{t+1}) : N_t \geq \hat{E}/e^+\}$, which is a convex set. Hence there is a strictly concave value function V^* that solves (A.12) on $\bar{\mathbf{\Omega}}_{con}$ with the restriction $N_{t+1} \in \mathcal{F}(N_t) \cap \bar{\mathbf{\Omega}}_{con}$. Second, for $N_t = \hat{E}/e^+$, an argument by contradiction shows that $H(\hat{E}/e^+) > \hat{E}/e^+$: otherwise $H(\hat{E}/e^+) = \hat{E}/e^+$ so $\hat{E}$ would not constrain emissions, which would make $n^+ > 1$ optimal. Third, $\frac{\partial^2 \upsilon}{\partial N_{t+1} \partial N_t} > 0$ holds for all N_t, because (A.2) is satisfied for $\theta = b$, so

Lemma 3(b) applies. Then the first-order condition $\frac{\partial \upsilon(N_t, N_{t+1})}{\partial N_{t+1}} + b_0 \frac{dV^*(N_{t+1})}{dN_{t+1}} = 0$, which holds because V^* is strictly concave and hence differentiable, implies that N_{t+1} is increasing in N_t. Thus $H(N_t) > \hat{E}/e^+$ for $N_t > \hat{E}/e^+$, so V^* solves (A.12) on $\bar{\Omega}_{con}$ without the restriction $N_{t+1} \in F(N_t) \cap \bar{\Omega}_{con}$. Fourth, because u^A is concave by Proposition 3(c), V^* is concave for all N_t, which implies that $H(N_t)$ is convex. If $N_t \in H(N_t)$ for any $N_t \leq \hat{E}/e^+$, one would again obtain a contradiction because $\hat{E}$ would not constrain emissions, which would make $n^+ > 1$ optimal. If $N_{t+1} < N_t$ for any $N_{t+1} \in H(N_t)$ and $N_t \leq \hat{E}/e^+$, convexity of $H(N)$ implies that $N \in H(N)$ must apply for some $N \leq N_t$, again leading to a contradiction. Thus $N_{t+1} > N_t$ for all $N_{t+1} \in H(N_t)$ and all $N_t \leq \hat{E}/e^+$, which means population converges to a value in Ω_{con} from any starting value N_0. Concavity of υ implies strict concavity of $V^*(N_t)$ on any set such that $V^*(N_{t+1})$ is strictly concave, so by backward recursion, V^* is strictly concave for all N_t. QED.

The following lemma is used to prove uniqueness of a steady state.

Lemma 5 (Condition for increasing H): If (A.2) holds on any interval $[N_t^1, N_t^2]$ with $0 < N_t^1 < N_t^2$, then the policy correspondence H is increasing in the sense that $\max H(N_t^1) \leq \min H(N_t^2)$.

Proof: Consider $N_{t+1}^1 = \max H(N_t^1)$ and $N_{t+1}^2 = \min H(N_t^2)$. Note that

$$\upsilon(N_t^1, N_{t+1}^1) + b_0 V^*(N_{t+1}^1) \geq \upsilon(N_t^1, N_{t+1}^2) + b_0 V^*(N_{t+1}^2)$$

because $N_{t+1}^1 \in H(N_t^1)$, and

$$\upsilon(N_t^2, N_{t+1}^1) + b_0 V^*(N_{t+1}^1) \leq \upsilon(N_t^2, N_{t+1}^2) + b_0 V^*(N_{t+1}^2)$$

because $N_{t+1}^2 \in H(N_t^2)$. Hence

$$\begin{aligned} \upsilon(N_t^1, N_{t+1}^1) - \upsilon(N_t^1, N_{t+1}^2) &\geq b_0 V^*(N_{t+1}^2) - b_0 V^*(N_{t+1}^1) \\ &\geq \upsilon(N_t^2, N_{t+1}^1) - \upsilon(N_t^2, N_{t+1}^2), \end{aligned}$$

which implies $\upsilon(N_t^2, N_{t+1}^2) - \upsilon(N_t^1, N_{t+1}^2) \geq \upsilon(N_t^2, N_{t+1}^1) - \upsilon(N_t^1, N_{t+1}^1)$ and

$$\Delta_{21} \equiv \int_{N_t^1}^{N_t^2} \left[\upsilon_1(N_t, N_{t+1}^2) - \upsilon_1(N_t, N_{t+1}^1)\right] dN_t \geq 0.$$

To establish a contradiction, suppose that $N_{t+1}^1 > N_{t+1}^2$. Because $\frac{\partial^2 \upsilon}{\partial N_{t+1} \partial N_t} > 0$ is continuous on a compact set, $M = \min_{[N_t^1, N_t^2] \times [N_{t+1}^2, N_{t+1}^1]} \frac{\partial^2 \upsilon}{\partial N_{t+1} \partial N_t}$ exists and $M > 0$. For any N_t, the mean-value theorem implies

$$\begin{aligned} \upsilon_1(N_t, N_{t+1}^1) - \upsilon_1(N_t, N_{t+1}^2) &= \frac{\partial^2 \upsilon}{\partial N_t \partial N_{t+1}}(N_t, N)(N_{t+1}^1 - N_{t+1}^2) \\ &\geq M(N_{t+1}^1 - N_{t+1}^2) \end{aligned}$$

for some $N \in [N_{t+1}^2, N_{t+1}^1]$. Therefore

$$\Delta_{12} \equiv \int_{N_t^1}^{N_t^2} \left[\upsilon_1(N, N_{t+1}^1) - \upsilon_1(N, N_{t+1}^2)\right] dN \geq M(N_{t+1}^1 - N_{t+1}^2)(N_t^2 - N_t^1) > 0,$$

which implies $\Delta_{21} = -\Delta_{12} < 0$, contradicting $\Delta_{21} \geq 0$. Hence $N_{t+1}^1 \leq N_{t+1}^2$, which implies $\max H(N_t^1) \leq \min H(N_t^2)$. QED.

Proposition 6 (Steady states):

(a) For any admissible (θ, b), there is a steady-state population $N_{ss}^* \in H(N_{ss}^*)$. Any steady state must satisfy $S^*(e(N_{ss}^*)) = 0$ with

$$S^*(e) = (1 - b_0)S(1, e) - b_0 u'(f(e) - \chi) f'(e)e.$$

Moreover, $e_{ss}^* = e(N_{ss}^*) \in (e_{ss}, e^+)$, so $N_{ss}^* \in (\hat{E}/e^+, N_{ss})$.

(b) Sufficient conditions for a unique N_{ss}^* and a unique e_{ss}^* are that $\xi \geq 0$ (equivalent to $\omega \leq 1$), or that $\xi > \Phi(e(N_t))$ for all $N_t \in (\hat{E}/e^+, N_{ss})$, or that $\Phi(e)$ is strictly monotone (increasing or decreasing), where $\Phi(e) \equiv -\frac{f''(e)e}{f'(e)} > 0$.

(c) If (A.2) and $\xi > -\varepsilon_{n_t, y_t} \varkappa(e(N_t))$ apply for all $N_t \in [\min\{N : N \in H(N)\}, N_{ss}]$, then $N^*_{ss} = \min\{N : N \in H(N)\}$ is the unique steady state, and $e^*_{ss} = e(N^*_{ss})$.

Proof: (a) Define

$$S^*_u(e) \equiv \frac{S^*(e)}{u'(f(e)-\chi)} = b_0 \left(\frac{1-b}{1-\theta}(f(e)-\chi) + \chi - f'(e)e \right) - \chi.$$

Because $u' > 0$, $S^*(e(N^*_{ss})) = 0$ is equivalent to $S^*_u(e(N^*_{ss})) = 0$. Note that $V^*(N) \geq \frac{1}{1-b_0} u^A(N, N)$ for all N because $N \in \digamma(N)$. Hence

$$\begin{aligned} V^*(N) &= \max_{N' \in \digamma(N)} \{\upsilon(N, N') + b_0 V^*(N')\} \\ &\geq \max_{N_{ss} \in \digamma(N)} \{\upsilon(N, N_{ss}) + \frac{b_0}{1-b_0} \upsilon(N_{ss}, N_{ss})\} \equiv V^*_{ss}(N) \end{aligned}$$

for all N, where V^*_{ss} can be interpreted as utility given a constant population starting one period ahead. Since υ is differentiable, the optimal N^*_{ss} must satisfy the first-order condition

$$\upsilon_2(N, N^*_{ss}) + \frac{b_0}{1-b_0} \left[\upsilon_1(N^*_{ss}, N^*_{ss}) + \upsilon_2(N^*_{ss}, N^*_{ss}) \right] = 0. \tag{A.14}$$

If $N^*_{ss} \in H(N^*_{ss})$ then $V^*(N^*_{ss}) = \frac{1}{1-b_0} \upsilon(N^*_{ss}, N^*_{ss})$, so $N_{ss} = N^*_{ss}$ must maximize $V^*(N^*_{ss})$ and hence satisfies the first-order condition (A.14).

When $N = N_{ss} = N^*_{ss}$, (A.14) reduces to $b_0 \upsilon_1(N, N) + \upsilon_2(N, N) = 0$, which means $b_0 \left[\xi(f(e) - \chi) + (f(e) - f'(e)e) \right] - \chi = S^*_u(e) = 0$. Steady states $N \leq \hat{E}/e^+$ are ruled out by $S^*_u(1) > 0$; steady states with $N > \hat{E}/f^{-1}(\chi)$ are ruled out by $N \notin \digamma(N)$ for $N > \hat{E}/f^{-1}(\chi)$; and $N = \hat{E}/f^{-1}(\chi)$ is ruled out because $\upsilon_1 \to \infty$ as $N^*_{ss} \to N^{\max}(N^*_{ss})$ contradicts (A.14). Therefore $N^*_{ss} \in (\hat{E}/e^+, \hat{E}/f^{-1}(\chi))$.

Existence of N^*_{ss}: Because H is u.h.c., $h(N) \equiv \min_{N'}\{N' \in H(N)\}$ exists for all N. The set $\Omega^{\leq}_{ss} \equiv \{N \geq \hat{E}/e^+ : h(N) \leq N\}$ is non-empty because $N^{\max}(\hat{E}/f^{-1}(\chi)) \leq \hat{E}/f^{-1}(\chi)$. Because H is u.h.c., $N^{\leq} \equiv \inf \Omega^{\leq}_{ss} \in \Omega^{\leq}_{ss}$. To show that $h(N^{\leq}) = N^{\leq}$, note that $h(N^{\leq}) > N^{\leq}$ is ruled out because $N^{\leq} \in \Omega^{\leq}_{ss}$. If $h(N^{\leq}) < N^{\leq}$, then by Lemma 5, $\max H(N) \leq h(N^{\leq})$ for $N \in (h(N^{\leq}), N^{\leq})$, so $N \in \Omega^{\leq}_{ss}$, contradicting $N^{\leq} = \inf \Omega^{\leq}_{ss}$.

Thus $h(N^{\leq}) = N^{\leq}$, so $N^{\leq}$ is a steady state.

(b) Uniqueness of N^*_{ss}: Note that $S^*_u(e^+) = \frac{S(1,e^+)(1-b_0)}{u'(f(e^+)-\chi)} > 0$ and $S^*_u(e_{ss}) = -f'(e)eb_0 < 0$. The continuity of S^*_u implies that $S^*_u(e) = 0$ has at least one root e^*_{ss} with $e_{ss} < e^*_{ss} < e^+$. Moreover,

$$\frac{\partial S^*_u(e)}{\partial e} = b_0\left(\xi f'(e) - f''(e)e\right) = b_0 f'(e)\left(\xi + \Phi(e)\right).$$

If Φ is either strictly increasing or decreasing, $\xi + \Phi(e) = 0$ has at most one root, denoted e_0, so S^*_u has at most one increasing and one decreasing segment on $[0, e^+]$. Because $S^*_u(e_{ss}) = b_0\left(-f'(e)e\right) < 0$ and $S^*_u(e^+) = \frac{S(1,e^+)(1-b_0)}{u'(f(e^+)-\chi)} > 0$, S^*_u must increase on a subset of $[e_{ss}, e^+]$. There are three possibilities: (i) if $\xi + \Phi(e) \neq 0$ for $e \in (0, 1)$, then S^*_u must be strictly increasing everywhere, implying $\xi + \Phi(e) > 0$, so $S^*_u(e) = 0$ has a single root $e^*_{ss} = e(N^*_{ss}) \in (e_{ss}, e^+) \subset [0, e^+]$; (ii) if $e_0 \in (0, 1)$ exists and Φ is strictly decreasing, then $\frac{\partial S^*_u(e)}{\partial e} > 0$ for $e \in [0, e_0)$ and $\frac{\partial S^*_u(e)}{\partial e} < 0$ for $e \in (e_0, e^+]$, so $S^*_u(e) \geq S^*_u(e^+) > 0$, which means $S^*_u(e) = 0$ has a single root on $(e_{ss}, e_0) \subset [0, e^+]$; (iii) if $e_0 \in (0, 1)$ exists and Φ is strictly increasing, then $\frac{\partial S^*_u(e)}{\partial e} < 0$ for $e \in [0, e_0)$ and $\frac{\partial S^*_u(e)}{\partial e} > 0$ for $e \in (e_0, e^+]$, so $S^*_u(e) = 0$ has a single root on $[e_{ss}, e^+]$. Uniqueness of $N^*_{ss} = e^{-1}(e^*_{ss})$ follows.

(c) Combining $\xi > -\varepsilon_{n_t,y_t}\varkappa(e)$ and (A.2), which implies $\varepsilon_{n_t,y_t} < (\frac{f'(e)e}{f(e)})^{-1}$, one obtains $\xi\frac{f'(e)e}{f(e)} > -\varepsilon_{n_t,y_t}\frac{f'(e)e}{f(e)}\varkappa(e) > -\varkappa(e)$, so $\xi > -\varkappa(e)(\frac{f'(e)e}{f(e)})^{-1} = -\Phi(e)$, so uniqueness of N^*_{ss} and e^*_{ss} follow from part (b). QED.

Remark: The production functions considered in the calibration section have strictly increasing $\Phi(e)$, so there is a unique steady state for any (θ, b).

Proposition 7 (Convergence and conditions for a unique optimal path):

(a) Suppose $N_0 < N^{*1}_{ss}$, where $N^{*1}_{ss} \equiv \min\{N : N \in H(N)\}$, and suppose (A.2) holds on $[N_0, N^*_{ss}]$ in that (A.2) holds for all $N_t \in [N_0, N^*_{ss}]$ and all $N_{t+1} \in H(N_t)$. Then population converges monotonely from N_0 to N^*_{ss}.

(b) If there is a unique steady state N^*_{ss} and (A.2) holds on a interval Ω that includes N^*_{ss}, then population converges monotonely to N^*_{ss} from an initial value $N_0 \in \Omega$.

(c) For any admissible (θ, b), if an interval Ω satisfies (b) and $\Omega \subset \{N_t : 1/\omega > 1 - \varepsilon_{n_t,y_t}\varkappa(e(N_t))$ for $N_{t+1} \in H(N_t)\}$, then V^* is strictly concave and differentiable on Ω, and H is single-valued and continuous.

Proof: (a) For any $N_t < N_{ss}^{*1}$, (A.2) implies that H is increasing as defined in Lemma 5, so $N_{t+1} \le N_{ss}^{*1}$ for all $N_{t+1} \in H(N_t)$. By construction $N_{ss}^{*1} = \min \Omega_{ss}^{\le}$, with $\Omega_{ss}^{\le}$ as defined in the proof of Lemma 5. Hence $N_t \notin \Omega_{ss}^{\le}$ for all $N_t < N_{ss}^{*1}$, which implies $N_{t+1} > N_t$. Thus $N_{t+1} \in (N_t, N_{ss}^{*1}]$ for all $N_t < N_{ss}^{*1}$, which proves monotone convergence.

(b) If there is only one steady state, $N_{t+1} < N_t$ for $N_{t+1} \in H(N_t)$ must hold for all $N_t > N_{ss}^*$ (otherwise $N^{\max}(\hat{E}/f^{-1}(\chi)) \le \hat{E}/f^{-1}(\chi)$ would imply a second steady state on $(N_{ss}^*, \hat{E}/f^{-1}(\chi)]$, a contradiction), so $N_{t+1} \in [N_{ss}^*, N_t)$. Thus population converges monotonely from above; for $N_t < N_{ss}^*$, part (a) applies.

(c) From proposition 3, $\xi > -\varepsilon_{n_t,y_t}\varkappa(e(N_t))$ implies strict concavity of u^A. From part (b) $H(N_t) \subset \Omega$ for all $N_t \in \Omega$, so (A.12) on Ω has a solution in the space of strictly concave functions, which implies differentiability and that H is single-valued and continuous. QED.

Remark: Because Proposition 6(b) shows that N_{ss}^* is unique for $\omega \le 1$, Proposition 7(b) implies convergence for $\omega \le 1$, which is asserted in the text.

Remark: If ω is in a neighborhood of one, then N_{ss}^* is unique from Proposition 6(c), and Proposition 7(b-c) apply. If $\omega \approx 1$, then $\varepsilon_{n_t,y_t} \approx 1$ so condition (A.2) holds for all e. If $\varkappa$ and Φ are bounded away from zero (which is true for the Cobb-Douglas and for the abatement-cost production functions used in the calibrations), then the conditions $\xi > -\Phi(e(N_t))$ and $\xi > -\varepsilon_{n_t,y_t}\varkappa(e(N_t))$ apply on $\Omega = [\hat{E}/e^+, N_{ss}]$, so N_{ss}^* is unique and V^* strictly concave. Thus $\omega = 1$ is not a borderline case.

Remark on general (δ, Θ): We claimed at the end of Section IV that for general (δ, Θ), a steady state for $(E_{ss}^*, \mathbf{X}_{ss}^*)$ implies a steady-state condition for e similar to S^*.

Proof: Denote the return function in the general problem (4) by

$$\text{(A.15)} \qquad v(N_t, N_{t+1}, E_t, \mathbf{X}_t) \equiv \beta(N_t)u\left(f(\frac{E_t}{N_t})(1-\delta(\mathbf{X}_t)) - \chi\frac{N_{t+1}}{N_t}\right),$$

If $(E_t, \mathbf{X}_t) = (E^*_{ss}, \mathbf{X}^*_{ss})$ is constant, (A.15) is identical to the the return function (A.11) under S1-S3, except that $f(e(N_t))$ in (A.11) is replaced by $f(\frac{E^*_t}{N_t})(1-\delta(\mathbf{X}^*_t))$. Making the same substitution in steady state, e^*_{ss} is uniquely defined by $S^*(e^\circ_{ss}|\delta(\mathbf{X}^*_{ss})) = 0$ for any given $\delta(\mathbf{X}^*_{ss})$, where

$$S^*(e|\delta) \equiv (1-b_0)S^\circ(e|\delta) - b_0 u'(((1-\delta)f(e)-\chi)(1-\delta)f'(e)e$$

is similar to the S^*-function in Section IV. For constant E^*_{ss}, unique $N^*_{ss} = E^*_{ss}/e^*_{ss}$ follows. QED.

Remark on interpreting the optimal tax: We claimed end of Section IV.B that terms in (20) other than the real externalities $f'(e(N^*_{t+1}))e(N^*_{t+1})$ can be interpreted as the number of descendants in a future period times products of single-period discount factors. In detail:

$$\frac{\beta(n^*_t)}{n^*_t}\left[\prod_{j=1}^{i-1}\beta(n^*_{t+j})\right]\frac{u'(c^*_{t+i})}{u'(c^*_t)} = \left(\prod_{j=1}^{i-1} n^*_{t+j}\right)\left[\prod_{j=0}^{i-1}\frac{\beta(n^*_{t+j})}{n^*_{t+j}}\frac{u'(c^*_{t+j+1})}{u'(c^*_{t+j})}\right],$$

where $\prod_{j=1}^{i-1} n^*_{t+j} = L^*_{t+i}/L^*_{t+1}$ is descendants at time $t+i$ per child born at time $t+1$. The terms $\frac{\beta(n^*_{t+j})}{n^*_{t+j}}\frac{u'(c^*_{t+j+1})}{u'(c^*_{t+j})}$ can be interpreted as single-period discount factors. Specifically, if individuals could trade consumption loans that are settled by their children, $\frac{\beta(n^*_{t+j})}{n^*_{t+j}}\frac{u'(c^*_{t+j+1})}{u'(c^*_{t+j})}$ would be the market-clearing price in period j of a loan that pays one consumption unit in period $j+1$.

G. *Steady States With Time Costs and Backstop Technology (Section V)*

In section V-A, we claim that e_{ss} and e^*_{ss} in the model with time costs exist under assumptions similar to those made in Section IV. In section V-B, we claim that $e_{ss} > 0$ and $e^*_{ss} > 0$ exist whenever $f(0) < f^B$. Because the calibration analysis allows for both time costs and backstops, we prove existence and uniqueness of $e_{ss} > 0$ and $e^*_{ss} > 0$ for any $f(0) < f^B$ in a setting with both time costs and backstops. Recall that $f^B = \frac{\chi}{\phi-\psi}$ and, by assumption, $\chi + \psi f(e^+) < \phi f(e^+)$, where $\phi = 1/\left(1 + \frac{1-\theta}{1-b}\frac{1-b_0}{b_0}\right)$.

Proposition 8: In an economy with time costs and backstops, if $f(0) < f^B$, then: (a) a natural steady state $e^\circ_{ss} > 0$ exists; (b) an optimal steady state $e^*_{ss} > 0$ exists.

Proof: (a) Define $S_u(n,e) \equiv S(n,e)/u'((1-\psi n)f(e) - \chi n)$. Then the steady-state condition $S(n,e) = 0$ holds iff $S_u(n,e) = 0$, so e°_{ss} must satisfy

$$\begin{aligned} S_u(1,e) &= -[\chi + w(e)\psi] + \frac{\beta(1)}{1-\beta(1)}\frac{1-b}{1-\theta}\left[(1-\psi)f(e) - \chi\right] \\ &= \frac{1}{1-\phi}\left[(\phi-\psi)f(e) + (1-\phi)\psi f'(e)e - \chi\right] \\ &= f'(e)e\psi + \frac{\phi-\psi}{1-\phi}[f(e) - f^B] = 0. \qquad \text{(A.16)} \end{aligned}$$

Assumption $\chi + \psi f(e^+) < \phi f(e^+)$ implies $(\phi-\psi)f(e^+) - \chi > 0$ so $S_u(1,e^+) > 0$. Because $f'(e)e \to 0$ as $e \to 0$ by concavity of production, $S_u(1,0) < 0$ if and only if $f(0) < f^B$. Thus given $f(0) < f^B$, continuity of S_u implies a root $e^\circ_{ss} \in (0,e^+)$ with $S_u(1,e^\circ_{ss}) = 0$.

(b) Define

$$\begin{aligned} S^*_u(e) &\equiv \frac{S^*(e)}{u'((1-\psi)f(e) - \chi)} = (1-b_0)S_u(1,e) - b_0(1-\psi)f'(e)e \\ &= f'(e)e(\psi - b_0) + (1-b_0)\frac{\phi-\psi}{1-\phi}[f(e) - f^B], \qquad \text{(A.17)} \end{aligned}$$

Then e^*_{ss} must satisfy $S^*_u(e^*_{ss}) = 0$. Note that $S^*_u(e^+) = (1-b_0)S_u(1,e^+) > 0$ and $S^*_u(0) = \frac{\phi-\psi}{1-\phi}[f(0) - f^B] < 0$. Hence by continuity, $S^*_u(e^*_{ss}) = 0$ for some $e^*_{ss} \in (0,e^+)$. QED.

Proposition 9: In an economy with time costs and backstops, suppose $f(0) < f^B$. Then:

(a) e°_{ss} is unique if $\Phi(e) \equiv \frac{(-f''(e))e}{f'(e)}$ is strictly monotone in e (increasing or decreasing).

(b) e^*_{ss} is unique if $\xi > -\frac{b_0-\psi}{b_0(1-\psi)}\Phi(e)$ for all $e \in (0,e^+)$, or if $\Phi(e)$ is strictly monotone in e (increasing or decreasing).

Proof: (a) From (A.16) and (A.17), $f(0) < f^B$ implies $S_u(1,0) < 0$ and $S^*_u(0) < 0$.

Recall that $S_u(1, e^+) = S_u^*(e^+) > 0$. From (A.16):

$$\begin{aligned}\frac{\partial}{\partial e} S_u(1,e) &= f''(e)e\psi + f'(e)\psi + \frac{\phi - \psi}{1-\phi} f'(e) \\ &= f'(e)\left[\psi + \frac{\phi-\psi}{1-\phi} - \psi\Phi(e)\right].\end{aligned}$$

Assumptions $\chi + \psi f(e^+) < \phi f(e^+)$ and $\psi \geq 0$ imply $\psi + \frac{\phi-\psi}{1-\phi} > 0$. There are three cases:

(a-i) If $\psi = 0$, then $\frac{\partial}{\partial e} S_u(1,e) > 0$ for $e < e^+$ follows from $\frac{\phi-\psi}{1-\phi} > 0$, so $S_u(1,e) = 0$ has at most one root (which exists by Proposition 8).

(a-ii) If $\psi > 0$ and Φ increases strictly, then $\psi + \frac{\phi-\psi}{1-\phi} - \psi\Phi(e)$ decreases strictly so $\frac{\partial}{\partial e} S_u(1,e)$ decreases strictly. Because $S_u(1,0) < S_u(1,e^+)$, $\frac{\partial}{\partial e} S_u(1,e) > 0$ for some $e \in (0,e^+)$. Thus either $\frac{\partial}{\partial e} S_u(1,e) > 0$ for all $e \in (0,e^+)$ so again $S_u(1,e) = 0$ has at most one root, or $\frac{d}{de} S_u(1,e) = 0$ has a unique root $e^{\circ +} \in (0,e^+)$. Then $\frac{\partial}{\partial e} S_u(1,e) > 0$ for $e < e^{\circ +}$ and $\frac{\partial}{\partial e} S_u(1,e) < 0$ for $e > e^{\circ +}$, so $S_u(1,e) \geq S_u(1,e^+) > 0$ for $e \geq e^{\circ +}$. Hence $S_u(1,e) = 0$ has at most one root e_{ss}°, which must lie in the interval $(0, e^{\circ +})$.

(a-iii) If $\psi > 0$ and Φ decreases strictly, then $\psi + \frac{\phi-\psi}{1-\phi} - \psi\Phi(e)$ increases strictly so $\frac{\partial}{\partial e} S_u(1,e)$ increases strictly. By reasoning similar to case (ii), $S_u(1,e) = 0$ has at most one root. (The argument differs only in that if $\frac{d}{de} S_u(1,e) = 0$ has a root $e^{\circ +} \in (0,e^+)$, then $\frac{\partial}{\partial e} S_u(1,e) < 0$ for $e < e^{\circ +}$, and $\frac{\partial}{\partial e} S_u(1,e) > 0$ for $e > e^{\circ +}$, so the root of $S_u(1,e) = 0$ lies in the interval $(e^{\circ +}, e^+)$.

Thus e_{ss} is unique in all cases.

(b) From (A.17),

$$\begin{aligned}\frac{\partial}{\partial e} S_u^*(e) &= f'(e)\left[(\psi - b_0)(1-\Phi(e)) + (1-b_0)\frac{\phi-\psi}{1-\phi}\right] \\ &= f'(e)\left[b_0(1-\psi)\xi + (b_0-\psi)\Phi(e)\right],\end{aligned} \tag{A.18}$$

using

$$\psi - b_0 + (1-b_0)\frac{\phi-\psi}{1-\phi} = (1-\psi)\frac{\phi-b_0}{1-\phi} = (1-\psi)b_0\xi.$$

If $\Phi(e)$ is strictly monotone, the argument is similar to the proof of (a). Namely, If $b_0 = \psi$, then $\psi < \phi$ implies $b_0 < \phi$ and $\xi > 0$, so $\frac{\partial}{\partial e} S_u^*(e) > 0$ for all e as in (a-i). If $b_0 \neq \psi$, then $(b_0 - \psi)\Phi(e)$ either increases strictly or decreases strictly, so the argument is analogous to either case (a-ii) or (a-iii).

Even if $\Phi(e)$ is not monotone, $\xi > -\frac{b_0-\psi}{b_0(1-\psi)}\Phi(e)$ is sufficient for a unique e_{ss}^* because it implies $b_0(1-\psi)\xi + (b_0-\psi)\Phi(e) > 0$ in (A.18) and hence $\frac{\partial}{\partial e} S_u^*(e) > 0$. QED.

Remark: Recall from Proposition 6(b) that without time costs, $\xi > -\Phi(e)$ implies uniqueness of e_{ss}^*. Condition $\xi > -\frac{b_0-\psi}{b_0(1-\psi)}\Phi(e)$ in Proposition 9(b) is a generalization. Note that $\Phi(e) \geq 0$, so Proposition 9(b) always holds for $\xi \geq 0$ and $\psi < b_0$.

Proposition 10: For the Cobb-Douglas technology defined in Section VI, e_{ss} and e_{ss}^* are unique.

Proof: By assumption, $f(e) = f_0 e^{f_1}(f_2 - e)^{1-f_1}$, where $f_1 \in (0,1)$ and $f_2 = 1/f_1 > 1$. The derivatives are $f'(e) = f(e)\left[\frac{f_1}{e} - \frac{1-f_1}{f_2-e}\right] = \frac{f(e)}{e}\frac{1-e}{f_2-e}$ and $f''(e) = -\frac{f(e)}{e^2}\frac{(1-f_1)f_2}{(f_2-e)^2} < 0$. Hence

$$\Phi(e) = \frac{(-f''(e))e}{f'(e)} = \frac{(1-f_1)f_2}{(1-e)(f_2-e)},$$

which is increasing in e. Thus e_{ss} and e_{ss}^* are unique by Proposition 9. QED.

Proposition 11: For the abatement-cost technology defined in Section VI, e_{ss} and e_{ss}^* are unique provided $g_0 = f(0) < f^B$.

Proof: By assumption, $f(e) = 1-(1-g_0)(1-e)^{g_1}$ with $0 \leq g_0 < 1$ and $g_1 > 1$. The derivatives are $f'(e) = g_1(1-g_0)(1-e)^{g_1-1}$ and $f''(e) = -g_1(1-g_0)(g_1-1)(1-e)^{g_1-2}$. Hence

$$\Phi(e) = \frac{(-f''(e))e}{f'(e)} = (g_1-1)\frac{e}{(1-e)},$$

which is increasing in e. Thus e_{ss} and e_{ss}^* are unique by Proposition 9. QED.

Proposition 12: In an economy with time costs and backstops, if $f(0) > f^B$, then: (i) a steady state $e_{ss} > 0$ does not exist; (ii) fertility converges to the unique root n_{ss} of $S(n_{ss}, 0) = 0$, which satisfies $n_{ss} > 1$; and (iii) optimal fertility converges to the same value $n_{ss}^* = n_{ss}$. Moreover, if $f(0) = f^B$, then: (iv) the steady-state conditions reduce to $S(1,0)=0$ with no population policy and $S^*(0)=0$ in the optimal economy.

Proof: (i) In (A.16), $f(0) > f^B$ and $f'(e)e \geq 0$ imply $S_u(1,e) > 0$ for all e, so $S(1,e) > 0$ and there is no e_{ss} that satisfies $S(1,e_{ss}) = 0$.

(ii) Recall that $S(n,e) = -u'(c)(\chi+\psi w(e)) + \frac{\beta'(n)}{1-\beta(n)}u(c)$ with $c = (1-\psi n)f(e) - \chi n$. For $e=0$, $S(n,0) \to -\infty$ as $n \to f(0)/(\psi f(0)+\chi)$ from below, because the latter implies $c \to 0$. When $\theta < 1$, $\beta'(n) \to \infty$ as $n \to 0$, which implies $S(n,0) \to \infty$. When $\theta > 1$, $\frac{\beta'(n)u}{1-\beta(n)} \to \infty$ as $n \to \beta^{-1}(1)$ from above, which implies $S(n,0) \to \infty$. Hence the continuity of S implies that n_{ss} satisfying $S(n_{ss},0) = 0$ exists. Moreover, $\frac{d}{dn}S(n,0) = u''(c)(\psi f(e)+\chi)^2 + \frac{\beta''(n)}{1-\beta(n)}u(c) < 0$ at n such that $S(n,0)=0$, so $S(n,0)$ can cross zero only once. Hence n_{ss} is unique with $S(n,0) > 0$ for $n < n_{ss}$ and $S(n,0) < 0$ for $n > n_{ss}$. To show $n_{ss} > 1$, note that $n_{ss} \leq 1$ would imply $S(1,0) \leq 0$, which contradicts the result in (i) that $S(1,e) > 0$ for all e. Therefore $S(1,0) > 0$. Given $S(n_{ss},0) = 0$ with $n_{ss} > 1$, derivations analogous to those in the proof of Proposition 2 imply that utility converges to $U_{ss} = \frac{1}{1-\beta(n_{ss})}u[(1-\psi n_{ss})f(0) - \chi n_{ss}]$ and that population follows a path $N_{t+1} = N(N_t)$ that approaches a ray with slope $n_{ss} > 1$, which means fertility converges to n_{ss}.

(iii) For $f(0) > f^B$, a necessary condition for optimal steady-state fertility is that $S^*_{nu}(n,e) = 0$, where

$$S^*_{nu}(n,e) \equiv (1-\beta(n))S_u(n,e) - \frac{\beta(n)}{n}\left[(1-\psi n)f'(e)e\right] \tag{A.19}$$

replaces $S^*_u(e)$ when $n \neq 1$. (Note that $S^*_{nu}(1,e) = S^*_u(e)$.) Because $f'(e)e \to 0$ as $e \to 0$, $S^*_{nu}(n,0) = S_u(n,0)$ for all n. Hence for $f(0) > f^B$, $n^*_{ss} = n_{ss} > 1$.

(iv) For $f(e) = f^B$, (A.16) and (A.17) imply $S_u(1,0) = S^*_{nu}(1,0) = S^*_u(0) = 0$, so $S(1,0) = S^*(0) = 0$. QED.

H. Numerical Procedures (Section VI)

Solutions to the Bellman equation (4) are computed straightforwardly with value function iteration. The most general version we require is the problem specified in section

VI-E, which includes growth-adjustments and has a bivariate value function with state variables $(\tilde{N}, X)$.

To compute solutions for this problem, we define a grid of pairs $(\tilde{N}, X)$ on $[0, \tilde{N}_{\max}] \times [0, X_{\max}]$, where $X_{\max} = \hat{E}/(1-\gamma)$ for case $\delta 1$, $X_{\max} = \hat{X}$ for case $\delta 2$, and $X_{\max} = \hat{X} + \Delta\hat{X}$ for case $\delta 3$; $\tilde{N}_{\max}$ is chosen high enough that iterations starting at $\tilde{N}_0$ stay well below $\tilde{N}_{\max}$. (Since $\tilde{N}_t > \tilde{N}_{ss}$ is possible, generally $\tilde{N}_{\max} \geq \tilde{N}^{\circ}_{ss}$ is determined numerically.) We also solve the first-order conditions in steady state numerically to obtain $(\tilde{N}^*_{ss}, X^*_{ss})$ and $(\tilde{N}^{\circ}_{ss}, X^{\circ}_{ss})$, and we include these pairs in the grid. We initialize $U_0(\tilde{N}, X) = \frac{1}{1-b_0} u(f(e((1-\tilde{\psi})\tilde{N}))(1-\delta(X)) - \tilde{\chi})$ and $V_0(\tilde{N}, X) = \beta(\tilde{N}) U_0(\tilde{N}, X)$, which is feasible for all relevant $(\tilde{N}, X)$ because $\tilde{n} = 1$ is feasible for all $\tilde{N} \leq \tilde{N}_{ss}$. To compute optimal policy we then iterate on (4), using cubic splines to interpolate values off the grid. To compute policy without child taxes, each iteration is divided into two steps and uses an additional grid of emissions ratios $e \in [0, e^+]$. Step 1 obtains individual fertilities $n = n^0(\tilde{N}, X, e)$ by solving (7) for given $(\tilde{N}, X, e)$; step 2 maximizes (4) by choice of e, recognizing the functional dependence of fertility through n^0.

To compute results in sections VI-A to VI-D, we use a separate algorithm that was developed for an earlier version of this paper that did not include section VI-E. This is detailed below and exploits that under S1-S3, the value function is univariate, as shown in (5), and that optimal emission policy is a cap at $\hat{E}$. (A feasible but computationally much slower alternative is to use the bivariate value function defined above and impose (δ, Θ) as implied by S1-S3. We used this alternative algorithm to confirm that the two separate approaches yield the same results, with numerical accuracy at least up to the digits reported, or better.)

To compute allocations in the main model (under S1-S3), we first solve the steady-state conditions $S^{\circ}(\tilde{e}^{\circ}_{ss}) = 0$ for $\tilde{e}^{\circ}_{ss}$ and $S^*(\tilde{e}^*_{ss}) = 0$ for $\tilde{e}^*_{ss}$. Because $\hat{E}$ is fixed, the values of $\tilde{e}^{\circ}_{ss}$ and $\tilde{e}^*_{ss}$ imply values of growth-adjusted population $\tilde{N}_{ss} = \frac{1}{1-\tilde{\psi}} \hat{E}/\tilde{e}_{ss}$ (for $^{\circ}$ and *, superscripts omitted when optimal and natural values are the same), incomes $\tilde{y}_{ss} = (1-\tilde{\psi}) f(\tilde{e}_{ss})$, and wages $w(\tilde{e}_{ss}) = f(\tilde{e}_{ss}) - f'(\tilde{e}_{ss})\tilde{e}_{ss}$.

To compute natural population in the main model, we use value-function iteration to solve for $U^\circ(\tilde{N})$. Specifically, we define a grid of values of $\tilde{N}$ (equally spaced on a log-scale) and compute initial utility values at each gridpoint of $U_0^\circ(\tilde{N}) = u(f(e((1-\tilde{\psi})\tilde{N})) - \tilde{\chi})/(1-\beta(1))$, which is feasible because $\tilde{n} = 1$ is feasible for all $\tilde{N} \leq \tilde{N}_{ss}$. At each iteration $i = 1, 2, ...$ we calculate equilibrium fertility $\tilde{n} = \eta_i(\tilde{N})$ at each gridpoint by Gauss-Newton iteration on the household first-order condition $U_n^\circ(\tilde{n}, w + TR, U) = 0$ evaluated at the equilibrium values $w = w(e((1-\tilde{\psi}\tilde{n})\tilde{N}))$, $TR = \hat{E} f'(e((1-\tilde{\psi}\tilde{n})\tilde{N}))/\tilde{N}$, and $U = U_{i-1}(\tilde{n}\tilde{N})$, where the latter is the value function from the previous iteration approximated using cubic-spline interpolation between gridpoints. Given values $\eta_i(\tilde{N})$ at iteration i, we then compute

$$\begin{aligned} U_i^\circ(\tilde{N}) &= u((1-\tilde{\psi}\eta_i(\tilde{N}))f(e((1-\tilde{\psi}\eta_i(\tilde{N}))\tilde{N})) - \tilde{\chi}\eta_i(\tilde{N})) \\ &\quad + \tilde{\beta}(\eta_i(\tilde{N}))U_{i-1}^\circ(\eta_i(\tilde{N})\tilde{N}), \end{aligned}$$

and proceed to the next iteration. When an i is reached at which $U_i^\circ(\tilde{N}) - U_{i-1}^\circ(\tilde{N})$ is sufficiently small at all gridpoints, we take $U^\circ(\tilde{N}) = U_i^\circ(\tilde{N})$ and $\eta^\circ(\tilde{N}) = \eta_i(\tilde{N})$ at gridpoints, and use cubic-spline interpolation to find values of $U^\circ(\tilde{N})$ and $\eta^\circ(\tilde{N})$ between gridpoints. To compute specific population paths, we iterate on $\eta^\circ(\tilde{N})$ from a starting value $\tilde{N}_0$. Finally we convert the growth-adjusted population sequence into the actual population sequence by reversing the transformations in Section V.

To compute optimal population in the main model, we iterate similarly on

$$U_i^*(\tilde{N}) \equiv \max_{n \in \{n | (1-\tilde{\psi}n) f(e((1-\tilde{\psi}n)\tilde{N})) \geq \chi n\}} \left\{ \begin{array}{c} u((1-\tilde{\psi}n)f(e((1-\tilde{\psi}n)\tilde{N})) - \chi n) \\ + \tilde{\beta}(n)U_{i-1}^*(n\tilde{N}) \end{array} \right\},$$

starting again from $U_0^*(\tilde{N}) = U_0(\tilde{N})$ on a grid of values of $\tilde{N}$ and using cubic-spline interpolation to evaluate U_{i-1}^* between grid points. When an i is reached at which $U_i^*(\tilde{N}) - U_{i-1}^*(\tilde{N})$ is sufficiently small at all gridpoints, we take $U^*(\tilde{N}) = U_i^*(\tilde{N})$ and $\eta^*(\tilde{N}) = \arg\max u((1-\tilde{\psi}n)f(e((1-\tilde{\psi}n)\tilde{N})) - \chi n) + \tilde{\beta}(n)U^*(n\tilde{N})$ at gridpoints, and use cubic-spline interpolation to find values of $U^*(\tilde{N})$ and $\eta^*(\tilde{N})$ between gridpoints.

I. *Details on the Calibration of Child Costs (Section VI)*

USDA (*Expenditures on Children by Families, 2007*, http://www.cnpp.usda.gov/Publications/CRC/crc2007.pdf) reports expenditures of \$204,000 per child by husband-wife families with two children and before-tax annual income between \$45,800 and \$77,100. Department of Education (*Digest of Education Statistics 2007 Tables)* reports total K-12 spending of \$599 billion (www.nces.ed.gov/programs/digest/d07/tables/dt07_26.asp), enrollment in 2-year and 4-years colleges of 17,922,000 (www.nces.ed.gov/programs/digest/d07/tables/dt07_177asp), and average undergraduate tuition, fees, and room and board of \$15,434 (www.nces.ed.gov/programs/digest/d07/tables/dt07_320.asp, 2007) for 2007. From the Census (www.census.gov/prod/2008pubs/p20-558.pdf and www.census.gov/popest/national/asrh/files/NC-EST2007-ALLDATA-R-File16.csv), the population aged 0-17 was 73,902,000, and actual and replacement U.S. fertility were 1.9 and 2.1, implying $n = 0.9$ for the U.S. From the Economic Report of the President (http://www.gpoaccess.gov/eop/2008/B26.xls), NNP was \$12,381 billion in 2007. Thus aggregate expenditures on children as a fraction of NNP in 2007 were

$$\left[(73,902,000(\frac{204,000}{18}) + 599 * 10^9 + (15,434)(17,922,000)\right]/(12,381 * 10^9)$$
$$= 0.138.$$

Participation rates are from the Bureau of Labor Statistics (*Employment Status of Civilian Population by Sex and Age*, www.bls.gov/news.release/empsit.t01.htm, 2008). Specifically, we take the ratio of the average participation rate (over males and females) to the male participation rate to equal the relative decrease in labor due to time costs, $.685/.76 = 1-\psi n$ and use $n = 0.9$ to infer $\psi = 0.11$ and therefore $\tilde{\psi} = \psi\lambda/\alpha = 0.098$.

We then recover $\chi = 0.138$ from[42]

$$n\chi = (.9)\chi = (0.138)(1 - \psi n)(1) = (0.138)(\frac{0.685}{0.76}).$$

Note that costs of K-12 education, which are typically provided free by government, amounted to 4.8 percent of NNP in the U.S. in 2007. Neglecting other subsidies and taxes on children, this suggests a preexisting tax on children of $-.048$. It would be possible to adjust the calibration to take account of a preexisting tax with the value $\tau = -.048$; such a treatment would be correct if the subsidy is a historical accident. If the existing subsidy instead corrects some other market failure, on the other hand, assuming a preexisting value $\tau = -.048$ without modeling the market failure would be inappropriate. To avoid this, we assume there is no preexisting tax.

J. Permanent Emission Rights (Footnote in Section I)

The population externality can be thought of as resulting from imperfect property rights: there would be no population externality if government were to issue permanent, bequeathable emissions rights instead of permits valid for only a single period. With single-period permits, a marginal birth means more people share a given total amount of emissions rights in the next period so others receive fewer emission rights. With permanent rights, on the other hand, total rights to emit in the next period are given so the new person gets no rights from others. Instead, when a household has an additional child, it is the emission rights of the household's earlier children that are reduced. This provides a disincentive to have children equal to that induced by optimal Pigou taxes.

To formalize this, suppose the government at $t = 0$ simply gives the household rights to emit $\pi_0 = \hat{E}/N_0$ units of greenhouse gases each period in perpetuity. A household in period t with permanent rights to emit π_t units rents these to the firm at the competitive price $p_t = f'$ in the period, then leaves an equal share to each child so emission rights

[42]The calculated value of χ is close to but not the same as the calculated fraction of NNP going to child expenditures; to three decimals, each is 0.138.

evolve as $\pi_{t+1} = \pi_t/n_t$. The latter captures the reduction in an earlier child's inheritance from a marginal child , and implies $\pi_t = e(N_t)$ for all t. The household earns wage $w_t = f - e_t f'$ and consumes $c_t = w_t + \pi_t p_t - \chi n_t = f(e_t) - \chi n_t$. Then:

Proposition 13: With permanent, bequeathable emissions rights, the population sequence with no population policy is the optimal sequence.

Proof: In maximizing utility, the household at t chooses n_t taking π_t as given and also taking the future path of aggregate population and hence of per-capita emissions as given. Denote the utility value to parents of a child who inherits π_t by $\upsilon(\pi_t|N)$. The fertility that individually maximizes utility must satisfy the Bellman equation

$$\upsilon(\pi_t|N_t) = \max_{n_t}\{u(c_t) + \beta(n_t)\upsilon(\frac{\pi_t}{n_t}|\bar{n}_t N_t)\}, \tag{A.20}$$

with $\bar{n}_t = N_{t+1}/N_t$ given and $c_t = f(e(N_t)) + [\pi_t - e(N_t)]f'(e(N_t)) - \chi n_t$. The Euler equation is

(A.21)

$$U_n^\pi(n_t|N_t, \bar{n}_t, \pi_t) \equiv -u'(c_t)\chi + \beta'(n_t)\upsilon(\frac{\pi_t}{n_t}|\bar{n}_t N_t) - \frac{\beta(n_t)}{n_t}\frac{\pi_t}{n_t}\upsilon_\pi(\frac{\pi_t}{n_t}|\bar{n}_t N_t) = 0.$$

Optimality follows if the household's optimal fertility choice n_t satisfies the optimality condition in the optimal-population problem,

$$U_n^* = -u'(f(e(N_t)) - \chi n_t)\chi + \beta'(n_t)U^*(N_{t+1}) + \frac{\beta(n_t)}{n_t}N_{t+1}\frac{dU^*}{dN}(N_{t+1}) = 0, \tag{A.22}$$

where U_n^* and

$$\frac{dU^*}{dN}(N_{t+1}) = -\frac{1}{N_{t+1}}\sum_{i=1}^{\infty}\left[\prod_{j=1}^{i-1}\beta(n_{t+j})\right]u'(c_{t+i})f'(e(N_{t+i}))e(N_{t+i}) \tag{A.23}$$

were derived in Section IV-A.

The first two terms in (A.21) and in (A.22) are the marginal child cost and marginal private benefit of children. These correspond to the first two (private) terms in the household's first-order condition in the economy with no population policy ($U_n^0 = 0$). In equilibrium, $\pi_t = \hat{E}/N_t = e_t$, $\bar{n}_t = n_t$, and $c_t = f(e_t) - \chi n_t$ for all $t \geq 0$. Hence c_t in (A.21) equals $f(e(N_t)) - \chi n_t$ in (A.22), so the first terms are equal. The second terms are equal if $\upsilon(\frac{\pi_t}{n_t}|\bar{n}_t N_t) = \upsilon(e(N_{t+1})|N_{t+1}) = U^*(N_{t+1})$, which is verified below. The third term in (A.21) is the dilution of children's inheritances of emissions rights from a marginal child. To re-express this term, the envelope theorem implies

$$\upsilon_\pi(\pi_t|N_t) = u'(c_t) f'(e(N_t)) + \frac{\beta(n_t)}{n_t}\upsilon_\pi(\frac{\pi_t}{n_t}|N_{t+1}). \tag{A.24}$$

Multiplying (A.24) at $t+1$ by $\pi_{t+1} = \pi_t/n_t = e_{t+1}$ and expanding as a sum:

$$\begin{aligned}\frac{\pi_t}{n_t}\upsilon_\pi(\frac{\pi_t}{n_t}|N_{t+1}) &= u'(c_{t+1}) f'(e(N_{t+1}))e_{t+1} + \frac{\beta(n_{t+1})}{n_{t+1}} e_{t+1}\upsilon_\pi(\frac{\pi_{t+1}}{n_{t+1}}|N_{t+2}) \\ &= \sum_{i=1}^{\infty}\left[\prod_{j=1}^{i-1}\beta(n_{t+j})\right]u'(c_{t+i}) f'(e_{t+i})e_{t+i},\end{aligned}$$

so, using (A.23), $\frac{\pi_t}{n_t}\upsilon_\pi(\frac{\pi_t}{n_t}|N_{t+1}) = -N_{t+1}\frac{dV^*}{dN}(N_{t+1})$. Thus the third terms in (A.21) and (A.22) are equal. Hence $U_n^\pi = 0$ is satisfied at the same fertility choice $n_t = n_t^*$ that satisfies $U_n^* = 0$, provided $\upsilon(e(N_{t+1})|N_{t+1}) = U^*(N_{t+1})$. To verify this, suppose $\upsilon(e(N)|N) = U^*(N)$ at some time t_1. Then if $\pi_t = e(N_t)$ at $t = t_1 - 1$, by the argument above, $n_t = n_t^*$ solves the Bellman equation (A.20). Hence the optimal value is $\upsilon(e(N_t)|N_t) = u'(c_t^*) + \beta(n_t^*)\upsilon(e(n_t^* N_t)|n_t^* N_t) = u'(c_t^*) + \beta(n_t^*)U^*(n_t^* N_t) = U^*(N_t)$, which confirms that $\upsilon(e(N)|N)$ satisfies the same Bellman equation as $U^*(N)$. QED.

Remark: The analysis ignores tradeability of permanent emissions rights, but in the representative-agent economy, net trades among households would be zero.

Remark: Permanent emission rights would be created by government and be rights to the revenue from one unit of emissions per-period, forever, which would effectively privatize the public revenue stream generated by a cap. Permanent emission rights would mean the government at $t = 0$ binds all future governments. Permanence could fail if future governments were to change the level of the cap or were to tax or reallocate emission rights, which may benefit a majority of voters. With heterogeneous agents, for instance, dynasties with a heritable preference to have more children would over time form a relatively impoverished majority that would gain from redistribution.