

Online Appendix to The First-Order Approach to Merger Analysis

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August 22, 2013

A. Model with General Strategies

Consider a market with N firms denoted $i = (1, \dots, n)$. Firm i produces m_i goods, and chooses a strategy vector $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \dots, \sigma_{im_i})$ from a m_i -dimensional strategy space. As in the text where strategies are restricted to be prices we allow only one strategic variable per-product. The strategy space could be the space of product prices or product quantities or a different choice parameter, but we require that each combination of strategies $\sigma = (\sigma_1, \dots, \sigma_n)$ generate a unique demand-equilibrium, defined as a vector of prices $\mathbf{P}(\sigma) = (P_{11}, \dots, P_{1m_1}, \dots, P_{n1}, \dots, P_{nm_n})$ and quantities $\mathbf{Q}(\mathbf{P}(\sigma)) = (Q_{11}, \dots, Q_{1m_1}, \dots, Q_{n1}, \dots, Q_{nm_n})$. Uniqueness places some restriction on the possible space of strategies: if a firm's strategy specifies a twister rather than a shifter (Bresnahan, 1982) of a firm's supply function in the Klemperer and Meyer (1989) model, this will not tie down equilibrium prices and quantities.¹ We additionally require that the map from strategies to quantities be locally invertible and that the demand system be well-behaved and twice-differentiable.

To allow for the possibility of non-Nash equilibria, we permit firms to conjecture changes in other firms' strategies in response to changes in their own (in the spirit of Bowley, 1924). A firm believes that when it changes its strategies, σ_i , its

¹While it may seem that there are reasonable cases in which strategies do not imply unique prices and quantities, such cases are rarely applied in the industrial organization literature. Multiple equilibria would not create a problem if the firms agreed (and were correct) on which equilibrium would occur.

competitors will change their strategies, σ_{-i} , by $\frac{\partial \sigma_{-i}}{\partial \sigma_i}$. Therefore, the total effect of a change in own strategies on a vector of interest is the sum of the direct (partial) effect and the indirect effect working through the effect on others' strategies: $\frac{d\mathbf{A}}{d\sigma_i} \equiv \frac{\partial \mathbf{A}}{\partial \sigma_i} + \left(\frac{\partial \mathbf{A}}{\partial \sigma_{-i}}\right)^T \frac{\partial \sigma_{-i}}{\partial \sigma_i}$. In the case of a Nash equilibrium in σ we have $\frac{d\mathbf{A}}{d\sigma_i} = \frac{\partial \mathbf{A}}{\partial \sigma_i}$ since $\frac{\partial \sigma_{-i}}{\partial \sigma_i} = \mathbf{0}$.

Premerger

Firm i 's profit π_i depends on both its strategy vector and its competitors' strategies:

$$\pi_i = (\mathbf{P}_i(\sigma))^T \mathbf{Q}_i(\mathbf{P}(\sigma)) - \mathbf{C}_i(\mathbf{Q}_i(\mathbf{P}(\sigma))),$$

where \mathbf{C} and \mathbf{Q} are the (vector-valued) cost and demand functions and $\mathbf{P}(\sigma)$ is the demand-equilibrium price vector generated by σ . For brevity we write \mathbf{P}_i for $\mathbf{P}_i(\sigma)$ and \mathbf{Q}_i for $\mathbf{Q}_i(\mathbf{P}(\sigma))$ and $\mathbf{m}\mathbf{c}_i$ for the vector of marginal costs. The firm's first-order conditions premerger can be written as:

$$(1) \quad \mathbf{f}_i(\sigma^*) \equiv \underbrace{-\left(\frac{d\mathbf{Q}_i^{-1}}{d\sigma_i}\right)^T \left(\frac{d\mathbf{P}_i}{d\sigma_i}\right)^T \mathbf{Q}_i}_{\text{Generalized inverse hazard rate/Cournot distortion}} - \underbrace{(\mathbf{P}_i - \mathbf{m}\mathbf{c}_i)}_{\text{Markup}} = \mathbf{0},$$

This formula is a natural generalization of the standard, single-product oligopoly first-order condition: the markup on each product is equated to the appropriate partial inverse hazard rate or Cournot distortion. Under single-product Nash-in-prices conduct this is the partial inverse hazard rate of demand, $\frac{Q_i}{\partial Q_i}$; for multi-product firms it generalizes to the inverse of the Slutsky matrix limited to the firm's products multiplied by that firm's quantities, $\left(\frac{\partial \mathbf{Q}_i^{-1}}{\partial \mathbf{P}_i}\right)^T \mathbf{Q}_i$. Under single-product Nash-in-quantities conduct it is the partial slope of price multiplied by quantity, $\frac{\partial P_i}{\partial Q_i} Q_i$, which generalizes to that firm's portion of the matrix of derivatives of inverse demand multiplied by its quantities, $\left(\frac{\partial \mathbf{P}_i}{\partial \mathbf{Q}_i}\right)^T \mathbf{Q}_i$. Note that the Nash-in-prices and Nash-in-quantities formulae differ only in what actions of *other* firms they hold fixed in the inversion and thus, as we discuss below, a firm playing either strategy can think of itself as choosing either prices or quantities.

For general conduct, where strategies can be arbitrary, both the price and quantity matrices come into play. It is the matrix “ratio” of these that is the generalized version of the inverse quantity slope or direct price slope.

Incentives created by a merger

In studying the impact of a merger on firms’ incentives, it is useful to define a generalization of the notion of a diversion ratio as employed in previous work on UPP and in the informal discussion in the introduction. In doing so we use the notation d^M to represent the “postmerger” total derivative, in which any within-merged-firm conjectures are taken to be zero. That is for any function \mathbf{A} , after i and j merge, $\frac{d^M \mathbf{A}_i}{d\sigma_i} = \frac{\partial \mathbf{A}_i}{\partial \sigma_i} + \frac{\partial \mathbf{A}_i}{\partial \sigma_{-ij}} \frac{\partial \sigma_{-ij}}{\partial \sigma_i}$. Then we can define the diversion ratio matrix for a pair of merging firms as

$$\mathbf{D}_{ij}^\sigma \equiv - \left(\frac{d^M \mathbf{Q}_i}{d\sigma_i} \right)^T \left(\frac{d^M \mathbf{Q}_j}{d\sigma_i} \right)^T.$$

That is, rather than simply being the ratio of the quantity gained by the former rival’s products to that lost by one’s own in response to a change in strategy, it is the ratio of these in the matrix sense. Furthermore, note that it is this matrix ratio *holding fixed the strategy of the merger partner and allowing all other strategies to adjust as they are expected to in equilibrium*. We include a superscript σ to indicate the strategy under which the diversion ratio is taken.

Definition A1. *Let a premerger equilibrium be defined by $\mathbf{f}(\sigma^*) = 0$ and a post-merger equilibrium be defined by $\mathbf{h}(\sigma^M) = 0$, where \mathbf{f} and \mathbf{h} are normalized to be quasi-linear in marginal cost (and price). Then we define $\mathbf{g} \equiv \mathbf{h}(\sigma^*) - \mathbf{f}(\sigma^*)$ to be the Generalized Strategic Pressure (GeSP) on that strategy σ created by the merger.*

Thus GeSP is the change in the first-order condition at the premerger strategies. It holds fixed the firms’ strategy space and conjectures about other firms’ reactions, thus capturing only the unilateral effects of a merger. The value of GeSP is given in the following proposition.

Proposition A1. *The GeSP on firm i 's strategy generated by a merger between firms i and j is*

$$(2) \quad \mathbf{g}_i(\boldsymbol{\sigma}) = \mathbf{D}_{ij}^{\sigma}(\mathbf{P}_j - \mathbf{mc}_j) - \left(\frac{d^M \mathbf{Q}_i}{d\boldsymbol{\sigma}_i} \right)^T \left(\frac{d^M \mathbf{P}_j}{d\boldsymbol{\sigma}_i} \right)^T \mathbf{Q}_j \\ - \Delta \left(\left(\frac{d\mathbf{Q}_i}{d\boldsymbol{\sigma}_i} \right)^T \left(\frac{d\mathbf{P}_i}{d\boldsymbol{\sigma}_i} \right)^T \right) \mathbf{Q}_i.$$

Here $\Delta(\cdot)$ denotes the change from pre to postmerger value of its argument; the change is due to the merger partner's strategy no longer reacting.

The first and second terms of equation (2) are the changes in firm j 's profits induced by a sale by firm i (caused by changing firm i 's strategy). Postmerger firm i takes into account the effect of a change in its strategies on the quantities (first term) and the prices (second term) of its merging partner's products. The last term is the change in firm i 's marginal profit due to the end of accommodating reactions: once the firms have merged, the firm no longer anticipates an accommodating reaction from its merger partner.

Proof of Proposition A1. After a merger of firms i and j , the newly formed firm takes into account the effect of $\boldsymbol{\sigma}_i$ on π_j and no longer expects $\boldsymbol{\sigma}_j$ to react to $\boldsymbol{\sigma}_i$ since the two are chosen jointly. The merged firm's first-order condition with respect to $\boldsymbol{\sigma}_i$ can be written:

$$\mathbf{0} = -(\mathbf{P}_i - \mathbf{mc}_i(\mathbf{Q}_i)) - \left(\left(\frac{d\mathbf{Q}_i}{d\boldsymbol{\sigma}_i} \right)^T - \frac{\partial \mathbf{Q}_i}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)^{-1} \left(\frac{d\mathbf{P}_i}{d\boldsymbol{\sigma}_i} - \frac{\partial \mathbf{P}_i}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)^T \mathbf{Q}_i \\ - \left(\left(\frac{d\mathbf{Q}_i}{d\boldsymbol{\sigma}_i} \right)^T - \frac{\partial \mathbf{Q}_i}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)^{-1} \\ \left[\left(\frac{d\mathbf{P}_j}{d\boldsymbol{\sigma}_i} - \frac{\partial \mathbf{P}_j}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)^T \mathbf{Q}_j + \left(\frac{d\mathbf{Q}_j}{d\boldsymbol{\sigma}_i} - \frac{\partial \mathbf{Q}_j}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)^T (\mathbf{P}_j - \mathbf{mc}_j(\mathbf{Q}_j)) \right]$$

Subtracting the pre-merger first-order conditions, the $\mathbf{f}_i(\boldsymbol{\sigma})$ from Equation (1), from these first-order conditions gives the Generalized Pricing Pressure, $\mathbf{g}(\boldsymbol{\sigma})$, so that

post merger $\mathbf{f}(\boldsymbol{\sigma}) + \mathbf{g}(\boldsymbol{\sigma}) = \mathbf{0}$. This is given by:

$$\begin{aligned} \mathbf{g}_i(\boldsymbol{\sigma}) = & \\ & - \left(\left(\frac{d\mathbf{Q}_i}{d\boldsymbol{\sigma}_i} \right)^\top - \frac{\partial \mathbf{Q}_i}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)^{-1} \cdot \\ & \left(\left(\frac{d\mathbf{P}_j}{d\boldsymbol{\sigma}_i} - \frac{\partial \mathbf{P}_j}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)^\top \mathbf{Q}_j + \left(\frac{d\mathbf{Q}_j}{d\boldsymbol{\sigma}_i} - \frac{\partial \mathbf{Q}_j}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)^\top (\mathbf{P}_j - \mathbf{m}c_j(\mathbf{Q}_j)) \right) \\ & - \left(\left(\left(\frac{d\mathbf{Q}_i}{d\boldsymbol{\sigma}_i} \right)^\top - \frac{\partial \mathbf{Q}_i}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)^{-1} \left(\frac{d\mathbf{P}_i}{d\boldsymbol{\sigma}_i} - \frac{\partial \mathbf{P}_i}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)^\top - \left(\frac{d\mathbf{Q}_i}{d\boldsymbol{\sigma}_i} \right)^{-1} \right)^\top \left(\frac{d\mathbf{P}_i}{d\boldsymbol{\sigma}_i} \right)^\top \right) \mathbf{Q}_i \end{aligned}$$

Using the convention $\frac{d^M \mathbf{Q}_i}{d\boldsymbol{\sigma}_i} = \left(\left(\frac{d\mathbf{Q}_i}{d\boldsymbol{\sigma}_i} \right)^\top - \frac{\partial \mathbf{Q}_i}{\partial \boldsymbol{\sigma}_j} \frac{\partial \boldsymbol{\sigma}_j}{\partial \boldsymbol{\sigma}_i} \right)$ and similarly for price, we get the formulation in Proposition A1.

If $\boldsymbol{\sigma}$ is a strategy other than price, then we can still think of the two merging firms as setting prices as long as the merging firms' strategies generate unique prices – no two strategy combinations generate the same set of prices. This, of course, requires that the map from strategies to prices be invertible.² Assuming this is true, we can always re-conceptualize the firm's problem as a choice of prices. A firm's conjectures as well as other firms' non-price choosing behavior can be viewed as jointly forming a conjecture on how other firms will adjust price. For example, if firms are actually choosing quantities, we can think of them as choosing prices and expecting the other firms to adjust their prices so as to keep their quantities fixed.³ The advantage of this approach is that it has a clearer concordance with UPP and the quantitative changes in price that impact welfare. In this subsection we pursue this dual strategy.

If strategies are prices then the second term on the right hand side of equation

²A standard condition to guarantee this is that $\sigma \in \mathbb{R}^{\sum_i m_i}$ and $\frac{\partial P}{\partial \sigma}$ is either globally a P-matrix (a matrix with all positive principal minors, see Hicks (1939)) or globally the negative of a P-matrix. While this may seem a strong condition, it is trivially satisfied in many contexts; for example, if the equilibrium is Cournot (Nash-in-quantities) and consumers have quasi-linear utility then this follows directly from the fact that the Slutsky conditions imply that the Slutsky matrix $\frac{\partial Q}{\partial P}$ and thus its inverse $\frac{\partial P}{\partial Q}$ is negative definite globally, as all negative definite matrices are the negative of P-matrices. Any other sufficient condition for invertibility would be equally suitable.

³See the Nash-in-quantities example in Section IID for a fleshing out of this idea.

(2) vanishes because firm i 's prices do not change firm j 's prices. GeSP simplifies to *Generalized Pricing Pressure* (GePP) found in the main text:

$$\mathbf{g}_i(\mathbf{P}) = \tilde{\mathbf{D}}_{ij}(\mathbf{P}_j - \mathbf{mc}_j) - \Delta \left(\left(\frac{d\mathbf{Q}_i}{d\mathbf{P}_i} \right)^{-1} \right)^{\mathbf{T}} \mathbf{Q}_i.^4$$

Here, $\tilde{\mathbf{D}}_{ij} \equiv \mathbf{D}_{ij}^P$ is the diversion matrix *holding fixed the price of the merger partner* and allowing *all other firms' prices to adjust as they are expected to in equilibrium*. This diversion ratio is the general conjectures and matrix equivalent of the commonly used ratio of the derivatives of demand.

B. General formula under Cournot

In this appendix we provide a general formula for GePP under Cournot competition and discuss pass-through in our simple symmetric, merger to monopoly example. In a (differentiated products) Cournot equilibrium each firm takes competitors' quantities as fixed. Instead of thinking of each firm as setting quantity, we can think of it as setting price with the expectation that other firms will adjust their prices so as to keep their quantities fixed. Using single-product firms for simplicity, premerger we have the first-order condition

$$Q_i + \frac{\partial Q_i}{\partial \mathbf{P}} \frac{d\mathbf{P}}{d\mathbf{P}_i} (P_i - mc_i) = 0.$$

We have $\frac{dP_i}{dP_i} = 1$ and can pin down $\frac{\partial \mathbf{P}_{-i}}{\partial \mathbf{P}_i}$ because

$$\frac{\partial Q_{-i}}{\partial \mathbf{P}_i} + \frac{\partial Q_{-i}}{\partial \mathbf{P}_{-i}} \frac{d\mathbf{P}_{-i}}{d\mathbf{P}_i} = 0,$$

which implies

$$\frac{d\mathbf{P}_{-i}}{d\mathbf{P}_i} = - \left(\frac{\partial Q_{-i}}{\partial \mathbf{P}_{-i}} \right)^{-1} \frac{\partial Q_{-i}}{\partial \mathbf{P}_i}.$$

⁴Note that in the single-product firm case this is exactly Equation (2) from the introduction.

This gives us a premerger condition of

$$f_i(\mathbf{P}) = -(P_i - mc_i) - \frac{Q_i}{\frac{\partial Q_i}{\partial P_i} - \frac{\partial Q_i}{\partial \mathbf{P}_{-i}} \left(\frac{\partial \mathbf{Q}_{-i}}{\partial \mathbf{P}_{-i}} \right)^{-1} \frac{\partial \mathbf{Q}_{-i}}{\partial P_i}} = 0.$$

After the firms merger, firm i starts taking firm j 's price as given, so, following the same logic as above, the GePP is

$$g_i(\mathbf{P}) = - \underbrace{\frac{\frac{\partial Q_j}{\partial P_i} - \frac{\partial Q_j}{\partial \mathbf{P}_{-ij}} \left(\frac{\partial \mathbf{Q}_{-ij}}{\partial \mathbf{P}_{-ij}} \right)^{-1} \frac{\partial \mathbf{Q}_{-ij}}{\partial P_i}}_{\text{Diversion Ratio}} (P_j - mc_j) - Q_i \underbrace{\left(\frac{1}{\frac{\partial Q_i}{\partial P_i} - \frac{\partial Q_i}{\partial \mathbf{P}_{-ij}} \left(\frac{\partial \mathbf{Q}_{-ij}}{\partial \mathbf{P}_{-ij}} \right)^{-1} \frac{\partial \mathbf{Q}_{-ij}}{\partial P_i}} - \frac{1}{\frac{\partial Q_i}{\partial P_i} - \frac{\partial Q_i}{\partial \mathbf{P}_{-i}} \left(\frac{\partial \mathbf{Q}_{-i}}{\partial \mathbf{P}_{-i}} \right)^{-1} \frac{\partial \mathbf{Q}_{-i}}{\partial P_i}} \right)}_{\text{End of Accommodating Reactions}}.$$

References

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